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ON SUPERSYMMETRIC WILSON LOOPS

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with Drukker, Giombi, Ricci

WILSON LOOPS PLAY AN IMPORTANT ROLE IN THE AdS/CFT CORRESPONDENCE

- "SOURCES" OF FUNDAMENTAL STRINGS
- EXACT RESULTS AT STRONG COUPLING

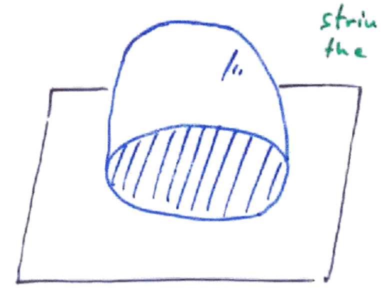
perturbation theory

matrix model

$= \langle \mathrm{Tr}\, e^{M} \rangle$

string in the bulk

$=$



LESS UNDERSTOOD THAN LOCAL OPERATORS ALBEIT MUCH PROGRESS IN THE PAST FEW YEARS

(Giant Wilson loops, bubbling geometries...)

OUTLINE

DEFINITION OF WILSON LOOPS IN $\mathcal{N}=4$ SYM_4

- SUSY ANALYSIS
- A NEW FAMILY OF LOOPS
- EXAMPLES

PERTURBATIVE ASPECTS

- A RELATION BETWEEN LOOPS IN $\mathcal{N}=4$ SYM_4 & LOOPS IN YM_2

STRING PICTURE

- PSEUDO-HOLOMORPHIC SURFACES & GENERALIZED CALIBRATIONS

I) DEFINITION AND SUSY ANALYSIS

$\mathcal{N}=4$ SYM_4

Euclidean

$$W_R(C) = \frac{1}{\dim R} Tr_R\, P\, e^{\oint_C d\tau (i A_\mu \dot{x}^\mu + \phi_I \dot{y}^I)}$$

gauge field $\mu=1,\dots 4$

scalars $I=5,\dots 10$

$R = \square$

$C = \begin{cases} x^\mu & \text{closed curve in } \mathbb{R}^4 \text{ (gauge invariance)} \\ y^I & \text{arbitrary curve in } \mathbb{R}^6 \end{cases}$

SUSY

10d $\mathcal{N}=1$ notation

$Q: \begin{cases} \delta_Q A_\mu = \bar{\Psi}\Gamma_\mu \epsilon_0 \\ \delta_Q \phi_I = \bar{\Psi}\Gamma_I \epsilon_0 \end{cases}$ $\qquad S: \begin{cases} \delta_S A_\mu = \bar{\Psi}\Gamma_\mu x^\nu \Gamma_\nu \epsilon_1 \\ \delta_S \phi_I = \bar{\Psi}\Gamma_I x^\nu \Gamma_\nu \epsilon_1 \end{cases}$

ϵ_0, ϵ_1 16-components MW spinors

Focus on Q's for the moment:

$$\delta_Q W = 0 \implies (i\Gamma^\mu \dot{x}_\mu + \Gamma^I \dot{y}_I)\epsilon_0 = 0$$

8 solutions if

$$(i\Gamma^\mu \dot{x}_\mu + \Gamma^I \dot{y}_I)^2 = 0 \underset{\{\Gamma^\mu,\Gamma^I\}=0}{\implies} \dot{x}^2 - \dot{y}^2 = 0$$

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Solutions to $\dot{x}^2 - \dot{y}^2 = 0$

1) if we were in Minkowski: $\dot{x}^2 + \dot{y}^2 = 0$

$\dot{y}^I = 0 \quad \forall I \qquad \dot{x}^2 = 0$ light-like loops

effectively non-susy

2) remain in Euclidean

$\dot{y}^I = |\dot{x}|\,\theta^I(\tau) \qquad \theta^I(\tau)\,\theta^I(\tau) = 1$

$$W(C) = \frac{1}{N}\,\mathrm{Tr}\,P\,e^{\oint d\tau\,(i A_\mu \dot{x}^\mu + |\dot{x}|\,\theta^I(\tau)\,\phi^I)}$$

only LOCALLY susy!

for GLOBAL susy $\theta^I = \text{constant}$

$\Rightarrow \ddot{x}^\mu = 0$ straight line

$$W(\text{line}) = \frac{1}{N}\,\mathrm{Tr}\,P\,e^{\int_{-\infty}^{+\infty} dt\,(i A_1 + \phi_5)}$$

