M-theory & the String Genus Expansion

David Berman
Queen Mary College
University of London
&
Malcolm Perry
DAMTP
University of Cambridge

Goal:
What is the origin of string perturbation theory?

The membrane origin of how the dilaton couples to the string world sheet.
What is String Perturbation theory?

\[ Z = \sum_{\text{Topologies}} g_s^2 \int \frac{dY dX}{w(\text{worldsheet})} e^{-S[Y, \phi]} \]

Sum over topologies of the worldsheet weighted by \( g_s \).

For \( g_s < 1 \) this is a perturbative expansion in the genus.

\[ \chi = 2 - 2g \]

Often written as

\[ e^{-\phi X} \]

in the partition function.

where \( g_s = e^\phi \)

or

\[ S_\phi = \phi X \]

in the action for each loop in field theory.
$M$-theory

Strong Coupling Limit of 10 string.

Its low energy effective action is 11 dimensional supergravity.

Extended objects are:

- Membrane
- Fivebrane

\[ M / \text{String Relation:} \]

\[ e^\phi = \left( \frac{R_{\mu\nu}}{4} \right)^{3/2} \]

Matching shown for classical solutions & dimensionally reduced world volume actions.
\[ S_{m2} = \frac{1}{4g^2} \int d^3 \sigma \sqrt{g} \left( g^{ij} \partial_i X^i \partial_j X^j - 1 + \delta^{\mu\nu} \partial_i X^i \partial_j X^j \delta^{ij} G_{ijk} G_{ikl} \right) \]

\( G_{i\sigma} \) \( \{ \) Background Bosonic Fields. \\
\( G_{i\sigma} \) \( \} \)

\( X^i(\sigma) \) \ World Volume Fields

\( \gamma_{\mu\nu} \) \ Auxiliary world volume metric

Reduction of the membrane action:

Wrap the M2

\( \sigma^3 = X'' \)

Gauge Fix:

\[ \gamma_{\mu\nu} = \begin{pmatrix} \tilde{\gamma}_{\mu\nu} & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \]
The Membrane becomes:

\[ S = \frac{1}{2 \kappa_0^2} \int d^2 \sigma \sqrt{g} \left( \bar{\Phi}^{\mu} \partial_{\mu} x^I \bar{\Phi}^I - B_{I J} \epsilon^{I J K} \partial_{\mu} x^I \partial_{\nu} x^J \right) + \text{boundary terms} \]

The action of the string

without the

\[ S_{\phi} = \phi X \quad \text{term} \]

Write \[ S_{\phi} = \phi X \]

in M-theory variables.

\[ S_{\phi} = \frac{3}{2} \phi \ln(R \sqrt{\eta}) \]

\[ \ln R \eta \] seems hard to lift to the membrane.

\[ X = 2d \] topological invariant

No unique lift to 3d.
Proposal:

Examine the Membrane
partition function for the

case relevant to the string:

\[ S' \times \Sigma \]

\[
\text{Fixed } R_{11} \quad \text{Riemann Surface}
\]

\[
\text{of genus } g
\]

\[ M_2 \text{ partition function:} \]

\[ Z = \int \frac{dX \omega X}{\text{Vol} (\text{off})} e^{-S_{M_2} [X, \omega]} \]

To evaluate the measure:

1. Gauge Fix

2. Write the space of metrics
   as an orthogonal decomposition
   into Pure Diff x Physical Metric
   deformations

3. Jacobian for the above
   (introduce ghosts).

4. Integrate over Pure Diff
   cancels Vol (Diff)

5. Integrate over Moduli
   with appropriate measure.
3d Diffeos $\rightarrow$ 2d Diffeos $\times$ Weyl

Be careful with double dimensional reduction a la Achucarro, Kapusta & Stelle.

$$X'' = \sigma^3 + f(\sigma, \tau)$$

Recall for diffeos

$$\delta X^m = \nu^k \partial_k X^m$$

$$\Rightarrow \delta X'' = \nu^3 (\sigma, \tau)$$

$$\therefore \nu^3 (\sigma, \tau) = \delta f(\sigma, \tau)$$

$$\delta_{ij} = \phi^{-3/2} \left( e^{-f(\sigma, \tau)} \delta_{ij} + \phi^2 A_i A_j \right)$$

Case $S' \times \Sigma$, with fixed radius for the $S'$. The moduli are as for a Riemann surface.

The 3-d diffeos $\rightarrow$ 2-d diffeos $\times$ Weyl.

