Gravitational Field of Relativistic Gyratons

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Motivations

- Gravitational field of a photon
- Spin effects in mini-black-hole production
- Interest in Penrose limit in string theory

Plan of the Talk

- Boosting of the extended rotating sources
- Gravitational field of gyraton
- Test particles and light in the field of gyraton
- Higher-dimensional gyratons
- Generalizations (charge, AdS background)

Based on:

- V.F. & D.Fursaev, PRD **71**, 104034 (2005); hep-th/0504027
- V.F. & W. Israel, and A. Zelnikov, PRD **72**, 084031 (2005); hep-th/0506001
- V.F. and A. Zelnikov, PRD **72**, 104005 (2005); hep-th/0509044
- V.F. and A. Zelnikov, gr-qc/0512124, CQG (to appear)
- V.F. and F.-L. Lin, hep-th/0603018

Gravitational Field of Rotating Objects

$$ds^2 = -d\bar{t}^2 + d\bar{\xi}^2 + d\mathbf{x}^2 + 2\bar{A}_{ab}x^b dx^a d\bar{t}$$

$$+\bar{\Phi}[d\bar{t}^2 + \frac{1}{D-3}(d\bar{\xi}^2 + d\mathbf{x}^2)]$$

➠ For a compact (point-like) object

$$T_{\bar{t}\bar{t}} = M\delta^{D-1}(\mathbf{X})\,, \qquad T_{\bar{t}a} = J_{ab}\partial_b\delta^{D-1}(\mathbf{X})$$

$$\bar{\Phi} \sim \frac{M}{\bar{r}^{D-3}}\,, \qquad \bar{A}_{ab} \sim \frac{J_{ab}}{\bar{r}^{D-1}}$$

$$\bar{r}^2 = \bar{\xi}^2 + r^2\,, \qquad r^2 = \mathbf{x}^2$$

➠ For an extended (line-like) object

$$\bar{\Phi} \sim \int \frac{d\bar{\xi}'\ \bar{\varepsilon}(\bar{\xi}')}{[(\bar{\xi} - \bar{\xi}')^2 + \mathbf{x^2}]^{(D-3)/2}}$$

$$\bar{A}_{ab} \sim \int \frac{d\bar{\xi}'\ \bar{j}_{ab}(\bar{\xi}')}{[(\bar{\xi} - \bar{\xi}')^2 + \mathbf{x^2}]^{(D-1)/2}}$$

➠ $\bar{\varepsilon}$ and $\bar{j}_{ab}$ are the mass and angular momentum density, respectively

Boosted Metric

- t and ξ are coordinates in the frame which is moving along $\bar{\xi}$ axis with the velocity β (in the negative direction) $[\gamma = (1-\beta^2)^{-1/2}]$

$$\bar{\xi} = \gamma(\xi - \beta t) = \frac{\gamma}{\sqrt{2}}[(1-\beta)v + (1+\beta)u]$$

$$\bar{t} = \gamma(t - \beta\xi) = \frac{\gamma}{\sqrt{2}}[(1-\beta)v - (1+\beta)u]$$

- $u = (\xi - t)/\sqrt{2}$ and $v = (\xi + t)/\sqrt{2}$ are null coordinates

- For $\beta \to 1$: $\bar{t} \sim -\sqrt{2}\gamma u\,, \qquad \bar{\xi} \sim \sqrt{2}\gamma u$

$$\bar{\varepsilon}(\bar{\xi}) = \frac{1}{\sqrt{2}\gamma^2}\varepsilon(u)\,, \quad E = \gamma\bar{M} = \int \varepsilon(u)du$$

$$\bar{j}_{ab}(\bar{\xi}) = \frac{1}{\sqrt{2}\gamma}j_{ab}(u)\,, \quad J_{ab} = \bar{J}_{ab} = \int j_{ab}(u)du$$

$$\frac{\gamma}{(\gamma^2 y^2 + r^2)^{m/2}} \Longrightarrow \frac{\sqrt{\pi}\Gamma((m-1)/2)}{\Gamma(m/2)}\frac{\delta(y)}{r^{m-1}}$$

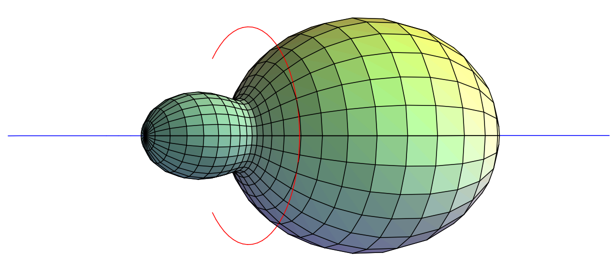
$$ds^2 = -2du\,dv + d\mathbf{x}^2 + 2A_{ab}x^b dx^a du + \Phi du^2$$

$$\Phi \sim \frac{\varepsilon(u)}{r^{(D-4)/2)}}\,, \qquad A_{ab} \sim \frac{j_{ab}(u)}{r^{(D-2)/2)}}$$

