23 August 2011
P. Sułkowski

Topological recursion, remodeling & quantization

(1) motivation (algebraic curves → quantum curves, invariants)

(2) topological recursion (statement, properties, origin, remodeling)

(3) re(constructing) quantum curves

(4) mirror curves & Faber-Pandharipande formula

Refs
- Eynard, Orantin math-ph/0702045
- Bouchard, Klemm, Mariño, Pasquetti 0709.1453
- Gukov, PS 1108.0002
- Bouchard, PS 1105.2052
- Bouchard, Mackal, PS 1108.2689

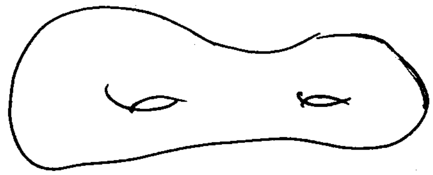
(1) Motivation

- algebraic curve

$$(u,v) \in \mathbb{C} \times \mathbb{C}$$

$$A(u,v) = 0$$

$$\begin{cases} u = u(p) \\ v = v(p) \end{cases} \quad p \in \Sigma$$



$$\omega \simeq \frac{i}{\hbar}\, du \wedge dv$$

$$(x = e^{u},\ y = e^{v}) \in \mathbb{C}^* \times \mathbb{C}^*$$

$$\omega \simeq \frac{i}{\hbar}\, \frac{dx}{x} \wedge \frac{dy}{y}$$

<u>"closed" invariants, F_g</u>

$$Z_{\text{closed}} = e^{\sum_{g=0}^{\infty} \hbar^{g-2} F_g}$$

<u>"open" invariants, $S_k(p)$</u>

$$W^g_n(p_1, \text{—}\, p_n)$$

$$\psi(p) = e^{\sum \hbar^{k-1} S_k} \qquad \left(\simeq \langle \det(u(p) - M) \rangle\right)$$

$$\hat{A}\,\psi(p) = 0 \qquad \hat{A}_0 = A$$

$$\boxed{\hat{A} = \hat{A}(\hat{u}, \hat{v}) = \hat{A}_0 + \hbar \hat{A}_1 + \hbar^2 \hat{A}_2 + \_\_\_}$$

$$\hat{x} = e^{\hat{u}},\ \hat{y} = e^{\hat{v}},\ \hat{y}\hat{x} = q\hat{x}\hat{y},\ q = e^{\hbar}$$

$$[\hat{v}, \hat{u}] = \hbar,\ \hat{v} = \hbar \frac{\partial}{\partial u};$$

Examples

① Airy curve

$$A(u,v) = u - v^2 = 0$$

$$u = p^2 \qquad \text{genus } 0$$
$$v = p$$

- eigenvalue statistics in matrix models
- $\mathcal{M}_{g,n}$ $\quad \psi_i = c_1(\mathcal{L}_i)$

$$\langle \psi_1^{d_1} \ldots \psi_n^{d_n} \rangle = \int_{\mathcal{M}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n} \neq 0 \text{ if } d_1 + \ldots d_n = 3g - 3 + n.$$

$$\psi(u) = Ai(u) = \int dz \; e^{\frac{1}{\hbar}(uz - \frac{z^3}{3})}$$

$$(\hbar^2 \partial_u^2 - u)\psi(u) = 0$$

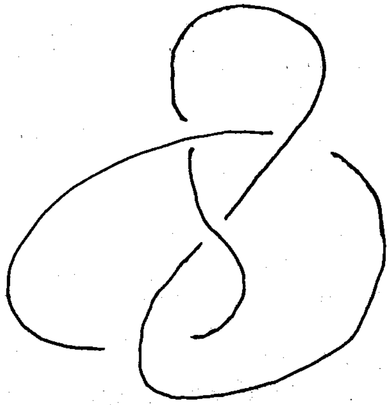
② c=1

$$A(u,v) = u^2 - v^2 + 2t$$

$$\hat{A}(\hat{u},\hat{v}) = \underbrace{\hat{u}^2 - \hat{v}^2 + 2t}_{\hat{A}_0} + \underbrace{\hbar}_{\hbar \hat{A}_1}$$

