Tetrahedra, 3-manifolds, and Gauge Theory: T. Dimofte

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1. 2d coord systems] → 3d geometry
   MCG actions
   { Dimofte, Gukov '11 }

2. 3 manifold $M \leftrightarrow$ 3d $\mathcal{N}=2$ gauge theory T_M
   { Dimofte, Gukov, Hollands }

3. Tetrahedra / toolbox

Coordinate transformations on $\mathcal{M}_H$

$$\mathcal{M}_H = \mathcal{M}_{flat}(G_{\mathbb{C}}, C)$$

($G_{\mathbb{C}} = SL(2,\mathbb{C})$; C ← punctured Riemann surface)

"J-hol coord^s" ↔ ~~ev~~ eigen values of holonomies

$$\Omega_J = \frac{1}{2\pi}\int_C \mathrm{Tr}(\delta A \wedge \delta A)$$

(A ↑ $\mathfrak{g}_{\mathbb{C}}$)

- Fenchel-Nielson / Darboux [NRS]

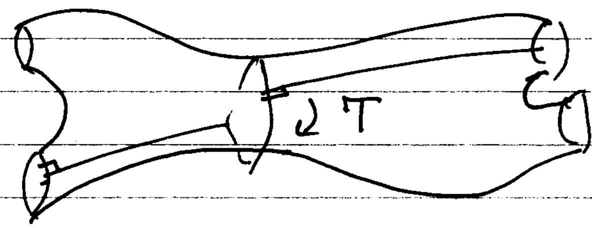
Cut C into pants γ

$SL(2,\mathbb{C})$ hol^s around γ

⇝ eigenvalues $\lambda = e^{\Lambda}$

duals $\tau \equiv e^{T}$

$SL(2,\mathbb{R})$ $\qquad$ C w/ hyp^c metric



Δ length
T twist

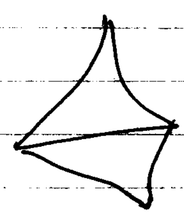
Teichmüller $\quad T$ not periodic

$$\Omega_J = \frac{1}{\hbar}\, dT \wedge d\Delta$$

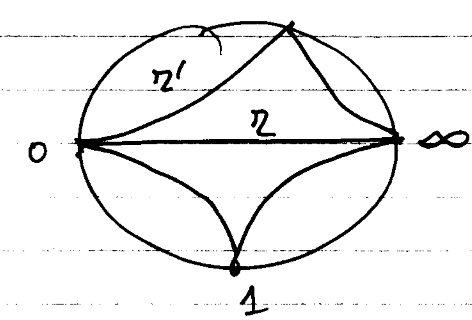
☒ <u>Shear coords</u>

Thurston, Penner, Fock
cx $\quad$ Fock-Goncharov, GMN

Surface w/ at least one puncture



real version
ideal



every edge $e \leftrightarrow \eta_e = e^{z_e}$

$\{ z_e, z_{e'} \} = \#$ faces shared by e, e' (oriented #)

$\{ \eta', \eta \} = 1$

$$\sum_{\text{puncture}} Z_s' = 2\Lambda \leftrightarrow \text{central element}$$

Q:
- transform shear $\leftrightarrow$ FN
- MCG of C acts
- quantize ?

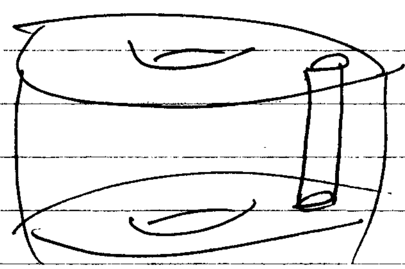
For $SL(2)$ all answered ✓

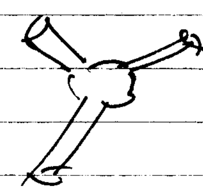
But consider flat $SL(2,\mathbb{C})$ connections on M's

