Geometry of fast moving strings.

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January 20, 2005

The talk is based on the following papers:

- S. Frolov, A.A. Tseytlin, "Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$", hep-th/0204226.
- M. Kruczenski, "Spin chains and string theory", hep-th/0311203.
- M. Kruczenski, A.V. Ryzhov, A. Tseytlin, "Large spin limit of $AdS_5 x S^5$ string theory and low energy expansion of ferromagnetic spin chains", hep-th/0403120.
- M. Kruczenski, A. Tseytlin, "Semiclassical relativistic strings in S^5 and long coherent operators in N=4 SYM theory", hep-th/0406189.
- G. Arutyunov, M. Staudacher, "Matching Higher Conserved Charges for Strings and Spins", hep-th/0310182.
- J. Engquist, "Higher Conserved Charges and Integrability for Spinning Strings in AdS_5 x S^5", hep-th/0402092.
- A.M., hep-th/0311019, 0402067, 0404173, 0409040, 0411178

Plan of the talk.

- Frolov-Tseytlin solutions.
  - fast moving "rigid" strings.
  - the energy reproduces the anomalous dimension on the field theory side.
- "Speeding strings".
  - Null-surfaces. Fast moving strings are perturbations of the null-surfaces.
  - Conjecture: the Yang-Mills perturbation theory corresponds to the perturbation theory around the null-surfaces.
- Classification of the null-surfaces. Supersymmetric null-surfaces. $U(1)_L$ symmetry.
- The $U(1)_L$ symmetry on the nearly-degenerate extremal surfaces; the action variable.
  - Definition of the first Pohlmeyer charge;
  - Special properties of $AdS_5 \times S^5$; the extention of $U(1)_L$ from the null-surfaces to the non-degenerate worldsheets.
  - Action variable and Pohlmeyer charges
- $U(1)_L$ in the BMN limit.
- Application of the action variable: anomalous dimension and local charges.

Frolov-Tseytlin solutions.

Classical string worldsheet theory on $AdS_5 \times S^5$ has so-called rigid solutions. These are the solutions for which the shape of the string does not depend on the time. This means: the profile of the string at a time $T = T_0$ is related to the profile of the string at $T = 0$ by an isometry of $AdS_5 \times S^5$.

Let us parametrize S^5 by three complex numbers Z_1, Z_2, Z_3 satisfying $|Z_1|^2 + |Z_2|^2 + |Z_3|^2 = 1$. The simplest examples of the rigid solutions are those worldsheets which project to the timelike geodesic in AdS_5. These strings "move only in S^5". The profile of the rigid string is given by the equation:

$$Z_I(\tau, \sigma) = e^{iw_i t} Z_I^{(0)}(\sigma)$$

where w_i, $i = 1, 2, 3$ are some real constants and $Z_I^{(0)}(\sigma)$ solves the differential equation:

$$\partial_\sigma^2 Z_I^{(0)} + Z_I^{(0)} \sum_{J=1}^{3} |\partial_\sigma Z_J|^2 = -w_I^2 Z_I^{(0)} + Z_I^{(0)} \sum_{J=1}^{3} w_J^2 |\partial_\sigma Z_J|^2$$

subject to the constraint

$$\sum_{J=1}^{3} w_J \overline{Z}_J \partial_\sigma Z_J = 0$$

The solutions should be periodic modulo the "overall phase" ϕ:

$$Z_J(\sigma = 2\pi) = e^{iw_J\phi} Z_J(\sigma = 0)$$

Properties of the rigid strings.

For each set $(w_1, w_2, w_3) \in \mathbf{R}^3$ there will be a discreet set of the periodic (modulo the "overall phase") trajectories, therefore a discreet set of string worldsheets.

For each worldsheet we can compute the momenta of $U(1)^3 \subset U(3) \subset SO(6)$, and parametrize the solution by the momentum (J_1, J_2, J_3). It turns out that the energy is given by

$$E = J \left[1 + \frac{\lambda}{J^2} c_1 + \left(\frac{\lambda}{J^2} \right)^2 c_2 + \ldots \right]$$

Frolov and Tseytlin conjectured that this expansion in powers of $\frac{\lambda}{J^2}$ correpsonds to the Yang-Mills perturbative expansion, and c_1, c_2, . . . are the coefficients of the anomalous dimension. They depend on the ratios $\frac{J_1}{J_2}$, $\frac{J_2}{J_3}$. These rigid strings correspond to very special operators in the $N = 4$ SYM: operators which extremize the anomalous dimension in the sector with the given charges.

It turns out that the Frolov-Tseytlin conjecture can be generalized to a more general class of operators.

Speeding strings.

One of the lessons from the BMN paper is that the perturbation theory for the long operators is often organized in powers of λ/L^2 (rather than powers of λ).

On the AdS side, to use the classical worldsheet theory, we need $\lambda >> 1$. For the YM perturbation theory to work, we need $\lambda/L^2 << 1$. Therefore, we need very large L to have an "overlap". Large number of partons means that the state has large R-charge, or from the point of view of the string theory the large momentum in S^5. When $L >> \sqrt{\lambda}$ the string is moving very fast. In the limit $L = \infty$ (for fixed large λ) every point on the string moves with the speed of light. The string worldsheet becomes a degenerate surface. A special class of degenerate surfaces obtained in this way is known as null-surfaces.

Fast moving strings are "nearly degenerate". It turns out that there is a perturbation theory in powers of $\sqrt{1-v^2}$ for the fast moving string as a perturbation (or "resolution") of the null-surface.

Conjecture: The Yang-Mills perturbation theory for the operators with the large R-charge corresponds to considering the worldsheet of the fast moving string as a perturbation of the null-surface.

Degenerate surfaces and null surfaces.

The surface is called degenerate if the induced metric is degenerate. When the string moves very fast, the worldsheet becomes a degenerate surface. The inverse is not quite true: not every degenerate surface can be obtained as a limit of a string worldsheet. Only the so-called null surfaces.

