Generalized Harmonic Evolutions
of Binary Black Hole Spacetimes

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Collaborators: Duncan Brown, Larry Kidder, Geoffrey Lovelace, Robert Owen, Oliver Rinne, Harald Pfeiffer, Mark Scheel, Saul Teukolsky

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- Constraint damping in the GH system.
- Moving Black Holes.
- Binary Black Hole Evolutions.
**Methods of Specifying Spacetime Coordinates**

We often decompose the 4-metric into its 3+1 parts:

\[ ds^2 = \psi_{ab}dx^a dx^b = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \]

The lapse \( N \) and shift \( N^i \) specify how coordinates are laid out on a spacetime manifold. Consider the unit timelike normal \( \vec{n} \):

\[ \vec{n} = \partial_\tau = (\partial t / \partial \tau) \partial_t + (\partial x^k / \partial \tau) \partial_k = (\partial_t - N^k \partial_k)/N. \]
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  \[ \vec{n} = \partial_\tau = \left( \frac{\partial t}{\partial \tau} \right) \partial_t + \left( \frac{\partial x^k}{\partial \tau} \right) \partial_k = \left( \partial_t - N^k \partial_k \right) / N. \]

- An alternate way to specify the coordinates is through the generalized harmonic gauge source function \( H^a \):

- Let \( H^a \) denote the function obtained by the action of the scalar wave operator on the coordinates \( x^a \):
  \[ H^a \equiv \nabla^c \nabla_c x^a = -\Gamma^a, \]
  where \( \Gamma^a = \psi^{bc} \Gamma^a_{bc} \) and \( \psi_{ab} \) is the 4-metric.
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  where \( \Gamma^a = \psi^{bc} \Gamma^a_{bc} \) and \( \psi_{ab} \) is the 4-metric.
- Specifying coordinates by the *generalized harmonic* (GH) method can be accomplished by choosing a gauge-source function \( H_a(x, \psi) = \psi_{ab} H^b \), and requiring that
  \[ H_a(x, \psi) = -\Gamma_a = -\Gamma_{abc} \psi^{bc}. \]
Important Properties of the GH Method

- The Einstein equations are manifestly hyperbolic when coordinates are specified using a GH gauge function:

\[
R_{ab} = -\frac{1}{2} \psi^{cd} \partial_c \partial_d \psi_{ab} + \nabla_{(a} \Gamma_{b)} + F_{ab}(\psi, \partial \psi),
\]

where \(\psi_{ab}\) is the 4-metric, and \(\Gamma_a = \psi^{bc} \Gamma_{abc}\). The vacuum Einstein equation, \(R_{ab} = 0\), has the same principal part as the scalar wave equation when \(H_a(x, \psi) = -\Gamma_a\) is imposed.
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- The Einstein equations are manifestly hyperbolic when coordinates are specified using a GH gauge function:

\[ R_{ab} = -\frac{1}{2} \psi^{cd} \partial_c \partial_d \psi_{ab} + \nabla (a \Gamma_b) + F_{ab}(\psi, \partial \psi), \]

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- Imposing coordinates using a GH gauge function profoundly changes the constraints. The GH constraint, \( C_a = 0 \), where

\[ C_a = H_a + \Gamma_a, \]

depends only on first derivatives of the metric. The standard Hamiltonian and momentum constraints, \( \mathcal{M}_a = 0 \), are determined by the derivatives of the gauge constraint \( C_a \):

\[ \mathcal{M}_a \equiv G_{ab} n^b = \left[ \nabla (a C_b) - \frac{1}{2} \psi_{ab} \nabla^c C_c \right] n^b. \]
Pretorius (based on a suggestion from Gundlach, et al.) modified the GH system by adding terms proportional to the gauge constraints:

\[ 0 = R_{ab} - \nabla_{(a}C_{b)} + \gamma_0 \left[ n_{(a}C_{b)} - \frac{1}{2} \psi_{ab} n^c C_c \right], \]

where \( n^a \) is a unit timelike vector field. Since \( C_a = H_a + \Gamma_a \) depends only on first derivatives of the metric, these additional terms do not change the hyperbolic structure of the system.
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Evolution of the constraints \( C_a \) follow from the Bianchi identities:

\[ 0 = \nabla^c \nabla_c C_a - 2\gamma_0 \nabla^c \left[ n_{(c} C_{a)} \right] + C^c \nabla_{(c} C_{a)} - \frac{1}{2} \gamma_0 n_a C^c C_c. \]