Suppose M 3-manifold

$$\partial M = C_1 \sqcup C_2 \sqcup \dots \sqcup C_k$$

3-valent network of Wilson loops





$$P_{\partial M} = \mathcal{M}_{flat}(G_{\mathbb{C}}, \partial M)$$

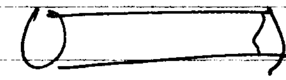
$$= P_{C_1} \times \dots \times P_{C_k}$$

$$\Omega_J$$

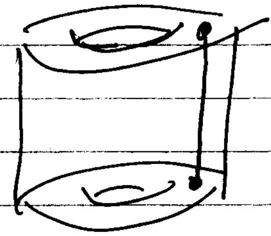
$$\cup$$

Lagrangian $\mathcal{L}_M = \mathcal{M}_{flat}(G_{\mathbb{C}}, M)$ [Gukov '03]

E.g.

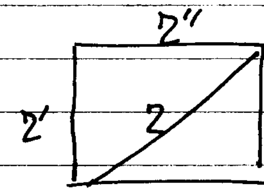


Slear



P_{C_2} $\quad z, z'$

$\mathcal{L}$ is diagonal



FN

P_{C_1} $\quad \lambda, T$

(handle body)

$$z + z' + z'' = 2A$$

$$\mathcal{L}_M \subset P_{C_1} \times P_{C_2}$$

encodes the coordinate transformations

change (A, B, A) frame

$$r = \sqrt{\frac{z}{z'}} \qquad s = \sqrt{z z'}$$

$$\mathcal{L}_M = \begin{cases} r + s + \frac{1}{s} - \lambda - \frac{1}{\lambda} = 0 \\ (s+\lambda)T + \lambda(1 - \lambda s) = 0 \end{cases}$$

Quantize

$$P_{\partial M} \to \mathcal{H}_{\partial M} \circlearrowleft \hat{A}_{\partial M}$$

$$M \to Z_M \subset \mathcal{H}_{\partial M}$$

$$\mathcal{L}_M \to \hat{\mathcal{L}}_M \cdot Z_M = 0$$

Add operator hats, and q's $\quad \boxed{q = e^{\hbar}}$

$$\hat{r} + \hat{s} + \frac{1}{s} - \hat{\lambda} - \frac{1}{\hat{\lambda}} = 0$$

$$(\hat{s} - q\hat{\lambda})\hat{T} + q$$

$$[\hat{R}, \hat{S}] = \hbar$$

$$\hat{r}\hat{s} = q\hat{s}\hat{r}$$

$$c_\hbar = i\pi + \frac{\hbar}{2}$$

$$Z_M(\Lambda, S) = e^{(\ )}\, \Phi_\hbar(\Lambda - S + c_\hbar)\, \Phi_\hbar(\tfrac{1}{2}\Lambda - S + c_\hbar)$$

Faddeev

$$\Phi_\hbar(p) = \prod_{r=1}^{\infty} \frac{1 + e^{(r - 1/2)\hbar + p}}{1 + e}$$

$$Z(\Lambda, S)\,\psi(S)$$

Choice of polarization $\quad \hat{\Lambda} = \Lambda, \quad \hat{T} = \hbar\,\partial_\Lambda$

Ideal Δ's

1. Any 3-manifold can be built from them
2. Induce either FN or shear coords on ∂M
3. There's a unique way to quantize them