- For Aichelburg-Sexl metric $\varepsilon = E\delta(u)$, $A_{ab} = 0$

Gyraton

- Finite energy E
- Finite length (time duration) L
- Finite angular momentum (spin) J
- • + axial symmetry



Example of a gyraton: Beam-pulse of spinning radiation

EM Gyratons in 4-D

- Coordinates $(u = t - \xi, v = t + \xi, x^3, x^4)$
- EM-beam-pulse in $\xi-$direction
  - monochromatic • circularly polarized
  - axisymmetric • gauge-choice: $a_u = a_v = 0$

$$a_\mu = b(r,u)\left[e^{\pm}_\mu \exp(-i\omega u) + \bar{e}^{\pm}_\mu \exp(i\omega u)\right]$$

- $e^{\pm}_\mu = \frac{1}{\sqrt{2}}(\delta^3_\mu \pm i\delta^4_\mu), \quad r = \sqrt{(x^3)^2 + (x^4)^2}$
- The amplitude $b(r,u)$ changes slowly
- After averaging

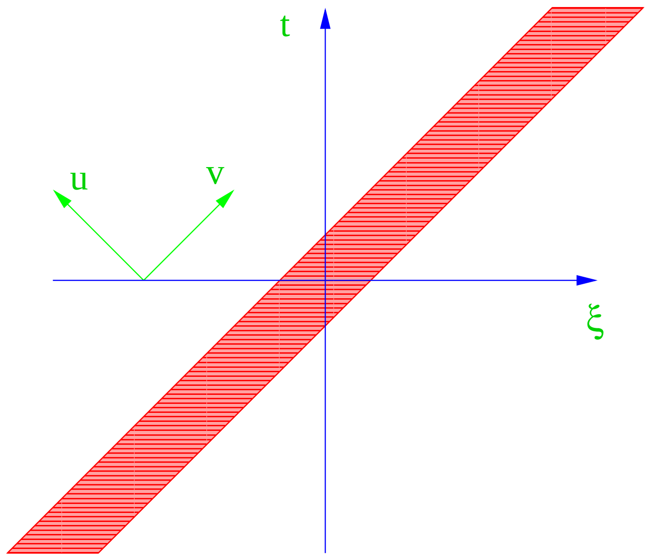
$$T_{uu} = 2\omega^2 b^2 \,,$$
$$T_{ua} = i\omega b b_{,b}(\bar{e}_a e_b - e_a \bar{e}_b), \quad a,b = 3,4$$

- Structure of the source-term (D-dim)

$$T_{uu}(u,\mathbf{x}), \quad T_{ua}(u,\mathbf{x}), \quad T_{ua}^{\ \ ;a} = 0$$

$$\varepsilon(u) = \frac{1}{\sqrt{2}} \int d\mathbf{x} T_{uu} \,, \qquad j_{ab}(u) = 2\int d\mathbf{x} T_{ua} x_b$$

Gyraton Metric: Exact Solutions



➠ Gyraton metric ansatz [Brinkmann (1925)]

$$ds^2 = -2\,du\,dv + d\mathbf{x}^2 + \Phi\,du^2 + 2\,(\mathbf{A}, d\mathbf{x})du$$

- $d\mathbf{x}^2 = \sum_{a=3}^{D}(dx_a)^2\,, \qquad (\mathbf{A}, d\mathbf{x}) = \sum_{a=3}^{D} A_a\,dx^a\,.$
- $\Phi = \Phi(u, \mathbf{x})\,, \qquad A_a = A_a(u, \mathbf{x})\,,$
- $\mathbf{l} = \partial_v$ is a null Killing vector
- The null Killing vector $\mathbf{l}$ is covariantly constant, $l_{\mu;\nu} = 0$. Plane-fronted gravitational waves with parallel rays, or briefly pp-waves
- Type III [Coley & Pelavas (2005)]

- Gauge invariance
  $v \to v + \lambda(u, \mathbf{x})\,, \;\; A_a \to A_a - \lambda_{,a}\,, \;\; \Phi \to \Phi - 2\lambda_{,u}$

- Scale transformations $u \to au$, $v \to a^{-1}v$,
  $\Phi \to a^2\Phi$, $\mathbf{A} \to a\mathbf{A}$

