Quantum integrability and gauge theory

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Vacua of supersymmetric gauge theories
and
Quantum integrable systems

two dimensional $\mathcal{N}=(2,2)$ super-Poincare invariance
4 supercharges

$\downarrow$

(twisted) chiral ring $\to$ commutative associative algebra for which the space $\mathcal{H}^{vac}$ is a representation

$\downarrow$

Quantum integrals of motion of some QIS

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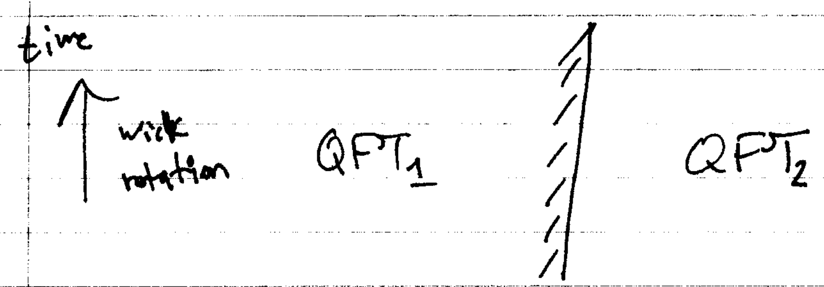
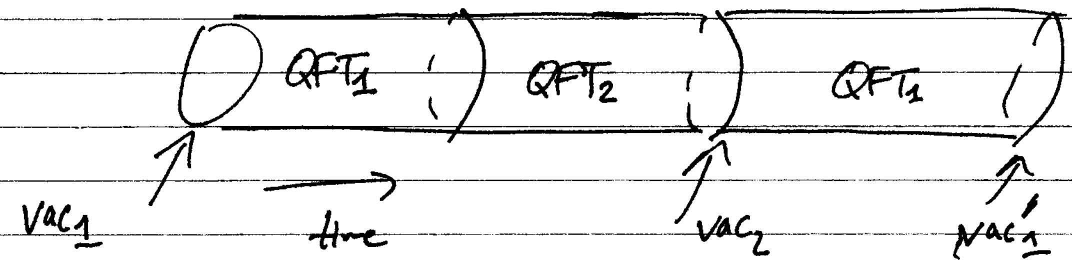
Infinite dimensional symmetry alg
Spectrum-generating
Yangian or $U_q(\widehat{sl}_n)$

get symmetries acting
(i) between vacua of a given QFT
(ii) map different QFTs to each other

Symmetries realized by domain walls





One ingredient of $\mathcal{N}=(2,2)$ supersymmetric gauge theory

twisted superpotential $\quad \widetilde{W}(\sigma)$

σ – $\mathbb{C}$-scalar in the vector multiplet

vacua are determined by $\quad \dfrac{\partial \widetilde{W}}{\partial \sigma_i} = 2\pi i n_j \qquad n_j \in \mathbb{Z}$, $\quad j = 1, \ldots, r$

at low energy $\sigma \in t_{\mathbb{C}}$

For chiral fields

$$W(x) \to V^{bos} = \left\| \frac{\partial W}{\partial x} \right\|^2$$

curvature of abelian gauge field

$$\cancel{V^{bos}} \quad \mathcal{L}^{bos} = \|F\|^2 + \mathrm{Im}\, \frac{\partial \widetilde{W}}{\partial \sigma_i} \dot{F}_i + \sum_i \left| \mathrm{Re}\, \frac{\partial \widetilde{W}}{\partial \sigma_i} \right|^2$$

$$\frac{1}{2\pi i} \int F_j = m_j \qquad \text{flux quantized}$$

$$e^{i n_j \int F_j} \qquad \text{Lagrange multiplier, resum}$$

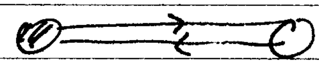
97' Losev + SS + NN

$$t = \frac{\theta}{2\pi} + i r$$ ↑ FI term

2d

<u>Ex</u> $U(N)$ gauge theory
$\mathcal{N} = (4,4)$ with L hypermultiplets in fundamental rep

color flavor



N $(\mu_1, \ldots, \mu_L)$

+ twisted masses, consistent w/ $\mathcal{N}=(2,2)$ SUSY
↑ global sym

$\mathcal{N}=4$ vector $= (\sigma, A_\mu)$ $= \mathcal{N}=2$ vector
Φ $+$ chiral in adjoint

$$W = \overset{-1}{\tilde{Q}}\,\overset{+2}{\Phi}\,\overset{-1}{Q}$$

Q_a	$-u - \mu_a$	$+\sigma_i$
$\tilde{Q}_a$	$-u + \mu_a$	$-\sigma_i$
Φ	$2u$	$+\sigma_i - \sigma_j$

