Branes in the Euclidean AdS_3

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hep-th/0112198

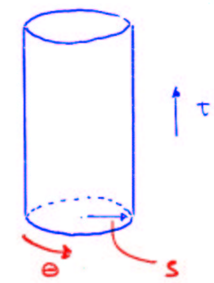
Motivation

- Branes on Minkowskian AdS_3
- Holography
- Generalization to other noncompact CFT with boundary (Cigar, Trumpet)

Plan

- Geometry, Semiclassics
- Quantum corrections from factorization constraint
- Analogues of the Cardy condition
- Outlook

AdS_3 :



"Wick-rotation" $\tau \to i\varsigma$

$\rightsquigarrow$ Euclidean $AdS_3 \equiv H_3^+$

Convenient model: Sp. of hermitian 2×2 matrices with $\det = 1$ and $\mathrm{Tr} > 0$.

Metric: $ds^2 = d\phi^2 + e^{2\phi} d\gamma d\bar\gamma$

if $h = \begin{pmatrix} e^{\phi} & e^{\phi}\bar\gamma \\ e^{\phi}\gamma & e^{\phi}\gamma\bar\gamma + e^{-\phi} \end{pmatrix}$

Conf. boundary: $\phi \to \infty$, parametrized by $\gamma, \bar\gamma$ ($\gamma = e^{\varsigma + i\theta}$)

$SL(2,\mathbb{C})$-symmetry acts via

$h \to g h g^+ \qquad g \in SL(2,\mathbb{C})$

③

To get a first idea of possible brane-configurations one may ask:

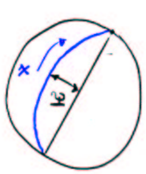
Which surfaces in H_3^+ preserve part of the $SL(2,\mathbb{C})$-symmetry?

Answer: Surfaces defd. by eqn. $\operatorname{tr} Ch = \text{const}$

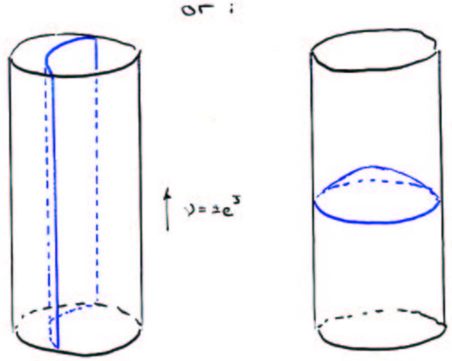
- $SU(2) \subset SL(2,\mathbb{C})$ preserved.
  $\to C = \mathcal{U}\mathcal{U}^\dagger \Rightarrow$ By $SL(2,\mathbb{C})$-Trsf. possible to map surface to surface $\operatorname{Tr} h = \text{const}$.
  $$\Leftrightarrow \quad e^{\phi}(1+|\gamma|^2) + e^{-\phi} = \text{const}.$$
  – Localized in the interior? –

- $SL(2,\mathbb{R}) \subset SL(2,\mathbb{C})$ preserved.
  $\Rightarrow C = \mathcal{U}\omega\mathcal{U}^\dagger$, $\omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By $SL(2,\mathbb{C})$-Trf:
  $$\operatorname{Tr} h\omega = \text{const}. \quad \Leftrightarrow \quad e^{\phi}(\gamma+\bar{\gamma}) = \text{const}.$$

"AdS_2-branes"



or:



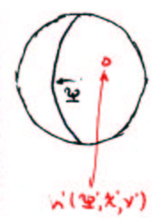
Coords. (Ψ, x, ν)