Indeed, let us introduce on the string worldsheet the coordinates ξ^+, ξ^- such that the induced metric is $\rho(\xi^+, \xi^-) d\xi^+ d\xi^-$. Let us denote $x^\mu(\xi^+, \xi^-)$ the embedding functions. Since the surface is extremal the embedding is a harmonic map:

$$\frac{D}{\partial \xi^+} \frac{\partial x^\mu}{\partial \xi^-} = 0 \tag{1}$$

In the limit when the string moves with the speed of light, the surface becomes isotropic. In this limit the two null-directions $\frac{\partial}{\partial \xi^+}$ and $\frac{\partial}{\partial \xi^-}$ coincide, and Eq (1) implies that the limiting null-directions are null-geodesics (= light rays).

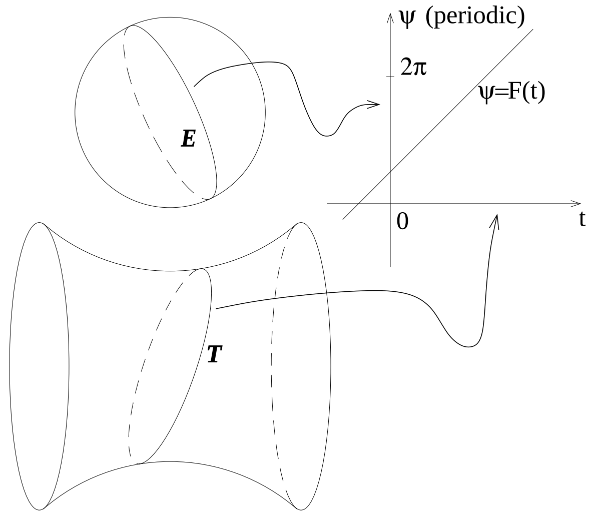
Degenerate surfaces and null surfaces.

Definition. A null-surface is a degenerate surface ruled by the light rays.

There are two types of light rays in $AdS_5 \times S^5$, therefore there are two types of the null-surfaces. The first type are null-surfaces ruled by the light rays totally inside AdS_5. Such null-surfaces extend to the boundary of AdS_5. We will not discuss this type of the null-surfaces here.

What we need now is the second type of null-surfaces. They are generated by the light rays which are obtained as a diagonal in the product of a timelike geodesic in AdS_5 and an equator in S^5.

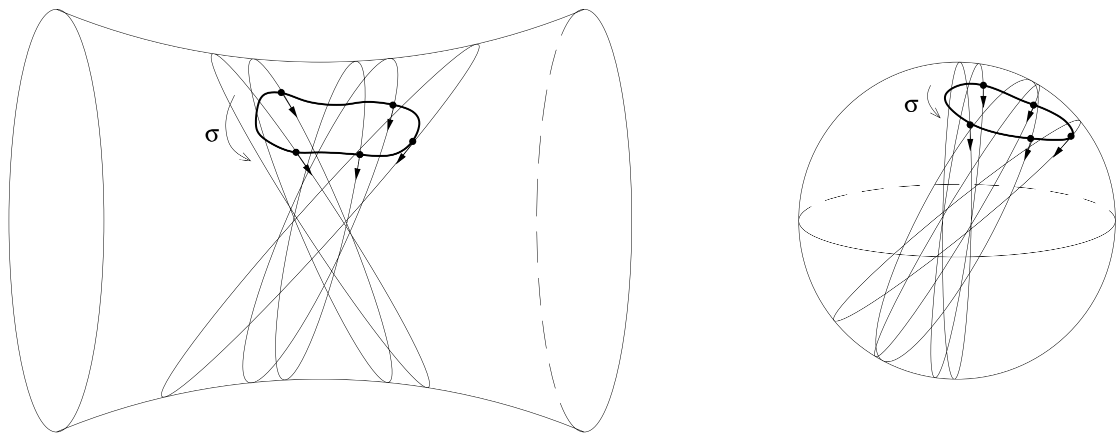
Classification of the null-geodesics of the second type in $AdS_5 \times S^5$:



The moduli space of null-geodesics:

$$\frac{SO(2,4)}{SO(2)\times SO(4)} \times \frac{SO(6)}{SO(2)\times SO(4)} \widetilde{\times} S^1$$

Classification of the null-surfaces of the second type in $AdS_5 \times S^5$:



The moduli space of collections of light rays:

$$\frac{\mathsf{Map}\left(S^1,\ \ \frac{SO(2,4)}{SO(2)\times SO(4)}\times\frac{SO(6)}{SO(2)\times SO(4)}\widetilde{\times}S^1\right)}{\mathsf{Diff}(S^1)}$$

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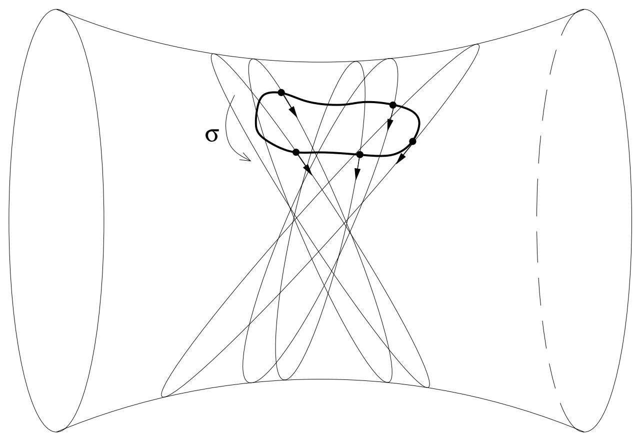
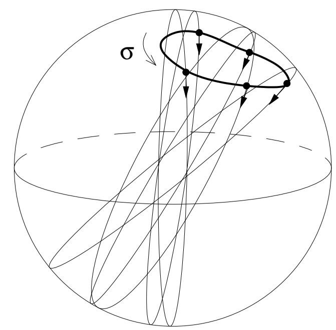




The moduli space of null-surfaces:

$$\frac{\mathrm{Map}_0\left(S^1,\quad \frac{SO(2,4)}{SO(2)\times SO(4)} \times \frac{SO(6)}{SO(2)\times SO(4)}\right) \widetilde{\times} S^1}{\mathrm{Diff}(S^1)}$$

Null-surfaces with fermions.