All the local scalar invariants constructed from the Riemann tensor and its covariant derivatives for the gyraton metric vanish. This statement is valid *off shell*, that is the metric needs not to be a solution of the vacuum Einstein equations

➠ Non-zero components of curvature: $R_{[au][bu]}$, $R_{[ab][cu]}$, $R_{[cu][ab]}$

➠ ∇_μ neither produces index v nor "kills" index u

➠ $g^{\mu u} = -\delta^\mu_v$

➠ This property is well known for 4-dimensional case, since pp-wave solutions are of Petrov type N

➠ Generalization of this result to higher-dimensional metrics with $\mathbf{A} = 0$ was given by [Amati & Klimcik (1989) and Horowitz & Steif (1990)

Vacuum Solutions

- The gyraton metric is Ricci flat iff

$$\partial_b F_a{}^b = 0\,, \quad F_{ab} = \partial_a A_b - \partial_b A_a$$

$$\triangle\Phi - 2\partial_u \text{div}\mathbf{A} = \frac{1}{2}\mathbf{F}^2$$

- In a special gauge $\text{div}\mathbf{A} = 0$

$$\triangle\mathbf{A} = 0\,, \quad \Phi = \varphi + \psi$$

$$\triangle\varphi = 0\,, \quad \triangle\psi = \frac{1}{2}\mathbf{F}^2$$

- Analogy of gravity with electromagnetism

- The problem of solving the $D-$dimensional vacuum Einstein equations for the gyraton metric is reduced to finding an electric potential φ and magnetic field F_{ab} created by a local source in the $(D-2)-$dimensional Euclidean space

- For these solutions the retarded time u plays a role of an external parameter which enters through the dependence of point-like sources on u

"Ground State" Gyratons

➠ "Lowest harmonic" solutions for Φ and $\mathbf{A}$

$$\Phi = \kappa\sqrt{2}\varepsilon(u) \begin{cases} -\frac{1}{2\pi}\ln r\,, & \text{if } n = 2 \\ \frac{g_n}{r^{n-2}}\,, & \text{if } n > 2\,, \end{cases}$$

$$A_a = \frac{\kappa g_n (n-2) j_{ab}(u) x^b}{r^n}$$

For n>2 , $g_n = \frac{\Gamma(\frac{n-2}{2})}{4\pi^{n/2}}$

For $n = 2 \;\to g_n(n-2) = 1/(2\pi)$

• $\psi(\mathbf{x}) = -\frac{1}{2}\int d\mathbf{x}'\mathcal{G}_n(\mathbf{x},\mathbf{x}')\mathbf{F}^2(\mathbf{x}')$

➠ Solutions which contain higher harmonics in Φ and $\mathbf{A}$ (excited states) are less symmetric

4-D Gyraton Metric

- $F_{ab} = F e_{ab}$, $\to F =$const $\to F = 0$
- The value of $j(u) \equiv \frac{1}{2}\epsilon^{ab} j_{ab} = \frac{2}{\kappa}\oint_C A_a dx^a$ does not depend on the choice of the contour C. (Aharonov-Bohm magnetic field)
- **A** in polar coordinates is $A_r = 0$, $A_\phi = \frac{\kappa}{4\pi} j(u)$
- A solution corresponding to a point-like charge is

$$\boxed{\varphi_0 = -\frac{\kappa\sqrt{2}}{2\pi}\varepsilon(u)\ln r}$$

- A decreasing at infinity solution is

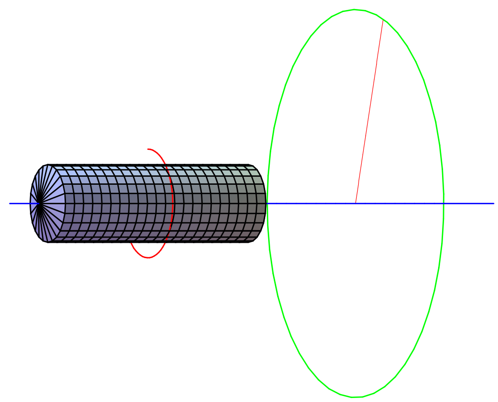
$$\Phi = \varphi_0 + \varphi' \,, \quad \varphi' = {\sum_{n=-\infty}^{\infty}}' \frac{b_n}{r^{|n|}} e^{in\phi} \,, \qquad \bar{b}_n = b_{-n}$$