④

Semiclass. boundary state

The boundary state $\langle B|$ is supposed to give the amplitude $\langle B|\Psi\rangle$ for absorption / emission of a cl. string state $|\Psi\rangle$ by the brane. In the point-particle limit $l_s \to 0$ one may consider states sharply localized at $h \in H_3^+$:

$$\Psi_h(h') = \delta_{H_3^+}(h,h')$$



If the closed str. is off the brane, it can't get absorbed:

$$\langle B|\Psi_h\rangle \propto \delta(\Psi-\Psi')$$

In CFT one rather considers states that diagonalize $L_0 \sim \Delta$. A complete set of eigenstates of Δ in H_3^+ is given by the $|j,u,\bar{u}\rangle$, $j \in -\frac{1}{2} + i\mathbb{R}^+$ with w.-f.:

$$\Phi^j_{u,\bar{u}}(h) = -\frac{2j+1}{\pi}\left(|u-\gamma|^2 e^{\phi} + e^{-\phi}\right)^{2j}$$

$\Rightarrow \langle B|j,u,\bar{u}\rangle$ can be read off from the expansion of $\langle B|\Psi_h\rangle \equiv \delta(\Psi-\Psi')$ w.r.t. $|j,u,\bar{u}\rangle$

$$\delta(\Psi-\Psi') = \int dj \int d^2u \, \underbrace{\langle B|j,u\rangle}_{\text{wanted}} \underbrace{\langle j,u|\Psi_h\rangle}_{\equiv (\Phi^j_{u,\bar{u}}(h))^*}$$

$\Rightarrow$
- for $SL(2,\mathbb{R})$-preserving branes:
  $${}_B\langle \Psi_0|j,u,\bar{u}\rangle = |u+\bar{u}|^{2j} \begin{cases} e^{-\Psi_0(2j+1)} & \operatorname{sgn}(u+\bar{u}) > 0 \\ e^{+\Psi_0(2j+1)} & \operatorname{sgn}(u+\bar{u}) < 0 \end{cases}$$
- for $SU(2)$-preserving branes:
  $${}_B\langle \Lambda_0|j,u,\bar{u}\rangle = (1+|u|^2)^{2j} \sinh \Lambda_0(2j+1)$$

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Stringy corrections

Brief review of the H_3^+-WZNW model:

- Spectrum organized by current algebras (generators $J^a(z)$, $\bar{J}^a(\bar{z})$, level k) that extend the class. $SL(2,\mathbb{C})$ symmetry:

$$\mathcal{V} = \int_S dj \; \mathcal{R}_j , \qquad S = -\tfrac{1}{2} + i\mathbb{R},$$

where $\mathcal{R}_j$: current algebra reprn. generated from the following zero mode reprn.:

$$\begin{aligned} J_0^a \, |j,u,\bar{u}\rangle &= \mathcal{D}^a_{j,u} \, |j,u,\bar{u}\rangle \\ \bar{J}_0^a \, |j,u,\bar{u}\rangle &= \mathcal{D}^a_{j,\bar{u}} \, |j,u,\bar{u}\rangle \\ J_n^a \, |j,u,\bar{u}\rangle &= 0 , \quad n>0 \end{aligned} \qquad \left| \qquad \begin{aligned} \mathcal{D}^+_{j,u} &= -u^2\partial_u + 2ju \\ \mathcal{D}^0_{j,u} &= -u\partial_u + j \\ \mathcal{D}^-_{j,u} &= -\partial_u \end{aligned} \right.$$

by acting with the J^a_{-n}, $n>0$ on the $|j,u,\bar{u}\rangle$.

- Primary fields $\Phi^j(u|z)$

$$J^a(z)\,\Phi^j(u|z') \sim \frac{1}{z-z'}\,\mathcal{D}^a_{j,u}\,\Phi^j(u|z')$$

Semiclassically:

$$\Phi^j(u|z) = -\frac{2j+1}{\pi}\Big((\gamma(z)-u)(\bar{\gamma}(\bar{z})-\bar{u})\, e^{\phi(z)} + e^{-\phi(z)} \Big)^{2j}$$

Conf. dimensions

$$\Delta_j = -\frac{1}{k-2}\, j(j+1)$$

Primary field $\Phi^j|_{j=\frac{1}{2}}$ related to the fundamental matrix-valued field h via

$$\Phi^{\frac{1}{2}}(u|z) = (-u,1)\cdot h(z)\cdot \binom{-\bar{u}}{1}$$

$$\Longleftrightarrow \quad \partial_u^2 \Phi^{\frac{1}{2}} = 0 = \bar{\partial}_{\bar{u}}^2 \Phi^{\frac{1}{2}} .$$