The Green-Schwarz superstring in $AdS_5 \times S^5$ has two Majorana-Weyl spinors $\theta^1(\tau,\sigma)$, $\theta^2(\tau,\sigma)$ modulo the κ-transformations. We will keep track only of the terms of the lowest order in $\theta^{1,2}$. The kappa-transformations are:

$$\delta_k \theta^1 = \widehat{\partial_+ x}\ k^1, \qquad \delta_k \theta^2 = \widehat{\partial_- x}\ k^2$$

and the equations of motion for fermions are:

$$\left\{ \begin{array}{l} \widehat{\partial_+ x}\ \mathcal{D}_- \theta^1 = 0 \\ \widehat{\partial_- x}\ \mathcal{D}_+ \theta^2 = 0 \end{array} \right. \quad \Longrightarrow \quad \text{exist} \ \ \eta_1, \eta_2 \ \ \text{such that} : \left\{ \begin{array}{l} \mathcal{D}_- \theta^1 = \widehat{\partial_+ x}\ \eta^1 \\ \mathcal{D}_+ \theta^2 = \widehat{\partial_- x}\ \eta^2 \end{array} \right.$$

Do the kappa-transformation with the parameters k^1, k^2 such that $D_{-,+} k^{1,2} = -\eta^{1,2}$

$$\mathcal{D}_- \theta^1 = \mathcal{D}_+ \theta^2 = 0$$

There are some "residual" kappa-transformations which preserve this condition.

In the null-surface limit $\mathcal{D}_+ = \mathcal{D}_-$. Therefore on each light ray of the null-surface 'live' two fermions θ_1 and θ_2 constant along this light ray.

'Two-time physics' representation of spinors in $AdS_5 \times S^5$.

$AdS_5 \times S^5$ can be emebedded into a flat twelve-dimensional space $\mathbf{R^{2+10}}$:
$AdS_5 \subset \mathbf{R}^{2+4}$ $(X_{-1}, X_0, \ldots, X_4)$ and $S^5 \subset \mathbf{R}^6$ $(X_5, \ldots, X_{10})$.

The spinor bundle on $AdS_5 \times S^5$ can be realized as a subbundle of the trivial bundle $\mathbf{C}^{32}$ which is a restriction to $AdS_5 \times S^5$ of the chiral spinors on $\mathbf{R}^{2+10}$. This subbundle is the image of the projector $\frac{1}{2}(1 + \Gamma^A \Gamma^S)$ where Γ^A is the Γ-matrix in the direction orthogonal to AdS_5 in $\mathbf{R}^{2+4}$ and Γ^S is in the direction orthogonal to S^5 in $\mathbf{R}^6$. Therefore a section of the spinor bundle on $AdS_5 \times S^5$ can always be represented in the form:

$$\psi = \frac{1}{2}\left(1 + \Gamma^A \Gamma^S\right) \Psi_{++}$$

where Ψ_{++} satisfies:

$$\left\{ \begin{array}{c} \Gamma_{-1}\Gamma_0 \cdots \Gamma_4 \Psi_{++} = i\Psi_{++} \\ \Gamma_5 \cdots \Gamma_{10} \Psi_{++} = i\Psi_{++} \end{array} \right. \qquad \text{this means that } \ \Psi_{++} \in \rho_{SU(2,2)} \otimes \rho_{SU(4)}$$

The covariant derivative modified by the Ramond-Ramond field strength is:

$$\mathcal{D}_i(\theta^1 + i\theta^2) = \left[D_i + \tfrac{1}{4}i(\hat{F}_A - \hat{F}_S)\Gamma_i\right](\theta^1 + i\theta^2)$$

The main advantage of considering ten-dimensional spinors as restrictions of twelve-dimensional spinors is a simple form of the covariant derivative:

$$\mathcal{D}_i\left[(1 + \Gamma^A\Gamma^S)\Psi_{++}\right] = (1 + \Gamma^A\Gamma^S)\partial_i\Psi_{++}$$

This means that covariantly constant spinors correspond to constant Ψ_{++}. The residual κ-symmetry:

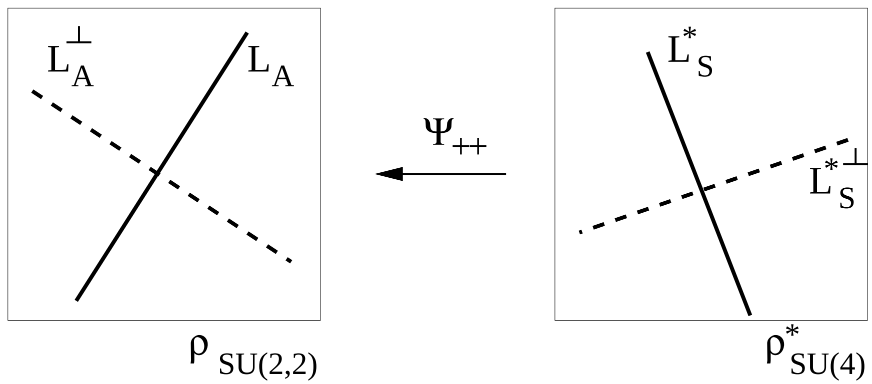
$$\delta_K\Psi_{++} = (\widehat{\partial_\tau x_A}\Gamma^A + \widehat{\partial_\tau x_S}\Gamma^S)K_{++}$$

The right hand side is constant on the light ray, because $\widehat{\partial_\tau x_A}\Gamma^A$ is the rotation in the equatorial plane of AdS_5 and $\widehat{\partial_\tau x_S}\Gamma^S$ is the rotation in the equatorial plane of S^5.

The equator of AdS_5 corresponds to the 2-plane $L_A \subset \rho_{SU(2,2)}$.

The equator of S^5 corresponds to the 2-plane $L_S \subset \rho_{SU(4)}$ (or $L_S^* \subset \rho^*_{SU(4)}$).