Preserves separately 8 Q's & 8 S's ½ BPS

Isometries $\begin{cases} SO(5,1) \longrightarrow SO(2,1) \times SO(3) \\ SO(6) \longrightarrow SO(5) \end{cases}$

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3) Zarembo's loops [Zarembo '02]

extract τ-dependence from projector

$(i \Gamma_\mu \dot{x}^\mu + |\dot{x}|\,\theta^I(\tau)\,\Gamma^I)\,\epsilon_0 = 0$

$x_\mu \mapsto x_\mu M^I_\mu \qquad \theta^I(\tau) = M^I_\mu \dfrac{\dot{x}_\mu}{|\dot{x}|}$

6x4 constant matrix

$M^I_\mu M^I_\nu = \delta_{\mu\nu}$

$\dot{x}^\mu (i \Gamma_\mu + \Gamma^I M^I_\mu)\,\epsilon_0 = 0$ 4 eqs for ϵ_0

Define 5 pairs of fermionic ops

$a_\mu \quad a^+_\mu$

$a_* \quad a^+_*$

$\{a^+_M, a^N\} = \delta^N_M \qquad M = (\mu, *)$

$2^5 = 32$ states $\to$ Spin(10)

Eqs become $a_\mu |\epsilon_0\rangle = 0$

- $x^\mu \in \mathbb{R}^4$ $\quad |\epsilon_+\rangle = |-----\rangle$

  $|\epsilon_-\rangle = a^+_* |-----\rangle = |----+\rangle$

  only one has chirality of $|\epsilon_0\rangle$ 1 Q

- $x_4 = 0 \quad a_i |\epsilon_0\rangle = 0 \qquad |---\pm\pm\rangle$ 2 Q's
- $x^\mu \in \mathbb{R}^2$ 4 Q's
- $x^\mu \in \mathbb{R}^1$ 8 Q's

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Straight line : Q's & S's separately
Zarembo's loops: only Q's

Turn out to have trivial expectation values

$$\langle W \rangle = 1$$

Allow now for loops to preserve combinations of Q's & S's

4) Circular loop

$$x^\mu = (\cos\tau, \ \sin\tau, 0, 0) \qquad \theta^I = \delta^{I5}$$

$$\delta_{Q+S} W = 0 \implies (i\Gamma_\mu \dot{x}^\mu + \Gamma_5)(\epsilon_0 + x^\nu \Gamma_\nu \epsilon_1) = 0$$

$$\Gamma_5 \epsilon_0 = i \Gamma_1 \Gamma_2 \epsilon_1 \qquad \text{½ BPS}$$

we'll see that in this case

$$\langle W \rangle \neq 1$$

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Can we construct other such loops?

$$x^\mu \in S^3 \subset \mathbb{R}^4 \qquad x^2 = 1$$

use the right 1-forms of $SU(2)$

$$U(x) = i x^i \tau_i + x^4 \mathbb{1} \qquad \sigma_i^R = -i \, \mathrm{Tr}(\tau_i \, U^\dagger dU)$$

$$\sigma_1^R = 2\left(x^2 dx^3 - x^3 dx^2 + x^4 dx^1 - x^1 dx^4\right)$$
$$\sigma_2^R = 2\left(x^3 dx^1 - x^1 dx^3 + x^4 dx^2 - x^2 dx^4\right)$$
$$\sigma_3^R = 2\left(x^1 dx^2 - x^2 dx^1 + x^4 dx^3 - x^3 dx^4\right)$$

then it is natural to define

$$W(C) = \frac{1}{N} \mathrm{Tr} \, P \, e^{\oint (iA + \frac{1}{2} \sigma_i^R M_I^i \phi^I)}$$

explicit computations $\quad M_5^1 = M_6^2 = M_7^3 = 1$

coupling to 3 scalars only

$$SU(4)_R \longrightarrow SU(2)_A \times SU(2)_B$$

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SUSY

From the susy variation $\delta_{Q+S} W = 0$ one gets for a generic curve

$(1+\Gamma_*)\,\epsilon_1 = 0$ $\qquad \Gamma_* = \Gamma_{1234}$

$\Gamma_5\,\epsilon_0 = i\,\Gamma_{12}\,\epsilon_1^-$

$\Gamma_6\,\epsilon_0 = i\,\Gamma_{23}\,\epsilon_1^-$ $\qquad \epsilon_1^- = \frac{1-\Gamma_*}{2}\,\epsilon_1$

$\Gamma_7\,\epsilon_0 = i\,\Gamma_{31}\,\epsilon_1^-$

↑ generators of $SU(2)_R \subset SO(4)$

$\Rightarrow \quad \epsilon_1^+ = \epsilon_0^+ = 0$ $\qquad$ chiral constraints

$(\Gamma_{12} + \Gamma_{67})\,\epsilon_1^- = 0$

$(\Gamma_{23} + \Gamma_{75})\,\epsilon_1^- = 0$ $\qquad$ only 2 are indep.