3 Chern-Simons

A-polynomial $\quad M = S^3 \setminus \{K\}$



4_1

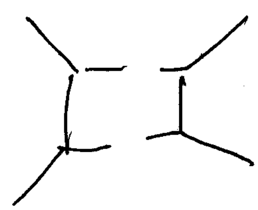
$x, y \in \mathbb{C}^*$

$$A(x,y) = (1-x^2-2x^4-x^6+x^8)y - x^4 - x^4y^2 = 0$$

$$\hat{A}(\hat{x},\hat{y}) = (1-q^4\hat{x}^4)(1-q^2\hat{x}^2-(q^2+q^4)\hat{x}^4+q^8\hat{x}^8)\hat{y} +$$
$$-q^3(1-q^4\hat{x}^4)\hat{x}^4 - q^5(1-q^2\hat{x}^4)\hat{x}^4\hat{y}^2$$

$\hat{A}$ (N-colored Jones polynomial)

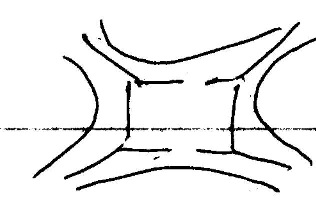
4 Top strings



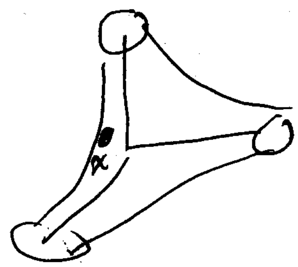
A-model top. strings on toric manifolds

$\Longleftrightarrow$ B-model $\quad z_1 z_2 = A(x,y)$

mirror curve: $A(x,y)=0$



e.g. $\mathbb{C}^3$



$$A_{\mathbb{C}^3} = 1 + x + y = 0$$

$$\psi(u) = (\text{B-brane amplitude}) = x^{\frac{1}{2}} \prod_{k=1}^{\infty} (1 + q^{k-\frac{1}{2}})^{-1}$$

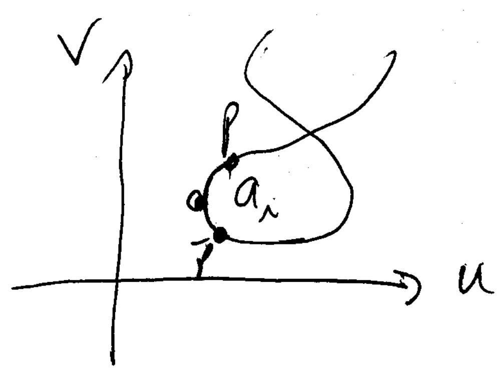
$$\hat{A}\psi = (1 + q^{\frac{1}{2}}\hat{x} + q^{-\frac{1}{2}}\hat{y})\psi(x) = 0$$

$$q = e^{g_s}$$

② Topological recursion

↳ associate, to a given curve $A(u,v)=0$, $\subset \mathbb{C}^* \times \mathbb{C}$
two sets of objects

$$\begin{cases} - \; W^g_n(p_1 \ldots p_n) \\ - \; F_g \end{cases} \qquad \begin{cases} u = u(p) \\ v = v(p) \end{cases}$$



- branch points: $du(a_i) = 0$
- conjugate point: $u(\bar{p}) = u(p)$
- Bergman kernel: $B(p,q) = \dfrac{dz(p)\,dz(q)}{(z(p)-z(q))^2}$ + finite

e.g. $g=0 \Rightarrow B(p,q) = \dfrac{dp\,dq}{(p-q)^2}$

$g=1 \Rightarrow B(p,q) = \left(\wp(p-q;\tau) + \dfrac{\pi}{\operatorname{Im}\tau} \right) dp\,dq$

– recursion kernel:

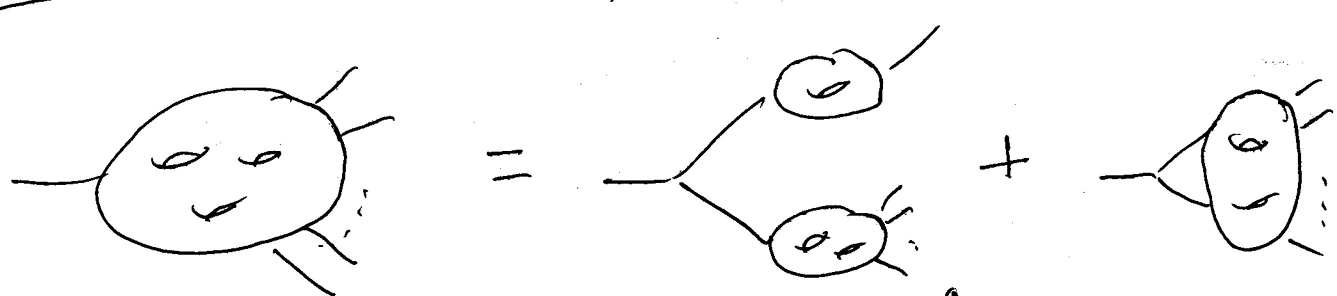
$$K_q(p) = \frac{\int_q^{\bar q} B(\xi, p)}{(y(q) - y(\bar q))\,dx(q)}$$

– primitive $\Phi(q) = \int^{q} y\,dx$

$W_1^0 = 0$

$W_2^0(p_1,p_2) = B(p_1,p_2)$

<u>Recursion</u> $\qquad \vec{P}_N = \{p_1\, p_2 \ldots p_n\}$



$$W_{n+1}^g(p, \vec{P}_N) = \sum_i \operatorname*{Res}_{q\to a_i} K_q(p) \Big[\sum_{m=0}^{g} \sum_{J\subset N} W^m_{|J|+1}(q, \vec{P}_J)\, W^{g-m}_{n-|J|+1}(\bar q, \vec{P}_{N\setminus J}) + W^{g-1}_{n+2}(q, \bar q, \vec{P}_N) \Big]$$

$$F_g = \sum_i \operatorname*{Res}_{q \to a_i} \phi(q) W_1^g(q)$$ — symplectic invariant $du \wedge dv$

BKMP: apply to curves in $\mathbb{C}^* \times \mathbb{C}^*$

$$\left\langle \left(\mathrm{Tr} \frac{1}{u(p_1) - M} \right) \cdots \left(\mathrm{Tr} \frac{1}{u(p_n) - M} \right) \right\rangle_c$$

$$= \sum_{g=0}^{\infty} \hbar^{2g-2+k} \frac{W_k^g(p_1 \cdots p_n)}{du(p_1) \cdots du(p_n)}$$

$$\sum_k \frac{1}{k!} \int^p \cdots \int^p (\;\;{}'' \;\;)$$

$$\langle \det (u(p) - M) \rangle = \exp \left(\sum_{h=0}^{\infty} \hbar^{k-1} S_k \right) \equiv \psi(p)$$

$$S_k = \sum_{n = 2g+k-1} \frac{1}{k!} \int^p \cdots \int^p W_n^g(p_1 \cdots p_n),$$

$k \geq 2$

$\uparrow$

$S_k(u, v)$

$$S_0 = \int v\,du$$

$$S_1 = -\frac{1}{2}\int^p \int^p \left(B(u(p_1) - u(p_2)) - \frac{dp_1\,dp_2}{(p_1-p_2)^2} \right)$$

genus 0 $\Rightarrow$ $S_1 = -\frac{1}{2} \ln \left(\frac{dy}{dp}\right)$ → torsion

③ quantum curves

$$\hat{A} = \hat{A}_0 + \hbar \hat{A}_1 + \text{———} \; ; \quad A_r = \hat{A}_r(\hat{u}, \hat{v})\Big|_{\substack{\hat{u} \to u \\ \hat{v} \to v}}$$

$$A_r(u,v)$$

Hierarchy of diff eq's

$\sim \hbar^n$: $\boxed{0 = \sum_{r=0}^{n} D_r A_{n-r}}$ The same form for curves in $\mathbb{C} \times \mathbb{C}$, $\mathbb{C}^* \times \mathbb{C}^*$

$$D_0 = 1$$

$$D_1 = \frac{S_0''}{2} \partial_v^2 + S_1' \partial_v$$

$$D_2 = \frac{(S_0'')^2}{8} \partial_v^4 + \text{———} + S_2' \partial_v$$

$\vdots$

$\sim\hbar^0$: $\quad 0 = A_0 = A \quad$ class. eq.