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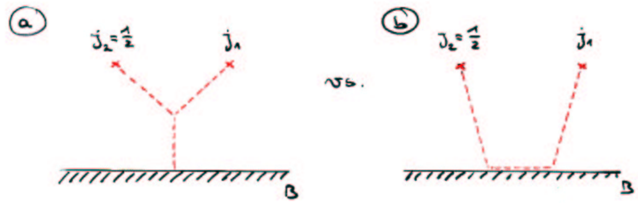
To determine the stringy corrections to the boundary states, let's consider the factorization of two point fcts.:



$$\langle \Phi^{\frac{1}{2}}(u_2|z_2)\, \Phi^{j_1}(u_1|z_1) \rangle_B$$

a symbolizes use of the OPE

$$\Phi^{\frac{1}{2}}(u_2|z_2)\,\Phi^{j_1}(u_1|z_1) \sim \sum_{\epsilon=\pm} |z_2-z_1|^{2(\Delta_{j_1+\frac{\epsilon}{2}}-\Delta_{j_1}-\Delta_{\frac{1}{2}})}\, |u_2-u_1|^{1-\epsilon} \times C_\epsilon(j_1)\,\Phi^{j_1+\frac{\epsilon}{2}}(u_1|z_1)$$

b symbolizes use of bulk-boundary OPE

$$\Phi^{\frac{1}{2}}(u|z) \underset{\mathrm{Im}\, z \downarrow 0}{\sim} A(\tfrac{1}{2},0|B)\,(\mathrm{Im}\, z)^{\frac{3}{2}b^2}\,(u+\bar{u})\cdot \mathbb{1} + A(\tfrac{1}{2},1|B)\,(\mathrm{Im}\, z)^{-\frac{b^2}{2}}\,\Psi^1(u|x)$$

Comparison of a and b $\Rightarrow$ Functional eqn. for the one-point function.

One point function vs. boundary state:

$$\langle \Phi^j(u|z) \rangle_B \;\xrightarrow[z\to\frac{i}{2}]{}\; \langle B|\, j,u,\bar{u} \rangle$$

(7)

The following expressions appear to be the most natural*) solutions to the factorization constraint:

- AdS$_2$ - branes:

also by: Lee, Ooguri, Park

$$\langle \Phi^j(u|z)\rangle_{\Psi_0} = |z-\bar{z}|^{2\Delta_j}\,|u+\bar{u}|^{2j} \times A_b\, \nu_b^{j+\frac{1}{2}}\,\Gamma(1+b^2(2j+1))\, e^{-\Psi_0(2j+1)\sigma}$$

$$\sigma := \mathrm{sgn}(u+\bar{u})$$

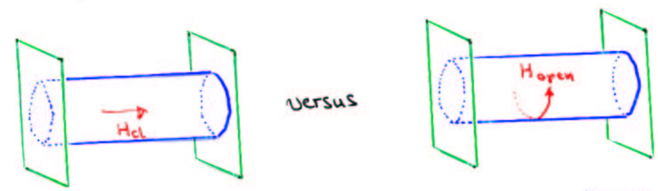
- Spherical branes:

Giveon, Kutasov, Schwimmer

$$\langle \Phi^j(u|z)\rangle_{\lambda_0} = |z-\bar{z}|^{2\Delta_j}\,(1+u\bar{u})^{2j} \times A_b\,\nu_b^{j+\frac{1}{2}}\,\Gamma(1+b^2(2j+1))\,\frac{\sin\lambda_0(2j+1)}{\sin\lambda_0}$$

*) Solutions are by no means unique, further constraints needed to show that the given expressions are indeed the physically relevant solutions!

