M-theory & the String Genus Expansion

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Goal:

What is the origin of string perturbation theory?

The membrane origin of how the dilaton couples to the string world sheet.

What is String Perturbation theory?

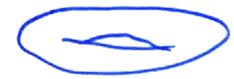
$$Z = \sum_{\text{Topologies}} g_s^{-\chi} \int \frac{\mathcal{D}\gamma\, \mathcal{D}X}{Vol(Diff \times Weyl)}\, e^{-S[X,\gamma]}$$

Sum over topologies of the world sheet weighted by $g_s^{-\chi(\text{worldsheet})}$.

For $g_s < 1$ this is a perturbative expansion in the genus.

$$\chi = 2 - 2\underbrace{g}_{\text{genus}}$$

$g \to g+1$



g_s^{+2} Weight

c.f.

g_{YM}^2 for each loop in field theory.

Often written as

$$e^{-\phi\chi}$$ in the partition fn.

where $g_s = e^{\phi}$

or $S_\phi = \phi\chi$ in the action

M- theory

Strong Coupling Limit of II_A string.

Its Low Energy Effective Action is 11 dimensional supergravity.

Extended objects are:

Membrane

Fivebrane

$$\begin{tikzcd}[column sep=small]
M \arrow[d, dash] & & M2 \arrow[dl] \arrow[d] & & M5 \arrow[dl] \arrow[d] & \text{Momentum} \\
S^1 \arrow[d] & F1 & D2 & D4 & NS5 & D0 \\
\mathrm{II}_A & & & & &
\end{tikzcd}$$

M / String Relation:

$$e^{\phi} = \left(\frac{R_{11}}{l_p}\right)^{3/2}$$

Matching Shown for Classical solutions & dimensionally reduced world volume actions.

$$S_{M2} = \frac{1}{l_p^3}\int d^3\sigma \sqrt{\gamma}\left(\gamma^{\mu\nu}\partial_\mu X^I \partial_\nu X^J G_{IJ} - 1 + \varepsilon^{\mu\nu\rho}\partial_\mu X^I \partial_\nu X^J \partial_\rho X^K C_{IJK}\right)$$

$$\left.\begin{array}{l} G_{IJ} \\ \\ C_{IJK} \end{array}\right\} \text{Background Bosonic Fields.}$$

$X^I(\sigma)$ World Volume Fields

$\gamma_{\mu\nu}$ Auxiliary world volume metric

Reduction of the membrane action:

Wrap the M2

$$\sigma^3 = X^{11}$$

Gauge Fix:

$$\gamma_{\mu\nu} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\tilde{\gamma}_{\mu\nu}} & \begin{array}{c} 0 \\ 0 \end{array} \\ \hline 0 & 0 & 1 \end{array}\right)$$

The Membrane becomes:

$$S = \frac{1}{2\pi\alpha'}\int d^2\sigma \sqrt{\gamma}\, \gamma^{\mu\nu}\partial_\mu X^I \partial_\nu X^J G_{IJ} + B_{IJ}\,\varepsilon^{\mu\nu}\partial_\mu X^I \partial_\nu X^J$$

The action of the string without the

$$S_\phi = \phi\,\chi \quad \text{term.}$$

Write $S_\phi = \phi\,\chi$

in M-theory variables.

$$S_\phi = 3/2\,\chi \ln(R_{11}/l_p)$$

$\ln R_{11}$ seems hard to lift to the membrane.

χ - 2d topological invariant

No unique lift to 3d.

Proposal:

Examine the Membrane partition function for the case relevant to the string:

$$\underbrace{S^1}_{\text{Fixed } R_{11}} \times \underbrace{\Sigma}_{\substack{\text{Riemann Surface} \\ \text{of genus } g}}$$

M2 partition fn.:

$$Z = \int \frac{\mathcal{D}X\, \mathcal{D}\gamma}{\mathrm{Vol}(\mathrm{Diff})}\, e^{-S_{M2}[X,\gamma]}$$

To evaluate the measure:

1. Gauge Fix
2. Write the space of metrics as an orthogonal decomposition into Pure Diffeos $\times$ Physical Metric deformations
3. Jacobian for the above (introduce ghosts).
4. Integrate over Pure Diffeos cancels Vol(Diff)
5. Integral over Moduli with appropriate measure.

3d Diffeos $\rightarrow$ 2d Diffeos $\times$ Weyl

Be careful with double dimensional reduction a la Achucarro, Kapusta & Stelle.

$$X^{11} = \sigma^3 + f(\sigma, \tau)$$

Recall for diffeos

$$\delta X^\mu = v^k \partial_k X^\mu$$

$$\Rightarrow \delta X^{11} = v^3(\sigma, \tau)$$

$$\therefore \quad v^3(\sigma,\tau) = \delta f(\sigma,\tau)$$

$$\gamma_{ij} = \phi^{-2/3} \begin{pmatrix} e^{-f(\sigma,\tau)} \tilde{\gamma}_{ij} + \phi^2 A_i A_j & \phi^2 A_j \\ \phi^2 A_i & \phi^2 \end{pmatrix}$$

Case $S^1 \times \Sigma$ with fixed radius for the S^1. The moduli are as for a Riemann surface.

The 3-d diffeos $\rightarrow$ 2d diffeos $\times$ Weyl.