$\Psi_{++} \in \rho_{SU(2,2)} \otimes \rho_{SU(4)}$ defines a linear map $\Psi_{++}: \ \rho^*_{SU(4)} \longrightarrow \rho_{SU(2,2)}$:



"Purely bosonic" light ray		Super light ray
$L_A \subset \rho_{SU(2,2)}, \quad L_S^* \subset \rho^*_{SU(4)}$	$\Rightarrow$	$L_A \oplus L_S^* \subset \rho_{SU(2,2)} \oplus \rho^*_{SU(4)}$

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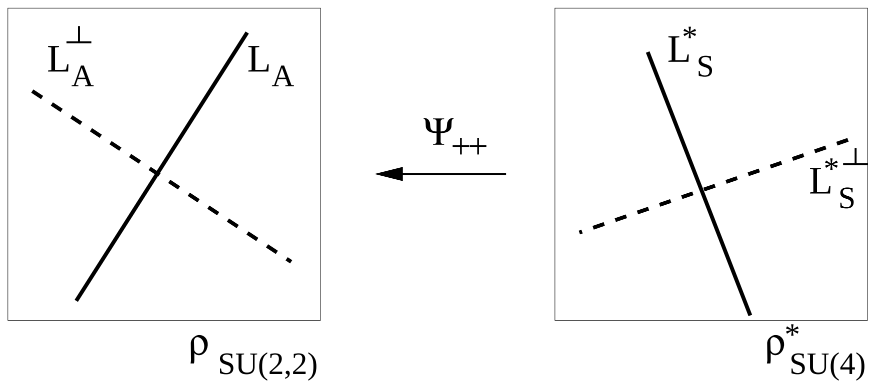

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Moduli space of supersymmetric null surfaces:

We see that turning on the fermionic degree of freedom on the light ray leads to:

$$\frac{SU(2,2)}{S(U(2)\times U(2))}\times\frac{SU(4)}{S(U(2)\times U(2))}\to\frac{PSU(2,2|4)}{PSU(2|2)\times PSU(2|2)\times U(1)^2}$$

In other words:

$$Gr(2,4)\times Gr(2,4)\to Gr(2|2,4|4)$$

This coset space is a super-symmetrization of the *future tube* in the complexified Minkowski space; it is also known as a *(4,2,2) analytic superspace.*

The moduli space of supersymmetric null-surfaces is:

$$\frac{\mathsf{Map}_0\left(S^1,\quad Gr(2|2,4|4)\right)\widetilde{\times}S^1}{\mathsf{Diff}(S^1)}$$

Comparison with the field theory.

Null-surfaces correspond to the long operators in the free Yang-Mills theory, which are "locally $\frac{1}{2}$-BPS". One can see that the moduli space of such operators (with the fixed length) is

$$\text{Map}_0\left(S^1,\ \ Gr(2|2,4|4)\right) \tag{2}$$

We can compare it to the moduli space of the supersymmetric null-surfaces:

$$\frac{\text{Map}_0\left(S^1,\ \ Gr(2|2,4|4)\right)\widetilde{\times}S^1}{\text{Diff}(S^1)} \tag{3}$$

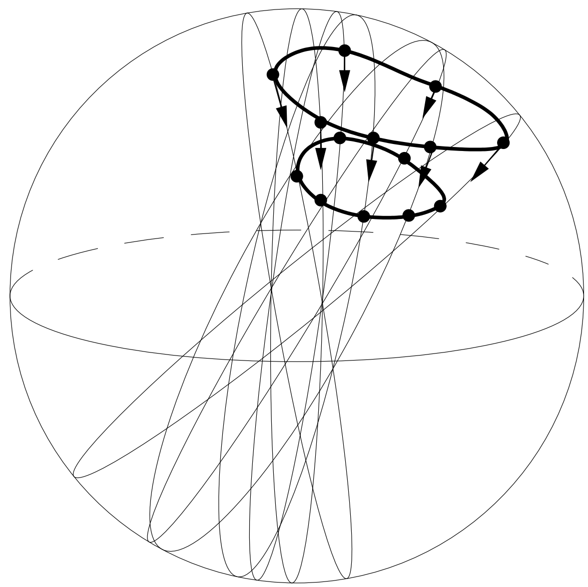
There is an apparent difference between (2) and (3). But in fact,

- The fiber S^1 is the degree of freedom dual to the *length* of the operator.
- we have to consider the *parametrized* null-surfaces; if we consider the null-surface as the limit of the worldsheet of the very fast moving string, it actually comes with a parametrization. This parametrization corresponds to the density of the conserved charge corresponding to the length of the operator. This removes Diff(S^1).

The moduli space of the null surfaces has a $U(1)$ symmetry.

For the continuous spin chain, the length is presumably conserved. Can we find the corresponding charge on the string theory side?

Consider the following $U(1)$ symmetry acting on the null surface:



We will call it $U(1)_L$.

The length of the spin chain on the string theory side.

Statement. There is a unique extension of $U(1)_L$ from the null-surfaces to the string phase space (at least to the fast moving strings). The charge of $U(1)_L$ corresponds to $L/\sqrt{\lambda}$ on the field theory side.

The action of $U(1)_L$ on the phase space of a classical string can be defined by the following characteristic properties:

- preserves the symplectic structure.
- has periodic orbits.
- commutes with $PSU(2,2|4)$
- acts on the null surfaces, as described above.
- does not change the projection of the worldsheet to AdS_5; moreover it preserves the projections to AdS_5 of the null-directions on the worldsheet.

Hamiltonian reduction; the statement of equivalence.

Consider the Hamiltonian reduction of the phase space of the classical string by $U(1)_L$ on the level set of $L = L_0 =$const.

- **There is an equivalence:**

Reduced phase space of the classical string reduced on the level set L_0 of $U(1)_L$.	$\Longleftrightarrow$	Phase space of the classical single trace state of the length L_0 in the Yang-Mills theory.

- **This equivalence commutes with the action of $PSU(2,2|4)$.**
- **The Yang-Mills perturbation theory corresponds to considering fast moving strings as perturbations of the null-surfaces; the small parameter is $\sqrt{1-v^2}$.**

This statement is a **conjecture**.

The construction of $U(1)_L$ in the perturbation theory.

