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1. point vs strings (Poisson algebras vs. Courant-Dorfman alg)
2. Examples of CD algebras
3. Formal quantization of these structures
4. List of exact results
5. Summary

Quantization of Poisson manifolds

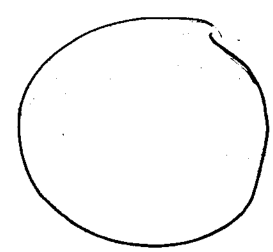
$$C^\infty(M), \ \cdot, \ \{ , \} \ \rightsquigarrow \ f *_\hbar g = fg + \hbar \{f, g\} + O(\hbar^2)$$

$\rightsquigarrow$ Poisson alg.

string / σ-model

$\{$conformal weight $0\} = R$ – commutative algebra

$\{$conformal weight $1\} = \mathcal{E}$ – R-module



σ

$$\tilde{\sigma} = f(\sigma)$$

$$\tilde{A}(\tilde{\sigma}) = \left(\frac{df}{d\sigma}\right)^{-h} A(\sigma)$$

h – conformal weight

$\partial : R \to \mathcal{E}$ $\quad A, B \in \mathcal{E}$ $\quad a, b \in R$

$$\{\underset{1}{A(\sigma)}, \underset{1}{B(\sigma')}\} = \underbrace{(A*B)(\sigma')}_{1}\,\underbrace{\delta(\sigma-\sigma')}_{1} + \underbrace{\langle A,B\rangle}_{0}\,\underbrace{\partial_\sigma \delta(\sigma-\sigma')}_{2}$$

$* : \mathcal{E}\otimes\mathcal{E} \to \mathcal{E}$

$\langle\ ,\ \rangle : \mathcal{E}\otimes\mathcal{E} \to R$ $\qquad A \star b = \langle A, \partial b\rangle$

$$\{\underset{1}{A(\sigma)}, \underset{0}{b(\sigma')}\} = \underbrace{(A\star b)(\sigma')}_{0}\,\underbrace{\delta(\sigma-\sigma')}_{1}$$

Poisson algebra

$$\{a(\sigma), b(\sigma')\} = 0.$$

~~Poisson algebra~~ Axioms:

(1) $A*(cB) = c(A*B) + \langle A, \partial c\rangle B$

(2) $\langle A, \partial\langle B,C\rangle\rangle = \langle A*B, C\rangle + \langle B, A*C\rangle$

(3) $A*B + B*A = \partial\langle A,B\rangle$

(4) $A*(B*C) = (A*B)*C + B*(A*C)$

(5) $(\partial b)*A = 0$

(6) $\langle \partial a, \partial b\rangle = 0$.

$(R, \mathcal{E}, \partial, \langle , \rangle, *)$ Courant-Dorfman algebra

(2009, D. Roytenberg)

<u>Remark</u> If $\langle , \rangle$ is nondegenerate, then (1), (5), (6) are redundant.

<u>Math</u>

<u>Th</u> PVA generated by objects of conformal weight $0, 1$ $\longleftrightarrow$ CD algebras

$\parallel$

(Poisson vertex alg)

$\updownarrow$

Poisson algebra

<u>Example</u>

① Kac-Moody

$$(\mathbb{C}, \mathfrak{g}, \partial = 0, \langle , \rangle, [,])$$

② M - smooth manifold

$R = C^\infty(M)$

$\mathcal{E} = \Gamma(TM + T^*M)$

$$\langle v_1+\beta_1, v_2+\beta_2\rangle \equiv \iota_{v_1}\beta_2 + \iota_{v_2}\beta_1$$

$$(v_1+\beta_1) * (v_2+\beta_2) = \{v_1, v_2\} + \mathcal{L}_{v_1}\beta_2 - \iota_{v_1} d\beta_1$$

(Dorfman bracket)

$$\partial = d : C^\infty(M) \to \Gamma(TM + T^*M)$$

<u>Example</u> (M, H), $H \in \Omega^3(M)$, $dH = 0$

$R = C^\infty(M)$, $\mathcal{E} = \Gamma(TM + T^*M)$

$\partial = d_H = d + H\wedge$

$\langle\, ,\rangle$ same as before

$[\;\;] = \cdots + \iota_{v_1}\iota_{v_2} H$

<u>In local coordinates</u>

<u>example 2</u>

$$\{X^\mu(\sigma), X^\nu(\sigma')\} = 0$$

$$\{X^\mu(\sigma), p_\nu(\sigma')\} = \delta^\mu{}_\nu\,\delta(\sigma-\sigma')$$

$$\{p_\mu(\sigma), p_\nu(\sigma')\} = 0$$

example 3 $\{P_\mu(\sigma), P_\nu(\sigma')\} = H_{\mu\nu\rho}\, \partial X^\rho(\sigma')\, \delta(\sigma-\sigma')$

Quantization

1. $\mathbb{R}^n$ no problem
2. $\tilde{X} = \tilde{X}(X)$, $\tilde{p}$ transform as $d\tilde{X}^\mu$ → in general, very hard