Weyl & Diffeos on the metric:

$$\delta \gamma_{\mu\nu} = (2\delta\sigma + \nabla^\rho \delta v_\rho)\gamma_{\mu\nu} + (P_1 \delta v)_{\mu\nu}$$

$$(P_1 \delta v)_{\mu\nu} = \tfrac{1}{2}(\nabla_\mu \delta v_\nu + \nabla_\nu \delta v_\mu - \gamma_{\mu\nu} \nabla_\rho v^\rho)$$

Also the conjugate:

$$(P_1^\dagger \gamma)_\mu = -2\nabla^\nu \gamma_{\mu\nu}$$

$\delta\sigma$ — Weyl

δv_μ — Diffeos

Case: $g \geq 2$ $\quad P_1$ has no zero modes

The dimension of Moduli Space is given by

$$\mathrm{Ker}(P_1^+) = 6g-6 = -3\chi$$

The Jacobian

$$J = (\det P_1^+ P_1)^{1/2}$$

Partition Function:

$$Z = \int \mathcal{D}X \underbrace{d^{6g-6}m}_{\text{Moduli}} \underbrace{(\det P_1^+ P_1)^{1/2}}_{\text{Jacobian}} \underbrace{\frac{d\sigma\, d\sigma}{\mathrm{Vol}(\mathrm{Diff}\times \mathrm{Weyl})}}_{=1} e^{-S[X,g]}$$

The measure of the moduli space integral $d^{6g-6}m$ is calculated from the norms of the quadratic differentials

ie.

$$\|\delta\gamma_{\mu\nu}\|^2 = \int \sqrt{\gamma}\,(\gamma^{\mu\rho}\gamma^{\nu\sigma} - c\,\gamma^{\mu\nu}\gamma^{\rho\sigma})\,\delta\gamma_{\mu\nu}\,\delta\gamma_{\rho\sigma}$$

c Non Physical constant (set to zero from now on).

Evaluate this for $S^1 \times \Sigma$ & write in terms of the norms of Σ.

$$\Rightarrow \|\delta\gamma_{\mu\nu}\| = \sqrt{R_{11}}\, \|\delta\tilde{\gamma}_{\mu\nu}\|$$

$$\Rightarrow \quad d^{6g-6} m = \left(R_{11}\right)^{3g-3} d^{6g-6} \tilde{m}$$

$$\Rightarrow$$

$$Z = \int DX \left(R_{11}\right)^{3g-3} d^{6g-6} \tilde{m} \left(\det P_1^{+} P_1\right)^{1/2} e^{-S}$$

Use $\quad \left(R_{11}\right)^{3/2} = e^{\phi}$

$\quad \& \quad \chi = 2 - 2g$

$$Z = \int DX \, d^{6g-6} \tilde{m} \left(\det P_1^{+} P_1\right)^{1/2} e^{-S - \phi \chi}$$

$S_{\phi} = \phi \chi$ is an effective action term from the membrane measure.

Remarkable:

The perturbative expansion comes from the membrane measure.

As it should for M-theory, the membrane provides the expansion (in the small R_{11} limit) it is not put in by hand.

Case $g=1$ & $g=0$.

P_1 has zero modes

"conformal killing vectors".

Modify the measure so as not to integrate over zero modes.

$$\mathcal{D}\gamma_{\mu\nu} = (\det{}' P_1^\dagger P_1)^{1/2}\, d'v^\mu\, d\sigma\, d\,\text{moduli}$$

$'$ $\Rightarrow$ not including zero modes.

This is equal to

$$\mathcal{D}\gamma_{\mu\nu} = \frac{1}{\text{Vol}(\text{Ker}P_1)} (\det{}' P_1^\dagger P_1)^{1/2}\, dv^\mu\, d\sigma\, d\,\text{moduli}$$

includes zero modes

Norm of the conformal Killing Vectors again scales like $\sqrt{R_{11}}$.

Thus

$$\underbrace{\text{Vol}(\text{Ker}\,P_1)}_{\text{M2 } P_1} = R_{11}^{\frac{1}{2}\text{Dim}(\text{Ker}\,P_1)}\, \underbrace{\text{Vol}(\text{Ker}\,\tilde{P}_1)}_{\text{String } P_1}$$

Recall Riemann Roch theorem:

$$\text{Dim Ker}P_1^\dagger - \text{Dim Ker}P_1 = -3\chi$$

The measure:

$$\int \frac{\mathcal{D}\gamma}{Vol(Diff)} = \int \frac{\mathcal{D}\tilde{\gamma}}{Vol(Diff)} R_{11}^{\,1/2(\mathrm{Dim\,Ker}P_1^{+} - \mathrm{Dim\,Ker}P_1)}$$

$$= \int \frac{\mathcal{D}\tilde{\gamma}}{Vol(Diff)} \, e^{-\phi\chi}$$

Again the measure scales as $e^{-\phi\chi}$ thus implying the effective string action

$$S_\phi = \phi\chi .$$

Discussion

This did not have to happen. The M/string relation could have been anomalous. That is although their classical actions matched their partition functions need not have.

Evidence to take the membrane seriously.

Look at other topologies S^3, $S^3/\mathbb{Z}_N$ etc.

Non Constant Dilaton

The dilaton need not be a constant.

The previous analysis works fine for $\phi(X)$ but not $\phi(\sigma)$ since

X — spacetime

σ — World Sheet

$$\int \phi R^{(2)} d^2\sigma$$

is no longer $\phi\chi$ and the coupling is not topological.

Note: In this case the dilaton coupling breaks κ symmetry

Including KK modes

From the string theory point of view these are D0 branes.

$\Rightarrow$ World Sheet coupling to dilaton with D0 branes present

Have Background topologically non-trivial

String Interpretation:

Background of D6 branes.

Are there any other topologies in 3-d that allow a similar expansion?

Need to know the moduli space or at least how the measure scales with some topological invariant.