- $\sum'$ indicates that the term $n = 0$ is excluded

$$\boxed{\begin{gathered} ds^2 = -2\,du\,dv + dr^2 + r^2 d\phi^2 \\ + \frac{\kappa}{2\pi} j(u) du d\phi + \left[-\frac{\kappa\sqrt{2}}{2\pi}\varepsilon(u)\ln r + \varphi' \right] du^2 \end{gathered}}$$

- Bonnor (1970): Spinning null fluid

Test Particles and Light in the Gyraton Field

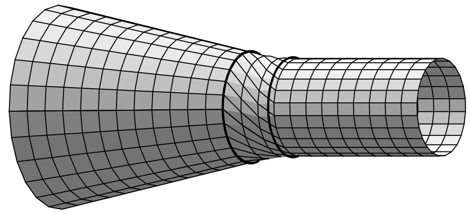


$$\frac{d^2 r}{du^2} = F_N + F_{CF}$$

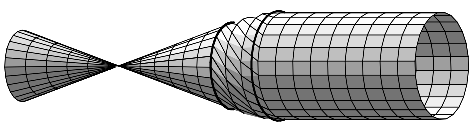
$F_N = -\frac{\mu}{r}$ 'Newton' force

$F_{CF} = \frac{p^2}{r^3}$ 'CF' force

➡ $\mu(u) = \frac{\kappa \varepsilon(u)}{2\pi\sqrt{2}}\,, \qquad p = \frac{\kappa j(u)}{4\pi}$



F_{CF} dominates



F_N dominates

$F_N \sim F_{CF}$ for $r_0 \sim \sqrt{G} J/\sqrt{EL}$

For $E \sim L^{-1}$, $J = s$, $\quad r_0 \sim s l_{Pl}$

5-Dimensional Gyratons

- To obtain a solution for the gravitational field of a 5-dimensional gyraton one needs to analyze electro- and magnetostatics in a flat 3-dimensional space
- "Magnetic field": $\mathbf{B} = \text{curl}\,\mathbf{A}\,, \qquad \text{curl}\,\mathbf{B} = 0$
- Scalar potential Υ: $\mathbf{B} = -\nabla\Upsilon\,,\ \triangle\Upsilon = 0$
- A general solution decreasing at infinity:

$$\Upsilon = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}\,, \qquad \bar{a}_{lm}(u) = a_{l\,-m}(u)$$

$$\text{curl}\,\mathbf{A} = -\nabla\Upsilon\,,\ \ \text{div}\mathbf{A} = 0$$

$$\mathbf{A} = \mathbf{A}_0 - \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\frac{a_{lm}}{l}\frac{\mathbf{r}\times\nabla Y_{lm}(\theta,\phi)}{r^{l+1}}$$

- "Electric potential:

$$\varphi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} b_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}\,, \qquad \bar{b}_{lm}(u) = b_{l\,-m}(u)$$

5-D Monopole Solution

- The magnetic potential Υ for the magnetic monopole:

$$\Upsilon = -\frac{\mu(u)}{r}$$

- The magnetic induction vector has components:

$$B_r = \frac{\mu}{r^2}\,, \qquad B_\theta = B_\phi = 0$$

- The vector potential: div$\mathbf{A} = 0$

$$A_r = A_\theta = 0\,, \quad A_\phi = -\mu\cos\theta$$

- The potential ψ: $\psi = \frac{\mu^2}{4r^2}$

- The monopole solution for the gyraton:

$$\begin{aligned} ds^2 &= -2\,du\,dv + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \\ &+ \left[\varphi(u,\mathbf{x}) + \frac{\mu^2(u)}{4r^2}\right] du^2 - 2\,\mu(u)\cos\theta du d\phi \end{aligned}$$

Higher-Dimensional Gyratons

- A general solution of the Laplace equation is

$$\varphi = \sum_{l=0}^{\infty} \sum_{q} \frac{\mathcal{Y}^{lq}}{r^{n+2l-2}}\,, \quad \mathcal{Y}^{l} = C_{c_1 \ldots c_l}(u) x^{c_1} \ldots x^{c_l}$$

- $C_{c_1 \ldots c_l}$ is a symmetric traceless rank-l tensor
- $q = 1, \ldots, d_0(n,l)$ enumerates linearly independent components of coefficients $C_{c_1 \ldots c_l}$

$$d_0(n,l) = \frac{(l+n-3)!(2l+n-2)}{l!(n-2)!}$$

- A solution of the gravitomagnetic equations

$$a_a = \sum_{l=1}^{\infty} \sum_{q} \frac{\mathcal{Y}_a^{lq}}{r^{n+2l-2}}$$

$$\mathcal{Y}_a^{l} = C_{abc_1 \ldots c_{l-1}}(u) x^{b} x^{c_1} \ldots x^{c_{l-1}}$$