$\sim\hbar^1$: $\quad 0 = \left(\frac{S_0''}{2}\partial_v^2 A + S_1'\partial_v A\right) + A_1$

$\sim\hbar^n$: $\quad 0 = \alpha_n A + \alpha_{n-1} A_1 + \cdots + A_n$

e.g. – Airy curve $\quad A = v^2 - u$

$\Rightarrow A_2, A_3 \ldots = 0$

– $c=1$ $\quad A = u^2 - v^2 + 2t$

$\Rightarrow A_1 = 1, \quad A_2 = A_3 = \ldots = 0$

$v = \pm\sqrt{2t - u^2}$

– CS 4_1

$\hat{A} = q\hat{x}^{12} + (q^2 + q^6)\hat{x}^{16} \ldots$

any function

– e.g. $A(x,y) = x + P(y)$

genus 0 $\quad S_1 = -\frac{1}{2}\ln\frac{du}{dp}$ $\quad \begin{cases} x = -P(p) \\ y = p \end{cases}$

$$A_1 = \frac{1}{2}\left(\frac{dy}{dp}\right)^{-2}\left(\frac{d^2y}{\partial p}\partial_r - \frac{dy}{dp}\frac{dr}{dp}\partial_r^2\right) A$$

$$= \frac{x}{2} - \frac{y}{2}\frac{d\underline{P}(y)}{dy}$$

$$\Rightarrow \boxed{\hat{A} = q^{\frac{1}{2}}\hat{x} + \underline{P}(q^{-\frac{1}{2}}\hat{y})}$$

e.g. $\mathbb{C}^3$, framed. $\quad x \to xy^f$

$$A_{\mathbb{C}^3} = 1 + x + y \to A^f_{\mathbb{C}^3} = 1 + y + xy^f$$

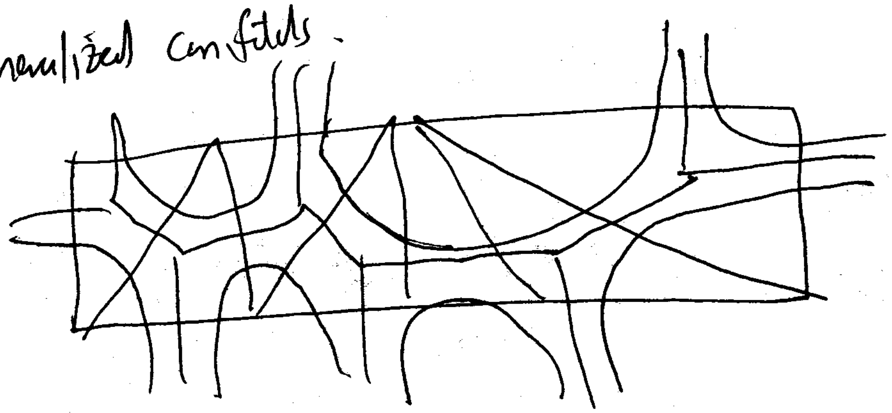
$$\sim x + \underbrace{\frac{1+y}{y^f}}_{\underline{P}(y)}$$

$$\hat{A}^f_{\mathbb{C}^3} = 1 + q^{-\frac{1}{2}}\hat{y} + q^{\frac{f+1}{2}}\hat{x}\hat{y}^f$$

$$\hat{A}^{f=0}_{\mathbb{C}^3} = 1 + q^{-\frac{1}{2}}\hat{y} + q^{\frac{1}{2}}\hat{x}$$

Compute $S_2, S_3, —$ $\Rightarrow$ $A_2, A_3, —$

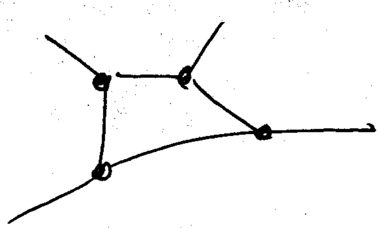
– generalized conifolds.



$$A = C(y) + x B(y) \quad \sim \quad x + \underbrace{\frac{C(y)}{B(y)}}_{P(y)}$$

$$\Rightarrow \hat{A} = C(q^{-\frac{1}{2}}\hat{y}) + q^{\frac{1}{2}} \hat{x} B(q^{\frac{1}{2}}\hat{y})$$

(4) $F_g^{\mathbb{C}^3} = $ F-P '98



$$Z = Z_{\beta=0} \, Z_{\beta \neq 0}$$

$$Z_{\beta=0} = M(q)^{\chi/2}$$