$(\Gamma_{31} + \Gamma_{56})\,\epsilon_1^- = 0$ $\qquad \epsilon_0^- = \epsilon_0^-(\epsilon_1^-)$

A generic curve on S^3 preserves 2 supercharges $\qquad$ 1/16 BPS

This number is enhanced if the curve has some symmetries

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SPECIAL CASES

- Equator of S^3 $\qquad x^\mu = (\cos\tau, \sin\tau, 0, 0)$

  $\sigma_1^R = 0 = \sigma_2^R \qquad \sigma_3^R = |\dot{x}|$

  couples only to a scalar

  1 non-chiral constraint $\qquad$ 1/2 BPS

- Hopf fibers $\qquad x^\mu = x^\mu(\theta, \phi, \psi)$

  non-intersecting circles all coupling to the same scalar $\qquad$ 1/4 BPS

- Latitude on S^2 $\qquad x^\mu = (\cos\theta_0 \cos\tau, \cos\theta_0 \sin\tau, \sin\theta_0, 0)$

  couples to 3 scalars $\qquad$ 1/4 BPS

  2 non-chiral constraints

- Loops on a great $S^2 \subset S^3$ $\qquad x^4 = 0$ $\qquad$ 1/8 BPS

- Zarembo's loop $\qquad x^4 \simeq 1$

  infinitesimal loops near a point

  (explains why $\langle W \rangle = 1$ ...)

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II) LOOPS IN PERTURBATION THEORY

Taylor expand

$$\langle W \rangle = \frac{1}{N} \langle \mathrm{Tr}\, P\, e^{\oint d\tau (i A_\mu \dot{x}^\mu + |\dot{x}| \theta^I \phi^I)} \rangle$$

$$= 1 + \frac{\mathrm{Tr}(T^a T^b)}{N} \int_0^{2\pi} d\tau_1 \int_0^{\tau_1} d\tau_2 \; \times$$

$$\times \left\{ -\dot{x}_1^\mu \dot{x}_2^\nu \langle A_\mu^a(x_1) A_\nu^b(x_2) \rangle + |\dot{x}_1||\dot{x}_2| \theta^I \theta^J \langle \phi_I^a(x_1) \phi_J^b(x_2) \rangle \right\}$$

In Feynman gauge

$$\langle A A \rangle = \frac{g^2}{4\pi^2} \frac{\delta_{\mu\nu} \delta^{ab}}{(x_1 - x_2)^2} \qquad \langle \phi \phi \rangle = \frac{g^2}{4\pi^2} \frac{\delta_{IJ} \delta^{ab}}{(x_1 - x_2)^2}$$

$$\Rightarrow \{ \cdots \} = \frac{g^2 \delta^{ab}}{4\pi^2} \frac{-\dot{x}_1 \cdot \dot{x}_2 + |\dot{x}_1||\dot{x}_2|}{|x_1 - x_2|^2}$$

$$= \begin{cases} 0 & \text{line} \\ \dfrac{g^2 \delta^{ab}}{8\pi^2} & \text{circle} \end{cases}$$

Then $\langle W(\text{line}) \rangle = 1$

but $\langle W(\text{circle}) \rangle \neq 1$ because of a conformal anomaly [Drukker-Gross '00]

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Graphs with internal vertices cancel for circle

$+ \quad = 0 \qquad \lambda^2$ order

conjectured to hold at all orders (maybe also proven...)

