Notes on higher spin symmetries.

**AdS$_5$/CFT$_4$**

\[
J_{\text{str}} \approx \frac{2}{g_{\text{YM}}}
\]

\[
\left( \frac{R}{L_{\text{str}}} \right)^4 \approx \frac{g_{\text{YM}}^2 N}{g_{\text{str}}^4}
\]

**R/L$_{\text{str}}$ → 0**: complicated in string theory

but:

Boundary $S$-matrix \( \approx \) correlation functions in the free field theory on the boundary.

Idea: learn about the string theory in the $R/L_{\text{str}} \to 0$ limit by looking at the boundary $S$-matrix.

(P. Haghi-Mani, B. Sundborg, hep-th/0002189.)

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1) Spectrum of primaries on the boundary \( \to \) field content in the bulk

2) Parameter $N$: when $N = \infty$, the correlation functions on the boundary factorize into the product of pair correlators.

$N = \infty$ corresponds to free theory in the bulk.

Considering \( 0 < \frac{1}{N} << 1 \) corresponds to turning on interactions.

The leading connected contribution is given by the classical theory with the coupling constant \( \frac{1}{N} \).

For example:

\[
\left\langle \frac{1}{N} \mathrm{tr} \phi^2(x_1) \ldots \frac{1}{N} \mathrm{tr} \phi^2(x_n) \right\rangle
\]

\[
= \sum_{\delta(i)=1} \frac{1}{N^{n-2}} \frac{1}{\|x_{\delta(i)} - x_{\delta(\delta(i))}\|^2} \ldots \frac{1}{\|x_{\delta(n)} - x_{\delta(\delta(n))}\|^2}
\]

Locality?
AdS pictures:

\[ n = 3 : \]

\[ n = 4 : \]

\[ \sum_{\text{perms}} \]

Some notations:

\[ \Delta(\Delta - 4) = m^2 R^2 \]

\[ D_{\Delta}(\nu) = \frac{1}{(\nu \cdot l)^\Delta} \]

For \( \text{tr} \phi^2 \), \( \Delta = 2 \),

\[ D_2(\nu) = \frac{1}{(\nu \cdot l)^2} \]

\[ = \int d^5 w \quad D_{\Delta = 2}(\nu, w) \frac{1}{(w \cdot l_1)^2 (w \cdot l_2)^2} = \]

\[ = \frac{1}{(\nu \cdot l_1) (\nu \cdot l_2) (l_1 \cdot l_2)} \]

(E. D'Hoker, D.Z. Freedman, L. Rastelli

hep-th/9905049)
Another suggestive formula:

\[ \frac{1}{(l_1 \cdot l_2 \cdot l_3)(l_3 \cdot l_4 \cdot (l_4 \cdot l_2))} = \sum_{n=0}^{\infty} \int d^5v \times \left( \frac{1}{(\partial^2 \partial_{(0)} - \partial_{(1)} \partial_{(2)} - \partial_{(2)} \partial_{(3)})} \right)^n \frac{1}{(v_1 l_2 v_2 l_3 v_4)^2} \]

(index \( n \) in \( \frac{1}{\partial_{(n)}} \) means that one has to differentiate w.r.t. \( v \) in \( (v \cdot l_1) \), etc.)

Nonlocal vertex comes from integrating out infinite series of higher spins.

Obvious obstacle: anomalous dimension of \( (\phi^2)^{\infty} \):

\[ \phi(\nu) = \phi^{(0)}(\nu) + \phi^{(1)}(\nu) \]

\[ (\square + m^2) \phi^{(0)}(\nu) = 0 \]

\[ \lim_{l_1, l_2} \phi^{(1)}(\nu) = \frac{1}{(v \cdot l_1)^4} \]

\[ \int d^5v \frac{1}{(v \cdot l_1)^4} \frac{1}{(v \cdot l_2)^4} \] is log-divergent

which means there are terms \( \propto \log(l_1 \cdot l_2) \)

Log-divergence should cancel when we sum over intermediate spins.
Idea: The free field theory has large symmetry group. \( \delta \phi = \partial \phi, \delta \phi = \partial_\mu \partial_\nu \phi, \ldots \)
Look for the theory in the bulk which has the same group of symmetries, as gauge symmetries.

Consistent truncation: The symmetries of the free field theory are generated by the currents bilinear in free fields.
The subset of operators, which are bilinears in free fields, is closed under the OPE.

\[
\begin{align*}
\text{bilinear } O_1 \\
\text{bilinear } O_2 \\
\text{bilinear } O_3
\end{align*}
\]

\[ \phi^a (u) = 0 \text{ unless if } O \text{ is bilinear in free fields} \]

This suggests that the theory in the bulk has consistent truncation to the set of operators which are bilinears in free fields.