This is a damped wave equation for \( C_a \), that drives all small short-wavelength constraint violations toward zero as the system evolves (for \( \gamma_0 > 0 \)).
Numerical Tests of the New GH System

- 3D numerical evolutions of static black-hole spacetimes illustrate the constraint damping properties of our GH evolution system.
- These evolutions are stable and convergent when $\gamma_0 = \gamma_2 = 1$.
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.

- Problems:
  - Re-gridding/interpolation is expensive.
  - Difficult to get smooth extrapolation at trailing edge of horizon.
  - Causality trouble at leading edge of horizon.

Solution:
- Choose coordinates that smoothly track the location of the black hole.
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.

Diagram showing the progression of a black hole over time with concentric circles representing the movement of the hole through the grid.
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Evolving Black Holes in Rotating Frames

- Coordinates that rotate with respect to the inertial frames at infinity are needed to track the horizons of orbiting black holes.
- Evolutions of Schwarzschild in rotating coordinates are unstable.

Evolutions shown use a computational domain that extends to \( r = 1000M \).

- Angular velocity needed to track the horizons of an equal mass binary at merger is about \( \Omega \approx 0.2/M \).
- Problem caused by asymptotic behavior of metric in rotating coordinates: \( \psi_{tt} \sim \rho^2 \Omega^2 \), \( \psi_{ti} \sim \rho \Omega \), \( \psi_{ij} \sim 1 \).

\[ \begin{align*}
\Omega &= 0.2/M \\
\Omega &= 0.02/M \\
\Omega &= 0.0002/M
\end{align*} \]
Dual-Coordinate-Frame Evolution Method

- Single-coordinate frame method uses the one set of coordinates, $x^{\bar{a}} = \{\bar{t}, \bar{x}^i\}$, to define field components, $u^{\alpha} = \{\psi_{\bar{a}\bar{b}}, \Pi_{\bar{a}\bar{b}}, \Phi_{\bar{i}\bar{a}\bar{b}}\}$, and the same coordinates to determine these components by solving Einstein’s equation for $u^{\bar{\alpha}} = u^{\bar{\alpha}}(x^{\bar{a}})$:

$$\partial_{\bar{t}}u^{\bar{\alpha}} + A^{\bar{k}\bar{\alpha}}_{\bar{\beta}}\partial_{\bar{k}}u^{\bar{\beta}} = F^{\bar{\alpha}}.$$
Dual-Coordinate-Frame Evolution Method

- Single-coordinate frame method uses the one set of coordinates, $x^\tilde{a} = \{\tilde{t}, x^\tilde{i}\}$, to define field components, $u^\tilde{\alpha} = \{\psi_{\tilde{a}\tilde{b}}, \Pi_{\tilde{a}\tilde{b}}, \Phi_{\tilde{i}\tilde{a}\tilde{b}}\}$, and the same coordinates to determine these components by solving Einstein’s equation for $u^\tilde{\alpha} = u^\tilde{\alpha}(x^\tilde{a})$:

$$\partial_\tilde{t} u^\tilde{\alpha} + A^\tilde{k}_{\tilde{\alpha}\tilde{\beta}} \partial_\tilde{k} u^\tilde{\beta} = F^\tilde{\alpha}.$$ 

- Dual-coordinate frame method uses basis vectors of one coordinate system to define components of fields, and a second set of coordinates, $x^a = \{t, x^i\} = x^a(x^\tilde{a})$, to represent these components as functions, $u^\tilde{\alpha} = u^\tilde{\alpha}(x^a)$. 