Analogs of the Cardy condition



$$\langle B|\,\tilde{q}^{\frac{1}{2}H_{cl}}\,|B\rangle \overset{?}{=} \mathrm{tr}_{\mathcal{H}_{open}}\left(q^{H_{open}}\right)$$

The l.h.s. diverges in the case of the AdS$_2$-branes. Let's take stock of these divergences:

$$\sum_{\substack{\text{representations}\\ \mathcal{R}_j}}\ \sum_{\substack{\text{zero mode}\\ \text{subspace of } \mathcal{R}_j}}\ \sum_{\text{descendants}} \langle B|j,u,\bar{u};I\rangle \cdot \langle j,u,\bar{u};I|B\rangle$$

$$\int_{\frac{1}{2\pi L}}^{\infty} dP \qquad \Lambda\, N_{ann}(P|B) \qquad \chi^j(\tilde{q}) \sim q^{b^2P^2}\,\eta^{-3}(-1/\tau)$$

$$j = -\tfrac{1}{2}+iP$$

cut-offs

$$\Rightarrow \langle B|\,\tilde{q}^{\frac{1}{2}H_{cl}}\,|B\rangle_{reg} = \int_{\frac{1}{2\pi L}}^{\infty} dP\ \chi^{-\frac{1}{2}+iP}(\tilde{q})\, N_{ann}(P|B),$$

where: $N_{ann}(P|\Psi_0) = (\ldots)\,\Lambda\,\dfrac{\cos 4\Psi_0 P + \cos 2\pi P}{\mathrm{sh}\, 2\pi P\ \mathrm{sh}\, 2\pi b^2 P}$

$$\underset{P\to 0}{\sim} (\ldots)\,\frac{1}{P^2}$$

Therefore:

$$\langle B | \tilde{q}^{\frac{\tilde{c}}{2}H} | B \rangle_{\text{reg}} = (\ldots)\,(\,L + \text{finite part}\,)$$

universal

In order to extract a nontrivial quantity, one needs to pick out the finite part, e.g. via

$$Z^{\text{reg}}_{\text{ann}}(B;B_*) \equiv \langle B | \tilde{q}^{\frac{\tilde{c}}{2}H_d} | B \rangle - \langle B_* | \tilde{q}^{\frac{\tilde{c}}{2}H_d} | B_* \rangle$$

After modular Trsf.:

$$\chi_P(\tilde{q}) = 2\sqrt{2}\,\tfrac{i}{3}\int_0^\infty dP'\,\cos 4\pi b^2 PP'\,\chi_{P'}(q)$$

One ultimately finds:

$$Z^{\text{reg}}_{\text{ann}}(B;B_*) = \int_0^\infty dP\,N_{\text{strip}}(P|B;B_*)\,\chi_P(q),$$

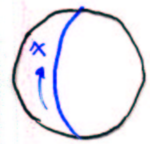
$$N_{\text{strip}}(P|B;B_*) = (\ldots)\frac{\partial}{\partial P}\log\frac{S_b(P+2\Psi_0)\,S_b(P-2\Psi_0)}{S_b(P+2\Psi_0^*)\,S_b(P-2\Psi_0^*)}$$

$$\log S_b(x) = i\int_0^\infty \frac{dt}{t}\left(\frac{\sin 2t b^2 x}{2\,\text{sh}\, b^2 t\;\text{sh}\, t} - \frac{x}{t}\right)$$

(10)

This story should be compared with $\text{tr}_{\mathcal{H}_{\text{open}}}(q^{H_{\text{open}}})$.

In the point particle limit: $H_{\text{open}} \rightsquigarrow \Delta_{\text{brane}}$



.... in suitable coords:

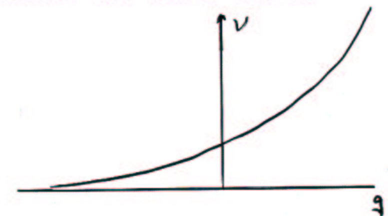
$$\Delta_{\text{brane}} = \partial_x^2 + \partial_x + e^{-2x}\partial_\nu^2$$

.... after Fourier-Transformation w.r.t. ν

$$\Delta_{\text{brane}} = \partial_x^2 + \partial_x - e^{-2x}k^2$$

$\Rightarrow$ Basically we have to deal with Liouville-q.m.

$$H = p^2 + e^{2q}$$



Reflection in the potential

$\hat{=}$ Reflection of signals sent from the bd. of AdS_2-brane in the interior?