- Interacting classical theories with the required symmetry group were constructed by E. S. Fradkin and M. A. Vasiliev:
  
  hep-th/9910036, 0104246, 0106200

  Formulation of the problem:

Plan:
- Bilinear operators and conserved currents
- Algebraic structure of the h.s. symmetries
- Free higher spin fields in AdS
  AdS/CFT for \( N=\infty \)
- Global symmetries on the boundary and gauge symmetries in the bulk
On shell, \( \Theta \phi = 0 \), we have
\[
\partial_i j^{i_2 \ldots i_d} = 0
\]

[For any tensor primary \( j^{i_1 \ldots i_d} \) of conformal dimension \( D-2+x \), it turns out that \( \Theta j^{i_1 \ldots i_d} \) is again a tensor primary. It has conformal dimension equal spin plus \( D \). But all primaries bilinear in free fields have conformal dimension equal spin plus \( D-2 \). Therefore, \( \Theta j^{i_1 \ldots i_d} = 0 \).]

Higher spin tensor currents are related to higher derivative symmetries:

Conformal Killing tensor \( g^{i_2 \ldots i_d} \):
\[
g^{ij_2 \ldots i_d} = g^{i_1 \ldots i_d}
\]

Given the Conformal Killing tensor \( g^{i_2 \ldots i_d} \), the contraction \( g^{ij_2 \ldots i_d} j^{i_1 \ldots i_d} \) is a conserved current in the usual sense of the word, i.e., it generates symmetry.
One can prove a converse statement:

any higher derivative symmetry of the action is related to the conformal Killing tensor \( \bar{g}^{i_2...i_d} \):

\[
\delta_\bar{g} \phi = (\bar{g}^{i_2...i_d} \partial_{i_2}...\partial_{i_d} + ...) \phi
\]

**Conclusion:** free theory has infinitely many symmetries, corresponding to the conformal Killing tensors.

These symmetries form an infinite-dimn nonabelian algebra.

We want now to describe, in some simple way, the structure of this algebra.

Suppose that \( \delta_\phi = L(x, \bar{\partial}_x) \cdot \phi \) is a symmetry of the action. Then, it preserves the Laplace equation: \( \Delta \phi = 0 \Rightarrow \Delta (L \cdot \phi) = 0 \).

Let us consider a special solution to the Laplace equation,

\[
\Phi_{\bar{q}, q}(x) = e^{9A \bar{q} \cdot A} x^{A\bar{A}}
\]

\( (\Delta \Phi_{\bar{q}, q}(x) = 0) \)

\[
L(x, \bar{\partial}_x) \cdot e^{9A \bar{q} \cdot A} x^{A\bar{A}} = P_L(x, q \bar{q}) \cdot e^{9A \bar{q} \cdot A} x^{A\bar{A}}
\]

\[
\Delta (P_L(x, q \bar{q}) \cdot e^{9A \bar{q} \cdot A} x^{A\bar{A}}) = 0
\]

\[
P_L(x, q \bar{q}) \cdot e^{9A \bar{q} \cdot A} x^{A\bar{A}} = \tilde{P}_L(q, \bar{q}, \partial_q, \partial_{\bar{q}}) \cdot e^{9A \bar{q} \cdot A} x^{A\bar{A}}
\]

Therefore, there is a one-to-one correspondence between the differential operators \( L \), preserving \( \Delta \phi = 0 \) and the polynomials \( \tilde{P}_L(q, \bar{q}, \partial_q, \partial_{\bar{q}}) \) which are invariant under \( U(1) \) generated by \( \delta q = i q \).

Consider the operators in the free theory which are linear in the free fields:

\[
\hat{\delta}_f = \int d^4x f(x) \phi(x)
\]

<table>
<thead>
<tr>
<th>( \phi(x) )</th>
<th>( \hat{f}(q, \bar{q}) = \int d^4x f(x) e^{9q \bar{q} x} )</th>
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</thead>
<tbody>
<tr>
<td>( \phi(0) )</td>
<td>1</td>
</tr>
<tr>
<td>( \partial_{A\bar{A}} \phi(0) )</td>
<td>( 9A \bar{q} \cdot A )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( L \cdot \hat{\delta}_f )</td>
<td>( \tilde{P}<em>L(q, \bar{q}, \partial_q, \partial</em>{\bar{q}}) \cdot \hat{f} )</td>
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</table>
Consider the algebra of oscillators, 4 coordinates and 4 momenta. Suppose that on the space of oscillators there is a complex structure \( I, I^2 = -1 \), such that:

- Commutation relations are invariant under the action of \( I \) (of type \( (1,1) \)).
- Kahler metric has signature \( (4^+,4^-) \).