Weyl & Diffeos on the metric:

$$6 \gamma_{\mu \nu} = (2 \delta \sigma + \nabla^\rho \delta \sigma_\rho) \gamma_{\rho \nu} + (P_\nu \delta \sigma_\rho)$$

$$\Rightarrow (P_\nu \delta \sigma_\rho) = \frac{1}{2} \left( \nabla_\mu \delta \sigma_\nu + \nabla_\nu \delta \sigma_\mu - \delta \sigma_\mu \delta \sigma_\nu \right)$$

Also the conjugate:

$$(p^\dagger \gamma)_\mu = -2 \nabla^\nu \gamma_{\mu \nu}$$

$$\delta \gamma = \text{Weyl}$$

$$\delta \nu_\mu = \text{Diffeos}$$
Case: \( g \geq 2 \), \( p_i \) has no zero modes.

The dimension of Moduli Space is given by

\[
\text{ker}(p_i^+) = 6g - 6 = -3X
\]

The Jacobian

\[
J = (\det p_i^+p_i)^{\frac{1}{2}}
\]

Partition Function:

\[
Z = \int dX \ d^{6g-6} (\det p_i^+p_i)^{\frac{1}{2}} \frac{d\mu d\nu}{\text{Vol}(\text{AdS}_5 \times \text{Sphere})} e^{-S[X]}
\]

The measure of the moduli space integral \( d^{3g-3} \) is calculated from the norms of the quadratic differentials

\[
|| \partial \phi ||^2 = \int \sqrt{g} \left( \partial_{\mu} \phi \right)^2 \left( \partial_{\nu} \phi \right)^2 - 2 \partial_{\mu} \phi \partial_{\nu} \phi \partial_{\mu} \phi \partial_{\nu} \phi \delta \phi^2
\]

Evaluate this for \( S^1 \times \Sigma \) and write in terms of the norms of \( \Sigma \).

\[
\Rightarrow \ || \delta \phi || = \sqrt{R^2 \ || \delta \phi \||}
\]
\[ d^{69-6} = (R_{\mu})^{39-3} d^{69-6} \]

\[ Z = \int dX \ (R_{\mu})^{39-3} d^{69-6} \text{ln} (\text{det} P^\mu P^\nu) e^{-S} \]

Use \( (R_{\mu})^{3/2} = e^\phi \)

& \( X = 2-2g \)

\[ Z = \int dX \ d^{69-6} \text{ln} (\text{det} P^\mu P^\nu) e^{-S - \phi X} \]

\[ S_\phi = \phi X \] is an effective action term from the membrane measure.

Remarkable: The perturbative expansion comes from the membrane measure for M-theory. As it should for M-theory, the expansion (in the small R limit) is not put by hand.
Case $g = 1 \land g = 0$.

$p_1$ has zero modes

"conformal Killing vectors".

Modify the measure so as not to integrate over zero modes.

$2 \gamma_{\mu \nu} = (\det^*_1 p_1^* p_1)^{1/2} \, d\nu^* d\nu$ do moduli
to not including zero modes.

This is equal to

$2 \gamma_{\mu \nu} = \frac{1}{\text{Vol} (\text{Ker} p_1)} (\det^*_1 p_1^* p_1)^{1/2} \, d\nu^* d\nu$ do moduli

Norm of the conformal Killing vectors again scales like $\sqrt{R_{\mu \nu}}$.

Thus

$\text{Vol} (\text{Ker} p_1) = R^{1/2} \text{Dim} (\text{Ker} p_1) \frac{\text{Vol} (\text{Ker} p_1)}{M_2 \, p_1}$

Recall Riemann Roch theorem:

$\text{Dim Ker} p_1^* - \text{Dim Ker} p_1 = -3X$
The measure:

\[
\frac{\mathcal{D} \sigma}{\text{Vol}(\mathcal{M})} = \frac{\mathcal{D} \tilde{\sigma}}{\text{Vol}(\mathcal{M}')} R, \\
= \frac{\mathcal{D} \tilde{\sigma}}{\text{Vol}(\mathcal{M}')} e^{-\phi x}
\]

Again the measure scales as \( e^{-\phi x} \) thus implying the effective string action

\[ S_{\phi} = \phi x. \]

**Discussion**

This did not have to happen. The \( M/\text{string} \) relation could have been anomalous. That is although their classical actions matched their partition functions need not have.

Evidence to take the membrane seriously.

Look at other topologies \( S^3, S^{3/2}, \text{etc.} \)
Non Constant Dilaton

The dilaton need not be a constant.

The previous analysis works fine for $\phi(X)$ but not $\phi(0)$ since

$$\int \phi \, R^{(2)} \, d^2 \sigma$$

is no longer $C \propto$ and the coupling is not topological.

Note: In this case the dilaton coupling breaks $K$ symmetry.
Are there any other topologies in 3-d that allow a similar expansion?

Need to know the moduli space or at least how the measure scales with some topological invariant.