$\langle W(\text{circle}) \rangle =$ sum over ladders

all propagators contribute the same factor

$\longrightarrow$ combinatorial problem [Erickson-Semenoff-Zarembo '00]

$d=0$ Gaussian Matrix model

$$\langle W(\text{circle}) \rangle = \frac{1}{N} \langle \mathrm{Tr}\, e^M \rangle_{\text{m.m.}} =$$

$$= \frac{1}{Z} \int [dM] \frac{1}{N} \mathrm{Tr}\, e^M \, e^{-\frac{2N}{\lambda} \mathrm{Tr} M^2} =$$

$$\underset{N \to \infty}{=} \frac{2}{\pi \lambda} \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} dm \sqrt{\lambda - m^2}\, e^m = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \xrightarrow[\lambda \to \infty]{} e^{\sqrt{\lambda}}$$

LOOPS ON A GREAT S^2 AND YM_2

$x^4=0$ $\quad \sigma_i^R = 2\,\epsilon_{ijk}\,x^j dx^k \quad i=1,\dots,3$

the combined "gauge + scalar" propagator is

$$\Delta^{ab}_{ij}(x-y) = \frac{g^2\delta^{ab}}{4\pi^2}\left(\frac{1}{2}g_{ij} - \frac{(x-y)_i(x-y)_j}{(x-y)^2}\right)$$

Feynman gauge as before $\xi=+1$

- not generically a constant
- dimensionless (rather than dimension 2)

Change coordinates

$$x_i \longmapsto z,\bar z \qquad ds^2 = \frac{4\,dz\,d\bar z}{(1+|z|^2)^2}$$

then

$$\Delta^{ab}_{ij} \longrightarrow \Delta^{ab}_{zz},\ \Delta^{ab}_{\bar z\bar z},\ \Delta^{ab}_{z\bar z}=0$$

with Δ^{ab}_{zz} annihilated by the Laplacian on S^2 in the gauge $\xi=-1$

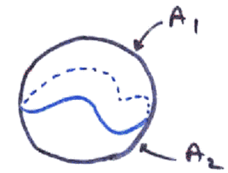
4d "gauge + scalar" $\xi=+1$ $\longrightarrow$ 2d vector $\xi=-1$

The equivalence of the propagators suggests the conjecture

Wilson loops on the great S^2 in $N=4$ SYM $\longleftrightarrow$ Wilson loops of non-susy YM on S^2

Start with comparing at weak coupling

$$\langle W\rangle \underset{\lambda\to0}{\simeq} 1+\frac{\lambda}{2}\frac{A_1A_2}{(4\pi)^2}$$



for any loop on the S^2

The total perturbative contribution to a Wilson loop in YM on $\mathbb{R}^2$ is

$$\langle W\rangle = \frac{1}{N}L^1_{N-1}\left(-\frac{g^2}{4\pi}A_1\right)\exp\left(\frac{g^2}{8\pi}A_1\right)$$

[Staudacher-Krauth '97]

generalize to S^2 by $A_1 \longrightarrow \frac{A_1A_2}{4\pi}$

$$\xrightarrow[N\to\infty]{} \frac{4\pi}{\sqrt{\lambda A_1A_2}}\, I_1\left(\frac{\sqrt{\lambda A_1A_2}}{2\pi}\right)$$

$\longrightarrow 1+\frac{\lambda}{2}\frac{A_1A_2}{(4\pi)^2} \quad \lambda\to0$

$\longrightarrow \exp\left(\frac{\sqrt{\lambda A_1A_2}}{2\pi}\right) \quad \lambda\to\infty$

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In some examples we can test this at all orders or for $\lambda \to \infty$

Note: same result of the matrix model for the circle modulo

$$\lambda \to \lambda' = \lambda \frac{A_1 A_2}{4\pi^2}$$

- subsector of a conformal theory invariant under area preserving diffeos

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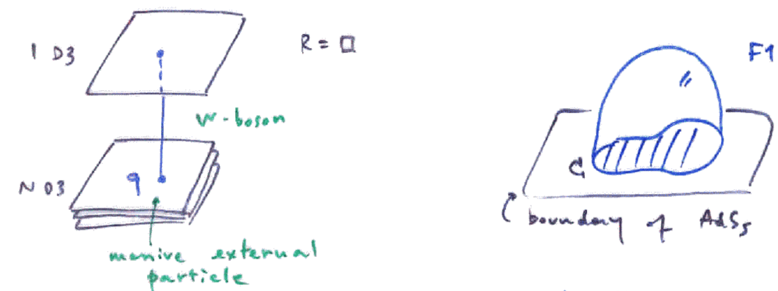
III) WILSON LOOPS IN THE BULK



gauge theory $\quad \mathcal{A} \underset{m\to\infty}{\simeq} e^{-m\,\ell(C)} \langle W(C)\rangle$

string theory $\quad \mathcal{A} = \int_{X|=C} [dX]\, e^{-\sqrt{\lambda}\, S(X)} \underset{\lambda\to\infty}{\simeq} e^{-\sqrt{\lambda}\,\text{Area}}$