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Dual-Coordinate-Frame Evolution Method

- Single-coordinate frame method uses the one set of coordinates, $x^\bar{a} = \{\bar{t}, x^\bar{i}\}$, to define field components, $u^\bar{\alpha} = \{\psi_{\bar{a} \bar{b}}, \Pi_{\bar{a} \bar{b}}, \Phi_{\bar{i} \bar{a} \bar{b}}\}$, and the same coordinates to determine these components by solving Einstein’s equation for $u^\bar{\alpha} = u^\bar{\alpha}(x^\bar{a})$:

$$\partial_{\bar{t}}u^\bar{\alpha} + A^k_{\bar{k} \bar{\alpha}} \partial_k u^\bar{\beta} = F^\bar{\alpha}.$$  

- Dual-coordinate frame method uses basis vectors of one coordinate system to define components of fields, and a second set of coordinates, $x^a = \{t, x^i\} = x^a(x^\bar{a})$, to represent these components as functions, $u^\bar{\alpha} = u^\bar{\alpha}(x^a)$.

- These functions are determined by solving the transformed Einstein equation:

$$\partial_t u^\bar{\alpha} + \left[ \frac{\partial x^i}{\partial t} \delta^\bar{\alpha}_{\bar{\beta}} + \frac{\partial x^i}{\partial x^k} A^k_{\bar{k} \bar{\alpha}} \bar{\beta} \right] \partial_i u^\bar{\beta} = F^\bar{\alpha}.$$
Testing Dual-Coordinate-Frame Evolutions

- Single-frame evolutions of Schwarzschild in rotating coordinates are unstable, while dual-frame evolutions are stable:

  Dual Frame Evolution

  ![Dual Frame Evolution Graph]

  Single Frame Evolution

  ![Single Frame Evolution Graph]

- Dual-frame evolution shown here uses a comoving frame with $\Omega = 0.2/M$ on a domain with outer radius $r = 1000M$. 
Horizon Tracking Coordinates

- Coordinates must be used that track the motions of the holes.
- For equal mass non-spinning binaries, the centers of the holes move only in the $z = 0$ orbital plane.
- The coordinate transformation from inertial coordinates, $(\bar{x}, \bar{y}, \bar{z})$, to co-moving coordinates $(x, y, z)$,

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{a(\bar{t})} \begin{pmatrix} \cos \varphi(\bar{t}) & -\sin \varphi(\bar{t}) & 0 \\ \sin \varphi(\bar{t}) & \cos \varphi(\bar{t}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix},
$$

with $t = \bar{t}$, is general enough to keep the holes fixed in co-moving coordinates for suitably chosen functions $a(\bar{t})$ and $\varphi(\bar{t})$.

- Since the motions of the holes are not known \textit{a priori}, the functions $a(\bar{t})$ and $\varphi(\bar{t})$ must be chosen dynamically and adaptively as the system evolves.
Horizon Tracking Coordinates II

- Measure the comoving centers of the holes: $x_c(t)$ and $y_c(t)$, or equivalently $Q^x(t) = [x_c(t) - x_c(0)]/x_c(0)$ and $Q^y(t) = y_c(t)/x_c(t)$.
- Choose the map parameters $a(t)$ and $\varphi(t)$ to keep $Q^x(t)$ and $Q^y(t)$ small.
Measure the comoving centers of the holes: $x_c(t)$ and $y_c(t)$, or equivalently $Q^x(t) = [x_c(t) - x_c(0)]/x_c(0)$ and $Q^y(t) = y_c(t)/x_c(t)$.

Choose the map parameters $a(t)$ and $\varphi(t)$ to keep $Q^x(t)$ and $Q^y(t)$ small.