The first Pohlmeyer charge. Suppose that the target space of the classical string is of the form $A \times S$. Then we can define the first Pohlmeyer charge:

$$Q^{[1]} = \oint_C d\tau^+ \sqrt{(\partial_{\tau^+} x_S, \partial_{\tau^+} x_S)}$$

Notations:

C is the closed contour on the string worldsheet;
τ^+, τ^- are the coordinates on the worldsheet; the worldsheet metric is $\simeq d\tau^+ d\tau^-$;
$x_S(\tau_+, \tau_-)$ is the projection of the worldsheet to S

Similarly, we define

$$\widetilde{Q}^{[1]} = \oint_C d\tau^- \sqrt{(\partial_{\tau^-} x_S, \partial_{\tau^-} x_S)}$$

The construction of $U(1)_L$ in the perturbation theory.

If S is the *sphere* S^5 then $Q^{[1]}$ generates periodic trajectories on the null surfaces.
The key properties of $Q^{[1]}$:

- $Q^{[1]}$ commutes with the reparametrizations $(\tau^+, \tau^-) \to (f^+(\tau^+), f^-(\tau^-))$
- $Q^{[1]}$ generates $U(1)_L$ *on the null-surfaces*

These two properties of $Q^{[1]}$ allow us to use $Q^{[1]}$ as a tool for extending $U(1)_L$ from the null-surfaces to the fast moving strings.

The construction of $U(1)_L$ in the perturbation theory.

Let M be the phase space of the classical string in $AdS_5 \times S^5$ It is the space of embeddings $(x_A(\tau^+, \tau^-),\ x_S(\tau^+, \tau^-))$ with the Virasoro constraints $(\partial_+ x_A)^2 + (\partial_+ x_S)^2 = 0$ and $(\partial_- x_A)^2 + (\partial_- x_S)^2 = 0$ satisfying the equations of motion $D_+ \partial_- x_{A,S} = 0$. Modulo conformal reparametrizations. The symplectic form is given by:

$$\omega(\delta_1 x, \delta_2 x) = \oint_C (\delta_1 x, * \overset{\leftrightarrow}{d}\ \delta_2 x)$$

Consider the phase space $\widehat{M}$ which consists of the embeddings $(x_A(\tau^+, \tau^-),\ x_S(\tau^+, \tau^-))$, $D_+ \partial_- x_{A,S} = 0$ *without the Virasoro constraints.* (Just harmonic maps.) Define the function K on $\widehat{M}$:

$$K = \int_{\tau=0} d\sigma |\partial_\tau x_S| = \int_{\tau=0} d\sigma |p_S(\sigma)|$$

The Hamiltonian vector field generated by K is manifestly periodic. But K does not preserve the Virasoro constraint. The idea of the construciton of $U(1)_L$ is: try to modify K so that the modified charge is periodic but also commutes with the Virasoro constraints. The modified charge will then descend on M and be the Hamiltonian of $U(1)_L$.

The construction of $U(1)_L$ in the perturbation theory.

Although K is not in the involution with the Virasoro constraints, there is a canonical transformation $F : \widehat{M} \to \widehat{M}$ such that F^*K is in the involution with the Virasoro constraints.

The first Pohlmeyer charge $Q^{[1]}$ is useful for constructing such a canonical transformation, in the following way. Notice that $\{K, Q^{[1]}\} \neq 0$. Let us try to find a canonical transformation F such that $\{F^*K, Q^{[1]}\} = 0$. Then, it turns out that

$$\{F^*K, Q^{[1]}\} = 0 \quad \Rightarrow \quad \{F^*K, \text{Virasoro}\} = 0 \tag{4}$$

Therefore, instead of considering invfinitely many conditions on F that $\{F^*K, \text{Virasoro}\} = 0$ we can consider just one condition $\{F^*K, Q^{[1]}\} = 0$.
The Virasoro constraints will follow.

We will now explain how to construct F such that $\{F^*K, Q^{[1]}\} = 0$ and prove (4).

We can construct F order by order in the perturbation theory. First, use the fact that $Q^{[1]}$ generates $U(1)_L$ on the null-surfaces, the same as K. Therefore:

$$Q^{[1]} = K + q_1 + q_2 + \dots$$

Under the rescaling $p_S \to tp_S$: $K \to tK$, $q_1 \to t^{-1}q_1$, $q_2 \to t^{-3}q_2$, $q_m \to t^{1-2m}q_m$.
The symplectic form is of the degree 1: $\omega \to t\omega$; the Poisson brackets are of the degree -1: $\{,\} \to t^{-1}\{,\}$.
Since K generates periodic trajectories, we can decompose q_m in the Fourier series:

$$q_m = q_{m,0} + \sum_{k\neq 0} q_{m,k} \quad \text{where} \quad \{K, q_{m,k}\} = ikq_{m,k}$$

Now we can take $F_{(1)} = \exp\left[\text{Hamiltonian vector field generated by} \left(\sum_{k\neq 0} \frac{1}{ik} q_{m,k}\right)\right]$ then $F_{(1)}^{-1*}Q^{[1]} = K + q_{1,0} + O(1/|p_S|^3)$ therefore $\{F^*_{(1)}K, Q^{[1]}\} \sim 1/|p_S|^3$. In the same way we can define $F_{(2)}$, $F_{(3)}$, ... so that $\{F^*_{(m)}K, Q^{[1]}\} \sim 1/|p_S|^{2m+1}$.

This procedure gives $F = F_{(\infty)}$, such that F^*K commutes with $Q^{[1]}$.

F^*K commutes with the Virasoro constraints.

After the canonical transformation we have:

$$F^{-1*}(Q^{[1]}) = K + q'_1 + q'_2 + \ldots + q'_m + \ldots$$

where for all k: $\{K,\ q'_k\} = 0$. The reparametrization invariance is manifestly preserved at each order, therefore the resulting charge F^*K will commute with $(p_S, \partial_\sigma x_S)(\sigma)$ for any σ. We can also prove that F^*K commutes with $|p_S|^2(\sigma) + |\partial_\sigma x_S|^2(\sigma)$ for any σ.