$T_{F1} \sim \sqrt{\lambda}$ from AdS/CFT dictionary

$$\langle W(C)\rangle \simeq e^{-[\sqrt{\lambda}\,\text{Area} - m\,\ell(C)]}$$

Then at large λ all reduces to finding the minimal surface with appropriate boundary conditions

Note: $\langle W(C)\rangle \underset{\lambda\to\infty}{\simeq} e^{\alpha\sqrt{\lambda}}$

α positive constant

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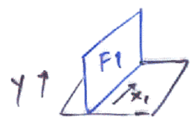
• Straight line



$$ds^2 = \frac{dy^2 + d\vec{x}^{\,2}}{y^2} \qquad WS = (y, x_1) \qquad x_2 = x_3 = x_4 = 0$$

$$\gamma_{\alpha\beta} = G_{\mu\nu}\,\partial_\alpha X^\mu \partial_\beta X^\nu$$

$$S = \frac{1}{2\pi\alpha'}\int dy\, dx_1 \sqrt{\gamma} = \frac{1}{2\pi\alpha'}\int_0^T dx_1 \int_{y_0}^{\infty} dy\, \frac{1}{y^2} = \frac{T\sqrt{\lambda}}{2\pi y_0}$$

After regularizing $\langle W(\text{line})\rangle = 1$

• Circle

$$ds^2 = \frac{1}{y^2}\left(dy^2 + dr^2 + r^2 d\varphi^2 + dx_3^2 + dx_4^2\right)$$

$$WS = (y, \varphi) \qquad r^2 + y^2 = 1 \qquad x_3 = x_4 = 0$$

$$S = \frac{1}{2\pi\alpha'}\int dy\, d\varphi \sqrt{\gamma} = \sqrt{\lambda}\int_{y_0}^{1} dy\, \frac{r}{y^2}\sqrt{1 + r'^2}$$

$$= \sqrt{\lambda}\left(\frac{1}{y_0} - 1\right) \qquad \text{finite remnant}$$

$$\langle W(\text{circle})\rangle = e^{\sqrt{\lambda}}$$

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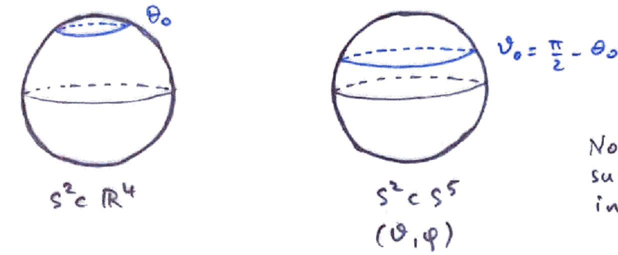
• Latitude on S^2



Now the string surface moves in $S^2 \subset S^5$!

$$ds^2 = \frac{1}{y^2}\left(dy^2 + dr^2 + r^2 d\phi^2 + dx_3^2\right) + d\vartheta^2 + \sin^2\vartheta\, d\varphi^2$$

$$y(\sigma) = \sin\theta_0 \tanh\sigma \qquad r(\sigma) = \frac{\sin\theta_0}{\cosh\sigma} \qquad x_3 = \cos\theta_0$$

$$\sin\vartheta(\sigma) = \frac{1}{\cosh(\sigma_0 \pm \sigma)} \qquad \phi = \varphi + \pi = \tau$$

two solutions (1 stable & 1 unstable) wrapping opposite sides of the $S^2 \subset S^5$

$$S = \mp\sqrt{\lambda}\,\sin\theta_0$$

Consistent with the YM_2 conjecture

$$\lambda' = \lambda\,\frac{A_1 A_2}{4\pi^2} \qquad A_1 = 2\pi(1 + \cos\theta_0) \qquad A_2 = 2\pi(1 - \cos\theta_0)$$

Actually this example gives an all-order result (same propagator of 1/2 BPS circle modulo this rescaling)

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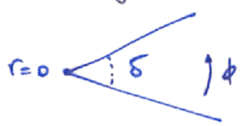
- Two longitudes at an angle