Changing the map parameters by the small amounts, $\delta a$ and $\delta \varphi$, results in associated small changes in $\delta Q^x$ and $\delta Q^y$:

$$\delta Q^x = -\delta a, \quad \delta Q^y = -\delta \varphi.$$
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Changing the map parameters by the small amounts, $\delta a$ and $\delta \varphi$, results in associated small changes in $\delta Q^x$ and $\delta Q^y$:

$$\delta Q^x = -\delta a, \quad \delta Q^y = -\delta \varphi.$$

Measure the quantities $Q^y(t)$, $dQ^y(t)/dt$, $d^2Q^y(t)/dt^2$, and set

$$\frac{d^3 \varphi}{dt^3} = \lambda^3 Q^y + 3\lambda^2 \frac{dQ^y}{dt} + 3\lambda \frac{d^2 Q^y}{dt^2} = -\frac{d^3 Q^y}{dt^3}.$$

The solutions to this “closed-loop” equation for $Q^y$ have the form $Q^y(t) = (At^2 + Bt + C)e^{-\lambda t}$, so $Q^y$ always decreases as $t \to \infty$. 

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Horizon Tracking Coordinates III

- In practice the coordinate maps are adjusted only at a prescribed set of adjustment times \( t = t_i \).
- In the time interval \( t_i < t < t_{i+1} \) we set:

\[
    \phi(t) = \phi_i + (t - t_i) \frac{d\phi_i}{dt} + \frac{(t - t_i)^2}{2} \frac{d^2\phi_i}{dt^2} \\
    + \frac{(t - t_i)^3}{2} \left( \lambda \frac{d^2Q^y_i}{dt^2} + \lambda^2 \frac{dQ^y_i}{dt} + \lambda^3 \frac{Q^y_i}{3} \right),
\]

where \( Q^x, Q^y \), and their derivatives are measured at \( t = t_i \), so these maps satisfy the closed loop equation at \( t = t_i \).
Horizon Tracking Coordinates III

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\[
\varphi(t) = \varphi_i + (t - t_i) \frac{d\varphi_i}{dt} + \frac{(t - t_i)^2}{2} \frac{d^2\varphi_i}{dt^2} + \frac{(t - t_i)^3}{6} \left( \lambda \frac{d^2 Q_i^y}{dt^2} + \lambda^2 \frac{dQ_i^y}{dt} + \lambda^3 Q_i^y \right),
\]

where \( Q^x, Q^y \), and their derivatives are measured at \( t = t_i \), so these maps satisfy the closed loop equation at \( t = t_i \).

- **This works!** We are now able to evolve binary black holes using horizon tracking coordinates until just before merger.
Evolving Binary Black Hole Spacetimes

- We can now evolve binary black hole spacetimes with excellent accuracy and computational efficiency through many orbits.

Head-on Merger Movie

Lapse-$\Psi_4$ Movie
Evolving Binary Black Hole Spacetimes II

- Gravitational waveform and frequency evolution for the equal mass non-spinning BBH evolution.

![Graphs showing gravitational waveform and frequency evolution for BBH evolution.](image-url)
Evolving Binary Black Hole Spacetimes III

- Initial steps in convergence testing the 15 orbit evolution:

![Graphs showing convergence of orbit phase and horizon area over time.](image)
Reducing Orbital Eccentricity

- Astrophysical BBH systems are expected to have almost circular orbits by the time of merger.
- Commonly used “quasi-circular” initial data approximate the small radiation reaction driven radial velocities by setting them to zero.
- Our group (Pfeiffer and Lovelace) have constructed better BBH initial data with radial velocities chosen to reduce the orbital eccentricity.
Comparing Waveforms for Low Eccentricity Orbits

- Orbital eccentricity has little effect on gravitational waveforms.
- Overlap integrals between the low eccentricity orbit waveforms and the “quasi-circular” waveforms are greater than 0.99.
- Graphs compare low eccentricity waveforms (black) with “quasi-circular” waveforms (red).