- $C_{abc_1 \ldots c_{l-1}}$ is a tensor obeying the properties: It is antisymmetric under interchange of a and b, and it is traceless under contraction of any pair of indices

➠ $q = 1, \ldots, d_1(n,l)$ enumerates different linearly independent vector spherical harmonics

$$d_1(n,l) = \frac{l(n+l-2)(n+2l-2)(n+l-3)!}{(n-3)!(l+1)!}$$

➠ Gauge condition $a_{a}{}^{;a} = 0$

➠ A general solution of the "gravitostatic" equation is

$$\Phi = \varphi + \psi$$

➠ φ is a general solutions for a point-like source and ψ is a special solution of the inhomogeneous equation

$$\triangle \psi = -J$$

➠ A special solution ψ can be presented as follows

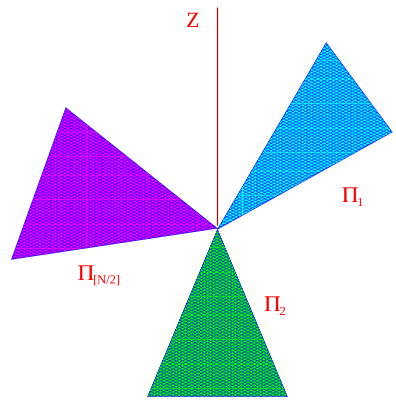
$$\psi(u, \mathbf{x}) = \int d\mathbf{x}' \mathcal{G}_n(\mathbf{x}, \mathbf{x}') J(u, \mathbf{x}') \,, \qquad J = -\frac{1}{2} f_{ab} f^{ab}$$

➠ $\mathcal{G}_n(\mathbf{x}, \mathbf{x}')$ is the Green's function for the $n-$dim Laplace operator

$$\triangle \mathcal{G}_n(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}')$$

$$\mathcal{G}_2(\mathbf{x}, \mathbf{x}') = -\frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}'| \,, \quad \mathcal{G}_{n>2}(\mathbf{x}, \mathbf{x}') = \frac{g_n}{|\mathbf{x} - \mathbf{x}'|^{n-2}}$$

Ground State of Higher Dim. Gyratons



$$ds^2 = -2dudv + \sum_{i=1}^{l}(dr_i^2 + r_i^2 d\phi_i^2) + \epsilon dz^2$$
$$+(2\Phi + \mathcal{B})du^2 + \frac{2}{r^n}\left[\sum_{i=1}^{l} j_i(u) r_i^2 d\phi_i\right] du$$

- $\Phi = \dfrac{\varepsilon(u)}{r^{2(l-1)}}\,, \quad \mathcal{B} = \dfrac{1}{r^{4l-2}}\left[\alpha_l \frac{P^2}{r^2} + \beta_l j^2\right]$
- $r^2 = \sum_{i=1}^{l} r_i^2 + \epsilon z^2\,, \quad n = D-2, \quad l = [n/2]$
- $P^2 = \sum_{i=1}^{l} j_i^2(u) r_i^2\,, \quad j^2 = \sum_{i=1}^{l} j_i^2(u)$
- $\varepsilon(u)$ is the energy density
- $j_i(u)$ is the "i"-th bi-plane component of the of angular momentum density

Summary and Conclusions

"Classical photon" $\Longrightarrow$ Gyraton

- Motion with the light velocity
- Energy E, angular momentum J_{ab}, duration L

Gravitational fi eld of a gyraton

- Vacuum field *outside* the gyraton
- Vacuum Einstein equations in D-dims reduce to linear problems in $(D-2)$-dim Euclidean space
  - Electric field for a point-like source
  - Magnetic field for a point-like source
- Recovering of the gyraton metric by using the solutions of the linear problems
- A general solution allows one to obtain the field outside a gyraton with *arbitrary* distribution of the energy and angular momentum inside it

Gyraton solutions

- Simplest ("ground state") gyraton solutions [VF & Fursaev '05]
  - Monopole solution for 'electric field'
  - Dipole solution for 'magnetic field'
- General gyraton solutions [VF, Israel & Zelnikov, '05]
  - Multipole decomposition
- Gyraton in asymptotically AdS spacetime [VF & Zelnikov, '05]
- Charged gyratons [VF & Zelnikov, '05]

GR spin-spin interaction

- Spin-spin and spin-orbit interaction in the strong field limit
- Field scattering by gyratons
- Interaction of gyratons with matter
- Role of the gravitational spin-spin and spin-orbit interactions in the mini-black hole formation