$$P_\Delta = \{ (\Pi, \Pi') \} \quad Z_\Delta(Z) = \Phi_\hbar(-Z + c_\hbar)$$

Claim

$M \rightsquigarrow T_M$ $\quad \mathcal{N}=2$ thy in 3d

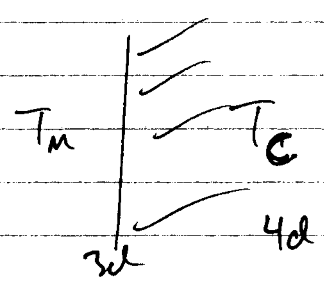
polarizations
for ∂M

- FN or shear
- $\wedge$ ↘ set of commuting edges

A, B cycles

M $\qquad \partial M = C$

↗ 3d $\mathcal{N}=2$ $\qquad$ ↑ 4d $\mathcal{N}=2$



global sym of T_M $\leftrightarrow$ gauge sym of T_C

Shear couple to IR
FN couple to UV

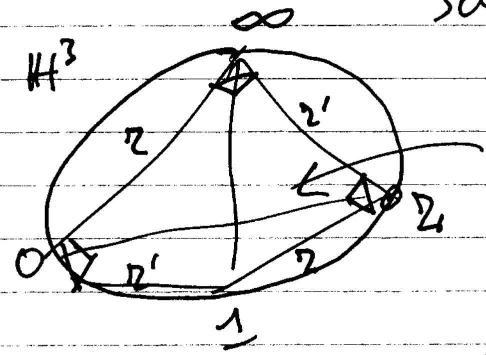
$\widehat{\mathcal{L}}_M$ algebra of loop operators

etc

$Z_M(\ldots, \hbar)$ $Z(S^3_b, \ldots)$ $\hbar = 2\pi i b^2$

Ideal Tetrahedra

3d hyp geom $\leftrightarrow$ $SL(2,\mathbb{C})$



geodesic surface

coord z $\leftrightarrow$ dihedral angles

z = torsion + i(angle)

$z z' z'' = -1$ $z + z' + z'' = i\pi$

$z + (z')^{-1} - 1 = 0$

sum of angles in euclidean triangles = 180° at bdy

shear coordinates on bdy (Kashaev, unpublished)
w/ unipotent monodromy

$\Rightarrow P_{\partial\Delta}$

$\Omega_J = \frac{1}{\hbar} dz \wedge dz'$ invariant under cyclic

$\mathcal{L}_M = \{ z + (z')^{-1} - 1 = 0 \}$ $\Leftrightarrow$ trivial holonomy q- correction ↙ difference equation

$$\hat{\mathcal{L}}_\Delta = (\hat{z} + \hat{z}'^{-1} - 1)\,\psi_\Delta(Z) = 0$$

$\Rightarrow$ ψ_Δ = QDL   quantum dilog

$\hat{z} = Z$

$\hat{z}' = \hbar\,\partial_Z$

$$\hat{Z} = Z + bm, \quad ib\partial_m = \hbar\,\partial_Z = \frac{1}{2}(\hat{Z}'' - \hat{Z}')$$

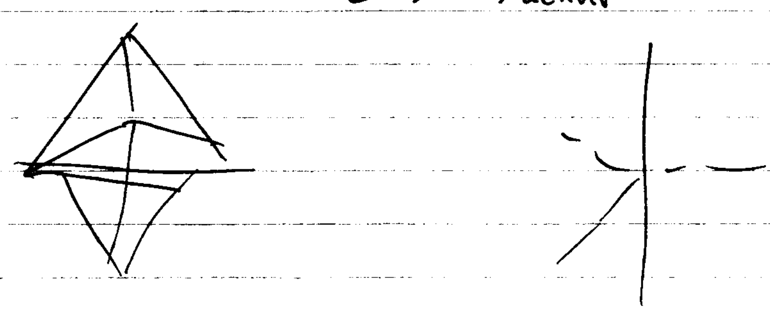
$$\psi_\Delta = s_b\left(m + \frac{i}{2}(b + b^{-1})\right)$$

$\leftrightarrow$ free chiral multiplet w/ U(1) flavor sym & real mass m $= T_\Delta$

Fix polarization ⇝ thy

$ST \in Sp(2, \mathbb{R})$

2-3 Pachner



U(1) SQED w/ $N_f = 1$ $\xleftrightarrow{MS}$ $W = XYZ$