Indeed, we know that $F^{-1*}Q^{[1]} = K + q'_1 + q'_2 + \ldots$ commutes with $F^{-1*}(|p_S|^2(\sigma) + |\partial_\sigma x_S|^2(\sigma)) = |p_S|^2 + \phi_0 + \phi_1 + \ldots$; therefore

$$\{K, \phi_0\} = \{|p_S|^2(\sigma), q'_1\} \Rightarrow \{K, \{K, \phi_0\}\} = 0 \Rightarrow \{K, \phi_0\} = 0$$

$$\{K, \phi_1\} + \{q'_1, \phi_0\} = \{|p_S|^2(\sigma), q'_2\} \Rightarrow \{K, \{K, \phi_1\}\} = 0 \Rightarrow \{K, \phi_1\} = 0$$

$etc.$ (here it was important that K generates periodic trajectories)

We see that F^*K commutes with the Virasoro constraints. It is a conserved charge, local at each order in the perturbation theory. It generates the "hidden" symmetry $U(1)_L$.

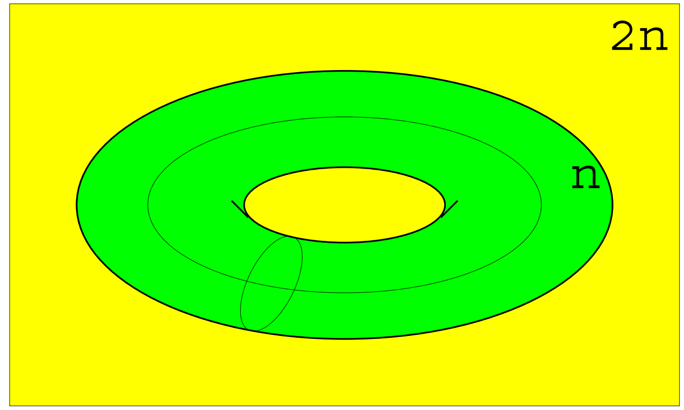
The role of the higher Pohlmeyer charges.

As we have seen, when S is the sphere the first Pohlmeyer charge $Q^{[1]}$ generates periodic trajectories on the null-surfaces. This is a manifestation of the integrability of the sigma-model. Another manifestation of the integrability is the existence of the infinite tower of the "higher" conserved charges. For example,

$$Q^{[2]} = \oint_C \left[\frac{d\tau^+}{|\partial_+ x_S|} \left(D_+ \frac{\partial_+ x_S}{|\partial_+ x_S|}, D_+ \frac{\partial_+ x_S}{|\partial_+ x_S|} \right) + 2 \frac{d\tau^-}{|\partial_+ x_S|} (\partial_- x_S, \partial_+ x_S) \right]$$

is the second conserved charge. (It is conserved only if S is a sphere.)

In classical mechanics, a mechanical system with $2n$-dimensional phase space is integrable if there are n functions $F_1, \ldots, F_n$ in involution with each other, and with the Hamiltonian. Then, there are n action variables $I_1, \ldots, I_n$, each generating a $U(1)$ symmetry. These action variables are special combinations of $F_1, \cdots, F_n$.



String theory is an infinite-dimensional integrable system, therefore this geometrical intuition is not directly applicable. But still, there is an infinite tower of the Pohlmeyer charges $Q^{[k]}$, in involution with each other. Can we find combinations of the Pohlmeyer charges generating periodic trajectories?

Notice that $Q^{[1]}$ generates periodic trajectories on the null-surfaces. On the fast moving string, the trajectries of $Q^{[1]}$ are not closed, the deviation from the periodicity is of the order $1/|p_S|$. Can we modify $Q^{[1]}$ by adding to it higher charges, so that the modified charge generates periodic trajectories?

It is certainly possible at the first order in $1/|p_S|$. Consider the following linear combination of the first two Pohlmeyer charges:

$$\frac{1}{16}\left[7(Q^{[1]}-\widetilde{Q}^{[1]})-\frac{1}{2}(Q^{[2]}-\widetilde{Q}^{[2]})\right]=$$
$$=\int d\sigma\left[|p_S|+\frac{1}{4|p_S|}\left[(\partial_\sigma x_S)^2-\left(D_\sigma\frac{p_S}{|p_S|},D_\sigma\frac{p_S}{|p_S|}\right)-\frac{(p_S,\partial_\sigma x_S)^2}{(p_S,p_S)}\right]+\ldots\right]$$

One can see immediately that the trajectories of this linear combination are closed up to the terms of the order subleading to $1/|p_S|$. Indeed, the leading term is just K, it gives periodic trajectories. And the subleading term averages to zero on the periodic trajectories of the leading term. Therefore the trajectories of of this Hamiltonian do not drift at the order $1/|p_S|$; the deviation from the periodicity is of the order $1/|p_S|^3$.

Conjecture: It is possible to add higher Pohlmeyer charges, so that the resulting charge will generate periodic trajectories to all orders in $1/|p_S|$. The resulting charge coincides with the generator of $U(1)_L$ which we described in perturbation theory (as F^*K). The classical string theory on $AdS_5\times S^5$ has an action variable (which corresponds to the length of the operator on the field theory side).

Pohlmeyer charges for rigid solutions were computed by G. Arutyunov, M. Staudacher and J. Engquist. The conserved charges have the following structure:

$$\mathcal{E}_n = \delta_{2,n}\mathcal{J} + \frac{\epsilon_n^{(1)}}{\mathcal{J}} + \frac{\epsilon_n^{(2)}}{\mathcal{J}^3} + \frac{\epsilon_n^{(3)}}{\mathcal{J}^5} + \dots$$

where $\mathcal{J} = J/\sqrt{\lambda}$, and J is a particular combination of the $U(1)^3 \subset U(3) \subset SO(6)$ momenta. The coefficients $\epsilon_n^{(m)}$ depend on what kind of a rigid string is considered (the ratio of spins). But Arutyunov, Staudacher and Engquist noticed that the coefficients $\epsilon_n^{(m)}$ for different values of n are not independent. For all the solutions they considered, they find that:

$$\mathcal{E}_{10} + \frac{74}{7}\mathcal{E}_8 + \frac{1898}{35}\mathcal{E}_6 + \frac{6922}{35}\mathcal{E}_4 + \frac{32768}{35}(\mathcal{E}_2 - \mathcal{J}) \sim \frac{1}{\mathcal{J}^9}$$