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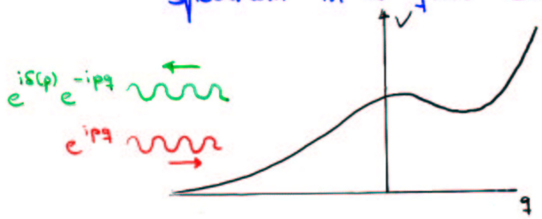
Let's recall the problem with the defn. of partition-fcts. in the presence of cont. spectrum in a q.m. context:



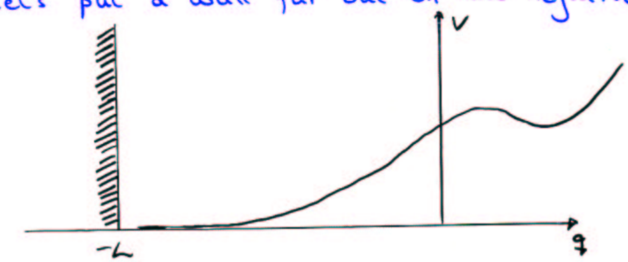
$H = p^2 + V(q)$

$R(p) \equiv e^{iS(p)}$ reflection amplitude

Then: $\mathrm{Tr}(e^{-tH}) \stackrel{"}{=} \int dE\, \rho(E)\, e^{-tE} = \infty$?

Why and how?

Let's put a wall far out on the negative axis:



If L is large enough, one may represent the wave-fct. near the wall by their asymptotic behavior:

$$\Psi_p(q) \sim e^{ipq} + e^{-ipq} R(p)$$

$\Rightarrow$ Quantization cond. $\Psi_p(q)\big|_{q=-L} = 0$ in terms of R: $-R(p) = e^{-2ipL}$.

($\Rightarrow$ connection betw. spectral data and scattering data)

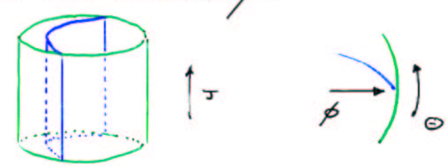
Vol-divergence, universal

$$\Rightarrow \quad \rho(E) = \frac{\delta N}{\delta E} \simeq_{L\to\infty} \frac{L}{\pi} + \frac{1}{2\pi i}\frac{\partial}{\partial p} \ln R(p)$$

V-dependent finite part

Back to the real thing. Now we need to discuss the H_3^+-WZNW model on the strip (or the UHP) with b.c. corresponding to the AdS_2-branes.

As in the closed string case it turns out that the worldsheet theory becomes free near the boundary.



$N: \tau, \phi$; $D: \sigma$

$$\to \mathcal{H}^{\text{free}} \simeq L^2(\mathbb{R}\times\mathbb{R};\, e^{\phi_0} d\phi_0 d\tau) \otimes \mathcal{F}$$

$$\stackrel{!}{\simeq} \int_{\mathbb{R}} dP\, \mathcal{P}_{-\frac{1}{2}+iP}$$

Zero mode space: $L^2(\mathbb{R}\times\mathbb{R}) \simeq \int dP\, \mathcal{P}_{-\frac{1}{2}+iP}$

with $\mathcal{P}_{-\frac{1}{2}+iP}$: Principal series of $SL(2,\mathbb{R})$

Complete reflection in the interior of AdS_3:

$$\Psi^j_u(\phi_0, \tau) \sim e^{-(j+1)\phi_0}\, \delta(u-\tau) + e^{j\phi_0} \frac{|u-\tau|^{2j}}{c(j)} R(j|\Psi_0)$$

The exact reflection amplitude $R(j|\Psi_0)$ will determine $N_{\text{strip}}(P|B,B^*)$ as

$$N_{\text{strip}}(P|B,B^*) = \frac{1}{2\pi i}\frac{\partial}{\partial P} \ln \frac{R(-\frac{1}{2}+iP\,|\,\Psi_0)}{R(-\frac{1}{2}+iP\,|\,\Psi_0^*)}$$

.... so the Task is To calculate $R(j|\underline{\Psi}_0)$.