Useful for realization of \( su(2,2) \):

\[
su(2,2) \subset sp(4,\mathbb{R}) \text{ - all quadratic hamiltonians invariant under } I \text{ (modulo center)}
\]

Infinite dimensional extension of \( su(2,2) \):

\[
hs(2,2) ; \quad i \chi^{(0)} + \sum_{i,j} Q^{i}Q^{j} + i \sum_{i,j} \Theta^{i}...\Theta^{j} + ...
\]

This is the algebra of symmetries of the free complex scalar field.

Doubleton representation:

\[
\begin{align*}
Q^I & \rightarrow q^I, \\
\bar{P}^I & \rightarrow \bar{q}^I.
\end{align*}
\]

Operators linear in \( \phi \) are in \( F \) and operators linear in \( \phi^* \) are in \( \overline{F} \).

This means that the bilinears are in \( F \otimes \overline{F} \).

There is a hermitean scalar product

\[
F \otimes \overline{F} \rightarrow \mathbb{C}
\]

Therefore \( \overline{F} \cong F^* \),

this means that the \( n \)-point function is the element of \( (F \otimes F^*) \otimes \mathbb{C} \), or

hs - invariant operator in \( F \otimes \mathbb{C} \).

It turns out that an arbitrary operator in \( F \) can be represented as a linear combination (perhaps infinite sum) of generators of \( hs(2,2) \).

Therefore, the hs - invariant operator in \( F \otimes \mathbb{C} \) should commute with any operator in \( F \) acting on \( F \otimes \mathbb{C} \) as symmetries act on tensor product:

\[
\begin{align*}
x, (\nu_1 \otimes \ldots \otimes \nu_n) &= \chi \nu_1 \otimes \nu_2 \otimes \ldots \otimes \nu_n + \\
&+ \nu_1 \otimes \chi \nu_2 \otimes \ldots \otimes \nu_n + \ldots + \nu_1 \otimes \ldots \otimes \chi \nu_n.
\end{align*}
\]
This implies that any hs-invariant operator in $F^n\Phi^n$ is in fact a linear combination of permutations.

Returning to correlation functions,

\[
\langle \phi^*(x_1)\phi(y_1) \cdots \phi^*(x_n)\phi(y_n) \rangle = \sum_{\sigma\in S_n} A_{\sigma} \frac{1}{|x_1-y_{\sigma(1)}|^2 \cdots |x_n-y_{\sigma(n)}|^2}.
\]

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Higher spin fields.

\[ h_{\mu_1 \cdots \mu_d} : \text{symmetric in } \mu_1 \cdots \mu_d; \]
\[ g_{\mu_2 \mu_3} g_{\mu_4 \mu_5} h_{\mu_1 \cdots \mu_d} = 0 \]

Gauge transformations:

\[ \delta h_{\mu_1 \cdots \mu_d} = \nabla_{\mu_1} \Lambda_{\mu_2 \cdots \mu_d}, \quad g_{\mu_2 \mu_3} \Lambda_{\mu_2 \cdots \mu_d} = 0 \]

Free eqs. of motion:

\[ \nabla_\nu \nabla^\nu h_{\mu_1 \cdots \mu_d} - d \nabla_\nu \nu_{\mu_1} h_{\mu_2 \cdots \mu_d} + \frac{d}{2} \nabla_\nu \nabla_\mu h_{\nu \mu_2 \cdots \mu_d} + + 2(d-1)(d+1) h_{\mu_1 \cdots \mu_d} = 0 \]

de Donder gauge:

\[ \Gamma_{\mu_2 \cdots \mu_d} [h] = \nabla^\nu h_{\mu_1 \mu_2 \cdots \mu_d} - \frac{d-1}{2} \nabla_{\nu} h_{\mu_1 \nu \mu_2 \cdots \mu_d} = 0 \]

Special gauge for solutions:

1) $\nabla^\nu h_{\nu \mu_2 \cdots \mu_d} = 0$
2) $g^{\nu_1 \nu_2} h_{\nu_1 \nu_2 \mu_3 \cdots \mu_d} = 0$
Boundary to bulk propagator.