$\updownarrow$

Aut of PVA

e.g. $X^\mu(\sigma) = \sum_n X^\mu_n e^{in\sigma}$

3. $R = C^\infty(M)$, $\mathcal{E} = \Gamma(TM + T^*M)$

$\sigma \to z = e^{i\sigma}$

$\text{PVA} \xrightarrow{\text{quantize}} \text{VA}$

1. formal calculus of distribution $[[z, z^{-1}]]$

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def V - vector space

$A(z) = \sum_n z^{-1-n} A(n)$, $A_n \in \text{End}(V)$

$\forall v \in V$, $A_{(n)} v = 0$, $n \gg 0$

$|0\rangle \in V$, ∂ - even end, $A \to Y(A,z) = A(z)$

1. $Y(|0\rangle, z) = \mathrm{id}$, $Y(A,z)|0\rangle = A + O(z)$, $\partial|0\rangle = 0$
2. $[\partial, Y(A,z)] = \partial_z Y(A,z)$
3. $(z-w)^n [Y(A,z), Y(B,w)] = 0$, $n \gg 0$.

$\boxed{\text{① formal calculus of distr } \mathcal{L}[[z, z^{-1}]]}$

$L = \beta \partial \gamma \quad c = 2n$

$\mathbb{R}^n \quad X^\mu, P_\mu \longleftrightarrow (\gamma, \beta)$-system

$$[\gamma^\mu(z), \beta_\nu(w)] = \delta^\mu_\nu \, \delta(z-w)$$

$$\alpha_\pm = \frac{\beta \pm \partial\gamma}{\sqrt{2}}$$

$\Psi_+, \Psi_- \longleftrightarrow (b,c)$ system

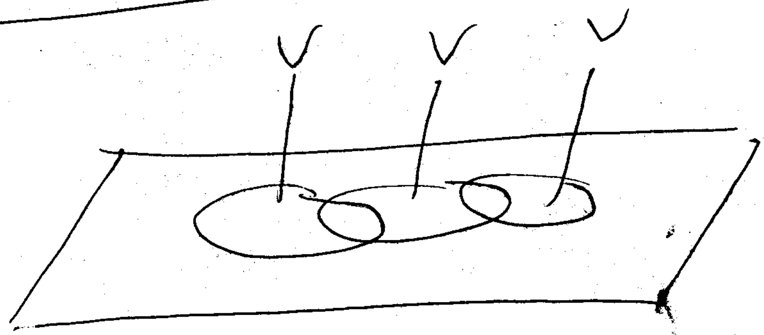
$$\Phi^\mu(z,\theta) = \gamma^\mu(z) + \theta c^\mu(z)$$

$$S_\mu(z,\theta) = b_\mu(z) + \theta \beta_\mu(z)$$

$$[\Phi^\mu(z,\theta), S_\nu(w,\bar\theta)] = \delta^\mu_\nu \hbar \, \delta^2(z-w, \theta-\bar\theta)$$

$$\left.\begin{array}{l} \tilde{\phi}^\mu = f(\phi) \\ \tilde{S}_\mu = :\dfrac{\partial f^\nu}{\partial \phi^\mu} S_\nu: \end{array}\right\} \text{ aut of vertex alg.}$$

Sheaf of susy VA



global sections

$$V_\hbar \hookrightarrow M$$

$$\Big\downarrow \hbar = 0$$

(PrA)

We can do coordinate-free

$R = C^\infty(M)$ $\quad \mathcal{E} = \Gamma(TM + T^*M)$ $\quad$ CD algebra

$\downarrow$

vertex algebra

Theorem (Heluani & M.Z.)

$V_\hbar \supset N=2$ superconformal $\Longleftrightarrow$ (M is generalized complex manifold)

$c = 3 \dim M$.

Theorem (Heluani)

$\left[V_\hbar \supset \begin{array}{c} \text{two commuting copies} \\ \text{of } N=2 \text{ superconformal} \\ c = \frac{3}{2}\dim M \end{array} \right] \longleftarrow$ (M is CY manifold with Ricci flat metric)

Theorem (Heluani & M.Z.)

$\left[V_\hbar \supset \begin{array}{c} \text{two commuting copies} \\ \text{of } N=2 \end{array} \right] \longleftarrow$ (M is general metric CY)

$c = \frac{3}{2}\dim M$

$\Uparrow$

$\underline{d=6}$ (g, H, ϕ) Type II $N=2$ SUSY background

Theorem (Heluani, Ekstrand, Källén & M.Z.)

$\left[V_\hbar \supset \begin{array}{c} \text{two commuting copies} \\ \text{of Odaka algebra} \end{array} \right] \longleftarrow \left[\begin{array}{c} \text{CY 3-fold with} \\ \text{Ricci flat metric} \end{array} \right]$

$$\begin{tikzcd}
\underline{\text{Bressler (2002)}} & & \underline{\text{Diff geom}} \\
V \arrow[d, shift right, "\text{truncation}"'] & & \text{Hitchin (2002)} \arrow[d] \\
\text{Courant algebra} \arrow[u, shift right, "0,1"'] \arrow[rr, leftrightarrow] & & \text{Courant brackets}
\end{tikzcd}$$