$\Rightarrow$ additional twisted mass u (corresponding to sym)

When $(\mu_1, \ldots, \mu_L, u)$ are generic

all matter fields are massive $\Rightarrow$ can be integrated out

$$\tilde{W}^{\mathrm{eff}}(\sigma_1, \ldots, \sigma_N; \mu, u) = \cancel{\varpi(x)}$$

$$= \sum_{i,j} \overline{\omega}(\sigma_i - \sigma_j + 2u)$$

$$+ \sum_a \overline{\omega}(\sigma_i - \mu_a - u) + \overline{\omega}(\mu_a - \sigma_i - u)$$

$$+ t \sum_i \sigma_i$$

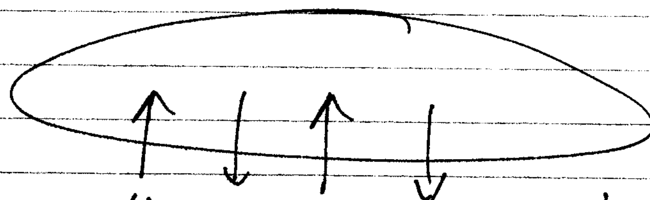
$$\overline{\omega}(x) = x(\lg x - 1)$$

Vacua

$$\prod_{a=1}^{L} \left(\frac{\sigma_i - \mu_a - u}{\sigma_i - \mu_a + u} \right) = e^{2\pi i t} \prod_{j \neq i} \left(\frac{\sigma_i - \sigma_j - 2u}{\sigma_i - \sigma_j + 2u} \right)$$

$$\varpi_\Lambda (x) = x \left(\log \left(\frac{x}{\Lambda} \right) - 1 \right)$$

Bethe equations for $XXX_{1/2}$ spin chain
inhomogeneous, twisted



μ_1 μ_2 L spins on a circle

local magnetic field $e^{2\pi i t \sigma_3}$
in a sector w/ N spin $\uparrow$

$$\mathcal{H}_{XXX} = (\mathbb{C}^2)^{\otimes L} = \bigoplus_{N=0}^{L} \mathcal{H}_N$$

$$\dim \mathcal{H}_N = \binom{L}{N}$$

SUSY vacua of $U(N)$ $\mathcal{N} = (4,4)$ + L flavors
$$\Updownarrow$$
$$\mathcal{H}_N$$

$\underline{Y}(sl_2)$ for which $\mathcal{H}_{XXX}$ is reducible (for generic μ's)

$$\mathcal{H}_{XXX} = V_{1/2}\left(\frac{\mu_1}{2\pi} \right) \otimes \cdots \otimes V_{1/2}\left(\frac{\mu_L}{2\pi} \right)$$

μ_i: generic irreducible rep of Yangian if $u_i - u_j = 2a$ $\mathcal{H}_{XXX}$ is not irreducible

level crossing between Q and $\tilde{Q}$

ex $u_a - u_{a+1}$ meson $\mathcal{O}_a = \tilde{Q}_{a+1} Q_a$ becomes massless

Can deform in two ways

$\langle \mathcal{O}_a \rangle \neq 0$ VEV

$W \to W + m\,\mathcal{O}_a$ deform by mass term

$\to$ generate couplings $\tilde{Q}\,\Phi^2\,Q$ $\rightsquigarrow$ spin 1 impurities in the spin chain

$$W^{gen} = \sum_a \tilde{Q}_a\,\Phi^{2s_a}\,Q_a$$

s_a half-integer

$-2s_a \quad +2 \quad -2s_a$

$$\tilde{Q}_{a+1}\,\Phi\,Q_{a+1} + \tilde{Q}_a\,\Phi\,Q_a + m\,\tilde{Q}_{a+1}\,Q_a$$

integrate out $\tilde{Q}_{a+1}$

$$\Rightarrow Q_a = m^{-1}\,\Phi\,Q_{a+1}$$