Boundary conditions:

Consider the traceless tensor field $V_{i_1 \ldots i_d}(x)$ on the boundary. Given $V_{i_1 \ldots i_d}(x)$ we can deform the action:

$$ S = \int d^d x \; g_{i_1 \ldots i_d}(x) V_{i_1 \ldots i_d}(x) $$

In the bulk, the corresponding solution of the free higher spin equations should have the following boundary behaviour:

$$ \left. \left( \frac{1}{r^2} h[V] \right) \right|_{i_1 \ldots i_d}(x) = V(x)_{i_1 \ldots i_d} $$

(Explanation: $h[V] = h[V]^{\mu_1 \ldots \mu_d}$ — contravariant tensor of rank $d$ in the bulk; we use natural restriction of vector fields to the boundary.)

It is not obvious that the restriction of $\frac{1}{r^2} h[V]$ to the boundary is traceless.

If one chooses the gauge $\partial_{\mu} h^{\mu_{1 \ldots d}} = 0$ then one can show that $h_{i_2 i_3 \ldots i_d} \sim x^2$, $h_{i_1 \ldots i_d} \sim x^2$.

Global transformations on the boundary and gauge transformations in the bulk:

$$ \frac{\partial}{\partial x^\mu} G_{\mu \nu} (x_0, x^\tau)_{\mu_1 \ldots \mu_d} \Lambda_{\nu_1 \ldots \nu_d} = \nabla_{\mu_1} \Lambda_{\mu_2 \ldots \mu_d} $$

where $\Lambda_{\mu_2 \ldots \mu_d} = \left( \frac{x_0}{x_0 + \frac{x^\tau}{2}} \right)^{d-1} \partial_{\mu_2} \frac{2 x_0}{x_0 + \frac{x^\tau}{2}} \ldots \partial_{\mu_d} \frac{2 x_0}{x_0 + \frac{x^\tau}{2}}$

Suppose that $\tilde{z}_{i_2 \ldots i_d}$ is a conformal Killing tensor on the boundary, $\tilde{z}_{i_2 \ldots i_d} G_{i_1 \ldots i_d} = 0$

$$ \tilde{z}_{i_2 \ldots i_d} \nabla_{\mu_1} \int d^d x \; \tilde{z}_{i_2 \ldots i_d}(x^\tau) \Lambda_{\mu_2 \ldots \mu_d} (x^\tau) = 0 $$

Therefore

$$ \Lambda(z) := \int d^d x \; \tilde{z}_{i_2 \ldots i_d}(x^\tau) \Lambda_{\mu_2 \ldots \mu_d} (x^\tau) $$

is a traceless Killing tensor in the bulk.
Free higher spin theory is invariant under
\[ \delta^\Lambda h^{\mu_1 \ldots \mu_3} = \partial^{[\mu_1} N_{\mu_2 \ldots \mu_3]} \]
Suppose that we have the triple interaction invariant under this \( \delta^\Lambda \) on shell; then
\[ \langle \bar{\theta}_i \bar{j}^{[\mu_1}(x_1) j^{\nu_2}(x_2) j^{\nu_3]}(x_3) \rangle = 0 \]
If all the three point functions of the theory on the boundary are correctly reproduced, then the triple interaction cannot possibly be invariant off shell.

To compensate for that, we need to deform \( \delta^\Lambda \):
\[ \delta^\Lambda = \delta^{(0)} + \lambda \delta^{(1)} + \ldots \]
\[ \nabla \delta^\Lambda = \lambda \nabla h \]
For \( \Lambda \) a Killing tensor, \( \delta^{(1)} \) should be a symmetry of the free action, and of the triple interaction.

Consider \( S \) free:
\[ S^{(0)}_{\text{free}} [h_{h_1}, h_{h_2}] = 0 \] because \( \Lambda \) goes either to 0 or to Killing near both \( x_1 \) and \( x_2 \)
\[ S^{(1)}_{\Lambda} h \in \Lambda_0 \left[ h_{h_1} + h_{h_2} \right] = S_0 \left[ \delta^{(0)}_{\Lambda} h_{h_1}, h_{h_2} \right] + S_0 \left[ h_{h_1}, \delta^{(0)}_{\Lambda} h_{h_2} \right] \]
\[ S_0 \left[ \delta^{(1)}_{\Lambda} h_{h_1}, h_{h_2} \right] = 0 \]
Conclusion: if the three point functions are correctly reproduced, and $S^{(2)}_\Lambda = S^{(3)}_\Lambda = \ldots = 0$ on shell, then the boundary S-matrix is the correlation functions of the free field theory.