To this aim let us observe that asymptotics

$$\underline{\Psi}^j_u(\phi_0,\bar{u}) \sim e^{-(j+1)\phi_0}\,\delta(u-\bar{u}) + R(j|\underline{\Psi}_0)\,e^{j\phi_0}\,\frac{|u-\bar{u}|^{2j}}{c(j)}$$

implies that the operators that create the states $\underline{\Psi}^j_u$ must satisfy a reflection property:

$$\underline{\Psi}^j_{\underline{\Psi}_0}(u|x) = R(j|\underline{\Psi}_0)\,(\mathcal{J}^j\,\underline{\Psi}^{-j-1}_{\underline{\Psi}_0})(u|x) \qquad (*)$$

unitary $SL(2,\mathbb{R})$-intertw. $P_{-j-1} \rightarrow P_j$

Moreover, there is a simple OPE between $\underline{\Psi}^j_{\underline{\Psi}_0}(u|x)$ and the spin 1 boundary field $\underline{\Psi}^1_{\underline{\Psi}_0}$

$$\left.\begin{aligned} &\underline{\Psi}^1_{\underline{\Psi}_0}(u_2|x_2)\,\underline{\Psi}^j_{\underline{\Psi}_0}(u_1|x_1) \\ &\quad\sim \sum_{s=-,0,+} e_s(j|\underline{\Psi}_0)\,(u_2-u_1)^{1-s}\,|x_2-x_1|^{\Delta_{j+s}-\Delta_j-\Delta_1} \\ &\qquad\qquad \cdot\,\underline{\Psi}^{j+s}_{\underline{\Psi}_0}(u_1|x_1) \end{aligned}\right\} (**)$$

Combination of (*) and (**) yields:

$$\frac{R(j+\frac{1}{2}|\underline{\Psi}_0)}{R(j-\frac{1}{2}|\underline{\Psi}_0)} = \frac{2j}{2j+1}\,e_-(-j-\tfrac{1}{2}|\underline{\Psi}_0),$$

which determines $R(j|\underline{\Psi}_0)$ uniquely given e_-.

Finally, calculation of e_- is possible, but tedious, with the help of the Cardy-Lewellen consistency conds.

Conclusions, outlook and questions

We believe to have learned something on the construction of branes on non compact backgrounds. Many qualitatively new issues arise due to the noncompactness. We find it particularly interesting to see what replaces the Cardy-Verlinde story in this context.

Our results will pave the way for the study of still more interesting examples like branes on the cigar $SL(2)/U(1)$ (under investig.)

Questions:

- The one-point function $\langle \Phi^j(u|z)\rangle_{\underline{\Psi}_0}$ has a jump at $u+\bar{u}=0$. What is the meaning of this jump in the space-time CFT?
- For the WZNW model describing open strings between two different branes we can prove nonexistence of a unitary reflection amplitude ($\Rightarrow$ no real spectral density). What does this mean?

We have therefore two independent means to calculate $N_{\text{strip}}(P \,|\, B, B^*)$

- From the annulus ampl. via modular trsf.
- From the reflection amplitude of the H_3^+ - WZNW model on the strip.

$$N_{\text{strip}}(P \,|\, B, B^*) = \frac{1}{2\pi i} \frac{\partial}{\partial P} \ln \frac{R(-\frac{1}{2}+iP \,|\, \Psi_0)}{R(-\frac{1}{2}-iP \,|\, \Psi_0)}$$

A perfect agreement is found!

Rem.: Comparison of the part that diverges like the volume yields only a rather weak constraint for the boundary state ($P \to \infty$ asymptotics $\propto \frac{1}{P}$)!