This means that up to the terms of the order $1/|p_S|^9$ we should have:

$$\frac{J}{\sqrt{\lambda}} = \mathcal{E}_2 + \frac{6922}{32768}\mathcal{E}_4 + \frac{1898}{32768}\mathcal{E}_6 + \frac{370}{32768}\mathcal{E}_8 + \frac{35}{32768}\mathcal{E}_{10} + \ldots \tag{5}$$

At first this formula looks rather strange, because it seems to imply that a certain combination of Pohlmeyer charges (which all commute with $SO(6)$) is equal to some component of the angular momentum (which transforms in the adjoint of $SO(6)$). We propose the following resolution of this puzzle. The right hand side of (5) is actually the action variable, which for a particular class of the solutions considered by Arutyunov *et al* happens to be equal to the $SO(6)$ charge J (because these particular solutions correspond to the chiral operators on the field theory side). In other words, this formula should be understood as follows:

$$\text{Generator of } U(1)_L \;=\; \mathcal{E}_2 + \frac{6922}{32768}\mathcal{E}_4 + \frac{1898}{32768}\mathcal{E}_6 + \frac{370}{32768}\mathcal{E}_8 + \frac{35}{32768}\mathcal{E}_{10} + \ldots$$

The BMN limit.

The worldsheet theory becomes quadratic in the Penrose limit:

$$S = \frac{1}{2\pi}\int d\tau d\sigma \sum_{i=1}^{4}\left[(\partial_\tau x_i)^2 + (\partial_\tau y_i)^2 - (\partial_\sigma x_i)^2 - (\partial_\sigma y_i)^2 - p_+^2(x_i^2 + y_i^2)\right]$$

The Hamiltonian is:

$$H = \frac{1}{2\pi}\int d\sigma \sum_{i=1}^{4}\left[p_i^2 + q_i^2 + (\partial_\sigma x_i)^2 + (\partial_\sigma y_i)^2 + p_+^2(x_i^2 + y_i^2)\right]$$

Here $p_i(\sigma)$ and $q_i(\sigma)$ are the momenta conjugate to x_i and y_i respectively:

$$\{p_i(\sigma), x_j(\sigma')\} = 2\pi\delta(\sigma - \sigma'), \quad \{q_i(\sigma), y_j(\sigma')\} = 2\pi\delta(\sigma - \sigma')$$

There is also a constraint:

$$\int d\sigma \left[(p, \partial_\sigma x) + (q, \partial_\sigma y)\right] = 0$$

The states of the quantum theory are constructed by acting on the vacuum $|0>$ by the operators α_n^i and β_n^i, $n \in \mathbf{Z}$, $i \in \{1, 2, 3, 4\}$:

$$\alpha_n^i = \int_0^{2\pi} d\sigma \; e^{-in\sigma} \left((n^2 + p_+^2)^{1/4} \; x^i + \frac{1}{(n^2 + p_+^2)^{1/4}} \frac{\partial x^i}{\partial \tau} \right)$$

$$\beta_n^i = \int_0^{2\pi} d\sigma \; e^{-in\sigma} \left((n^2 + p_+^2)^{1/4} \; y^i + \frac{1}{(n^2 + p_+^2)^{1/4}} \frac{\partial y^i}{\partial \tau} \right)$$

The length of the state is the total number of the β-oscillators in this state. For example the length of $(\beta_1^i)^3 \; \beta_7^j \; |0>$ is equal to $3 + 1 = 4$.

The explicit expression for the length is:

$$\mathcal{E} = \frac{1}{2\pi}\sum_{i=1}^{4}\int d\sigma \left[y_i(\sigma)\sqrt{p_+^2 - \partial_\sigma^2}\; y_i(\sigma) + q_i(\sigma)\frac{1}{\sqrt{p_+^2 - \partial_\sigma^2}}\; q_i(\sigma) \right]$$

This charge generates the following $U(1)$ symmetry:

$$\{\mathcal{E}, y_j(\sigma)\} = \frac{1}{\sqrt{p_+^2 - \partial_\sigma^2}}\; q_j(\sigma), \quad \{\mathcal{E}, q_j(\sigma)\} = -\sqrt{p_+^2 - \partial_\sigma^2}\; y_j(\sigma)$$

The length can be expanded in the inverse powers of p_+:

$$\mathcal{E} = \frac{1}{2\pi}\sum_{i=1}^{4}\int d\sigma \left[p_+ y_i y_i - \sum_{k=1}^{\infty} p_+^{1-2k}\frac{(2k-3)!!}{2^k k!}\; y_i \partial_\sigma^{2k} y_i + \right.$$

$$\left. + \sum_{k=1}^{\infty} p_+^{1-2k}\frac{(2k-3)!!}{2^{k-1}(k-1)!}\; q_i \partial_\sigma^{2k-2} q_i \right]$$

$$\mathcal{E} = \sum_{k=1}^{\infty} p_+^{1-2k} \frac{(2k-3)!!}{2^{k-1}(k-1)!} I_k \tag{6}$$

where

$$I_k = \frac{1}{2\pi} \sum_{i=1}^{4} \int d\sigma \left[q_i \partial_\sigma^{2k-2} q_i - y_i \partial_\sigma^{2k} y_i + p_+^2 y_i \partial_\sigma^{2k-2} y_i \right]$$

is the local conserved charge of the free theory. We see how the action variable is expanded in the sum of the local conserved charges. Eq. (6) is the Penrose limit of the Arutyunov-Staudacher-Engquist formula.

Anomalous dimension and local charges.

As an example of the application of the action variable, we will now answer the following question: given the classical string worldsheet, how to compute the anomalous dimension of the corresponding operator?

The anomalous dimension is usually defined as the deformation of the particular generator of the conformal algebra — the dilatation operator. But in fact the anomalous dimension parametrizes the deformation of the *representation* of the conformal algebra, rather than the deformation of a particular generator. It is natural to characterize the deformation of the representation by the action of the center of the group. Therefore we will define the anomalous dimension through the action of the center of the conformal group. This definition is manifestly conformally invariant.