1/4 BPS

Can be solved by mapping to a single cusp on the plane (Zarembo type)



$$ds^2 = \frac{1}{y^2}\left(dy^2 + dr^2 + r^2 d\phi^2\right) + d\varphi^2$$

$$\text{WS} = \{r, \phi\} \qquad y = r\, v(\phi) \qquad \varphi = \varphi(\phi)$$

Then map back to S^2

$$S = -\frac{1}{\pi}\sqrt{\lambda\, \delta\,(2\pi - \delta)}$$

Again consistent with the conjecture

$$A_1 = 2\delta \qquad A_2 = 2(2\pi - \delta)$$

The matrix model interpretation is not obvious a priori (the rungs are not constant)

GENERALIZED CALIBRATIONS

k-form φ : $d\varphi \neq 0$

A generalized calibrated mfld is a k-dim mfld M which is a minimum of the energy functional

$$E(M) = \text{Vol}(M) - \int_M \varphi$$

A minimum of $E(M)$ is not necessarily a minimal-volume mfld (e.g. unstable latitude)

e.g.: D-branes with WZ terms (torsions, background fluxes...)

We have $A(\Sigma) = \int_\Sigma J$

$$J = J_0 + d\Omega$$

$$A(\Sigma) = \int_\Sigma J_0 + \int_\Sigma d\Omega$$

divergent part

then J_0 is the analogue of φ

CALIBRATIONS IN $AdS_5 \times S^5$

Pseudo-holomorphic surface

$$\Sigma\,(j^\alpha{}_\beta) \longrightarrow M\,(J^M{}_N)$$

$$V^M_\alpha = \partial_\alpha X^M - J^M{}_N\, j_\alpha{}^\beta\, \partial_\beta X^N = 0$$

generalization of Cauchy-Riemann eqs

$$\Sigma = M = \mathbb{R}^2 \qquad j = J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Solutions to this eq. are calibrated by J

$$P = \int_\Sigma \sqrt{g}\, g^{\alpha\beta}\, G_{MN}\, V^M_\alpha\, V^N_\beta \geq 0$$

$$= \text{Area}(\Sigma) - \int_\Sigma J \qquad J = \tfrac{1}{2} J_{MN}\, dX^M \wedge dX^N$$

fundamental 2-form

For a pseudo-holomorphic surface $P = 0$

$$\text{Area}(\Sigma) = \int_\Sigma J$$

If J is closed, Σ has vanishing (regularized) area

If not $\to$ "generalized calibration" non-vanishing area

Our loops live in a $AdS_4 \times S^2$ subspace

$$ds^2 = \frac{1}{z^2}\, dx^\mu dx^\mu + z^2\, dy^i dy^i \qquad i = 1, \dots 3$$

$$x^\mu x^\mu + z^2 = 1 \qquad y^i y^i = 1/z^2$$

one can define an almost complex structure on this subspace (similarities with $S^6 \subset \mathbb{R}^7$)

$$J = \begin{pmatrix} z^2 \begin{pmatrix} 0 & y_3 & -y_2 & -y_1 \\ -y_3 & 0 & y_1 & -y_2 \\ y_2 & -y_1 & 0 & -y_3 \\ y_1 & y_2 & y_3 & 0 \end{pmatrix} & z^2 \begin{pmatrix} -x_4 & -x_3 & x_2 \\ x_3 & -x_4 & -x_1 \\ -x_2 & x_1 & -x_4 \\ x_1 & x_2 & x_3 \end{pmatrix} \\ z^{-2} \begin{pmatrix} x_4 & -x_3 & x_2 & -x_1 \\ x_3 & x_4 & -x_1 & -x_2 \\ -x_2 & x_1 & x_4 & -x_3 \end{pmatrix} & z^2 \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix} \end{pmatrix}$$

$J^2 = -1$ (on the tangent space)

Our string solutions are calibrated w.r.t. J

Can prove that preserve same susies of the dual gauge theory operator and obey e.o.m.

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SUMMARY & OUTLOOK

- new family of Wilson loops on S^3
  $1/2$ - $1/16$ BPS
  non-trivial expectation value
- restricted to S^2 seem equivalent to the loops of YM_2
  "consistent truncation"
  area preserving diffeos
- string duals are calibrated surfaces

  - Rigorous proof of the YM_2 conjecture
  - loops on S^3 ?
  - Topological YM
  - D-brane picture