The center of the superconformal group.

The group $PSU(2,2|4)$ is not simply connected; the superconformal group of the conformal field theory is actually a covering group which we will denote $\widetilde{PSU}(2,2|4)$. The bosonic part of $\widetilde{PSU}(2,2|4)$ is $[\widetilde{SU}(2,2) \times SU(4)]/\mathbf{Z}_2$ where $\widetilde{SU}(2,2)$ is the universal covering of $SU(2,2)$. Let c denote the generator of the center. The action of c can be understood in the following way. Consider the conformal field theory on $\mathbf{R} \times S^3$ where $\mathbf{R}$ is the time and the radius of S^3 is 1. Let t denote the time and $\vec{n}$ denote the unit vector parametrizing S^3. Then c acts as the combination of the conformal transformation:

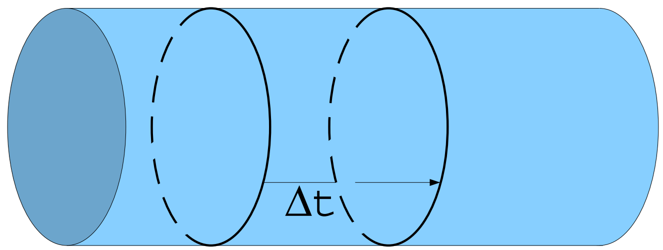
$$c: \ (t, \vec{n}) \to (t + \pi, -\vec{n})$$

with the R-symmetry $i\mathbf{1} \in SU(4)$. This transformation commutes with the generators of $so(2,4)$ and therefore it is in the center of the conformal group. It also commutes with the fermionic generators of the supersymmetry, therefore it is in the center of $\widetilde{PSU}(2,2|4)$.

We will define the anomalous dimension through the action of c^2:

$$c^2 = e^{2\pi i \Delta}$$

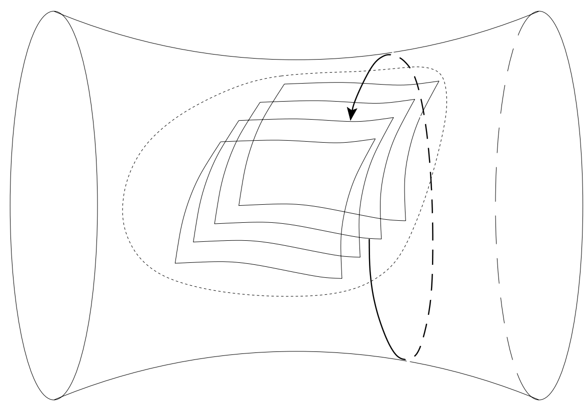
c^2 acts as a shift in time by $\Delta t = 2\pi$ (radius of S^3) and $(-1)^F$:



In the free theory the action of the centre is trivial; but in the interacting theory $c \neq 1$; the logarithm $\log c$ is called the anomalous dimension. We can expand $\log c$ in powers of λ: $\log c = \lambda d_1 + \lambda^2 d_2 + \ldots$.

Center of $PSU(2,2|4)$: the string theory side.

The AdS space is the universal covering space of the hyperboloid and c^2 acts as a deck transformation exchanging the sheets. We can visualize the action of this deck transformation on the string phase space in the following way:



The string worldsheet looks locally like a deck of cards. Going around the noncontractible cycle in the hyperboloid exchanges the sheets. This is how c^2 acts on the phase space of the classical string.

Deck transformation and local charges.

We have seen that the conserved charge corresponding to the length of the operator is an infinite linear combination of the Pohlmeyer charges:

$$L = \sqrt{\lambda}\,[\mathcal{E}_2 + a_1\mathcal{E}_4 + a_2\mathcal{E}_6 + a_3\mathcal{E}_8 + \ldots]$$

The corresponding Hamiltonian vector field ξ_L has periodic trajectories:

$$e^{2\pi\xi_L} = \text{identical transformation}$$

In this expansion $\mathcal{E}_{2k}$ are the Pohlmeyer charges for the S^5 sigma-model. But the classical string in $AdS_5 \times S^5$ essentially splits into two systems: the sigma-model with the target space AdS_5 and the sigma-model with the target space S^5. The AdS_5 sigma-model also has Pohlmeyer charges. Let us denote them $\mathcal{F}_{2k}$. How can we use them?

Consider the conserved charge $M = \sqrt{\lambda}\,[\mathcal{F}_2 + a_1\mathcal{F}_4 + a_2\mathcal{F}_6 + a_3\mathcal{F}_8 + \ldots]$ defined with the same coefficients a_k.

Notice that $e^{-2\pi\xi_M}$ acts as the deck transformation: $e^{-2\pi\xi_M} = c^2$

Since ξ_M commutes with ξ_L, we can also write:

$$c^2 = e^{2\pi(\xi_L - \xi_M)}$$

But $\mathcal{E}_2 = \mathcal{F}_2$ because of the Virasoro constraints. Therefore we can identify

$$\frac{1}{2\pi}\log c^2 = \sqrt{\lambda}\,[a_1(\mathcal{E}_4 - \mathcal{F}_4) + a_2(\mathcal{E}_6 - \mathcal{F}_6) + a_3(\mathcal{E}_8 - \mathcal{F}_8) + \ldots] \qquad (7)$$

This expression is a perturbative expansion of the anomalous dimension of the fast moving string in the perturbation theory around the null surface. The small parameter is the relativistic factor $\sqrt{1-v^2} \sim \frac{\sqrt{\lambda}}{L}$, where v is the typical velocity of the string. One can define the local conserved charges in such a way that $\mathcal{E}_{2k}, \quad \mathcal{F}_{2k} \ \simeq (1-v^2)^{k-3/2}$. Therefore (7) is an expansion in powers of $\frac{\lambda}{L^2}$. The first term is of the order $\frac{\lambda}{L}$, the second term is of the order $\frac{\lambda^2}{L^3}$ and so on.