Approaches to Numerical Relativity

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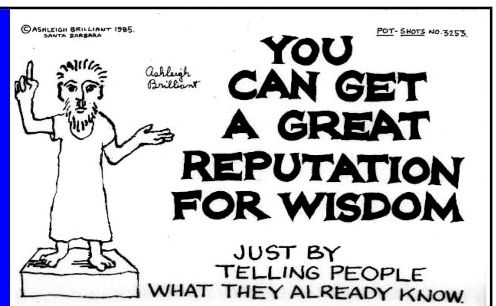
Outline



- Issues to deal with…
- Standard Cauchy approach
- Standard characteristic approach
- Conformal approach

Einstein eqns in the computer

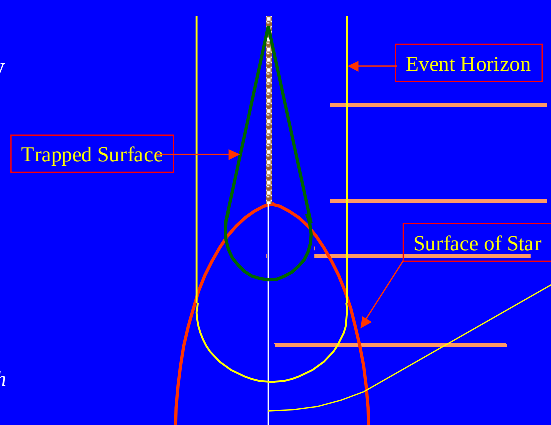
- Solve $G_{\mu\nu} = \kappa T_{\mu\nu}$ through simulations
  - nonlinear PDE system
  - *Singularities*
  - *'evolution' (?)*
  - Gauge issues
  - Initial and boundary data
- Fluid?
  - Shocks
  - Initial and boundary data

\+ Computat Resources:

AlGORITHM ISSUES…

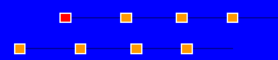
Black Holes

- Either there or will be there in a # of interesting problems
- 'Tricky' to handle analytically
- Computational nightmare!
- What to do?
  - Excise (in principle, OK, if cosmological censorship)
- *(I) Black hole 'found' a-posteriori*
  - *Look for trapped surfaces*
- *(II) Yet another boundary to deal with*
  - *(ab) use the one-way flow*

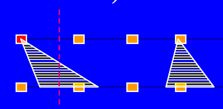


Excision

- Why works? (or would work)
  - Physically: spacelike surface, (ie. moving faster than c)

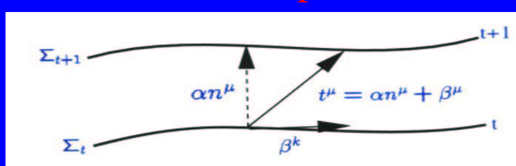


  - Mathematically: 'in-flow' boundary. (ie. All characteristics towards it)



- *Why worry?*
  - Coordinates *can* move faster than c!
  - At a given boundary, *even inside the event horizon*, characteristics might not be all incoming!
    - Example: x=const surface, Scwarzschild in PG or KS coordinates isn't spacelike if $x>2/3^{3/2}M$ isn't always spacelike even inside the black hole.

3+1 Decomposition of Spacetime



$g_{ij} \equiv$ 3-metric
$K_{ij} \equiv$ Extrinsic curvature
$\alpha \equiv$ Lapse function
$\beta^k \equiv$ Shift vector

- 3+1 Decomposition splits the spacetime metric into a 3-metric on each slice, an extrinsic curvature of the hypersurface, a lapse function and a shift vector.
- Vacuum Einstein field equations decompose:

Evolution Equations

$$(\partial_t - \beta^k \partial_k) g_{ij} = -2\alpha K_{ij} + g_{kj}\partial_i \beta^k + g_{ik}\partial_j \beta^k$$
$$(\partial_t - \beta^k \partial_k) K_{ij} = -D_i D_j \alpha + \alpha \left(R_{ij} + K K_{ij} - 2K_{il}K^l{}_j\right) + K_{kj}\partial_i \beta^k + K_{ik}\partial_j \beta^k$$

↔ Maxwell Analogy

$$\partial_t \mathbf{E} = \nabla \times \mathbf{B}$$
$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E}$$

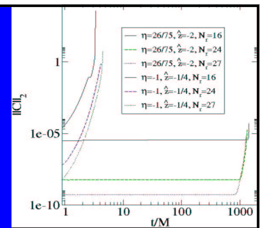
Constraint Equations

$$R + K^2 - K_{ij}K^{ij} = 0$$
$$D^j (K_{ij} - g_{ij} K) = 0$$

↔ Analogy to Maxwell

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$

Observations and questions



- Instabilities abound!
  - Exponential solutions commonly found
  - Constraints (when used as monitors) exponentially diverge.
  - 'Equivalent' reformulations of the equations give drastically different behavior, even when these have 'nice' properties [Kidder et.al]
- YET: Under very conveniently chosen coordinates, things look quite OK [Yo,et.al.; Alcubierre et.al, Kelly et.al]….*but for which systems?*
- Qns:
  - Is there a way to eliminate/minimize exponential growth?
  - Are constraints to blame for all/most of problems?
  - Are equations conveniently written?
  - Are problems due to using 'simplistic' boundary conditions?
  - {all of the above and then some?}

Partial answers

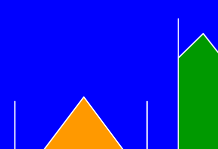
- Simulations with ADM, free evolution, typically unstable.
  - Note: except under specialized coordinates, system only *weakly hyperbolic*
  - Weakly/strongly/symmetric hyperbolic distinction obtained by analyzing *solely* principal part of equations:

$$u_{,t} = A^i u_{,i} + Bu + C$$

  - Weakly hyperbolic: *unstable* under generic lower order perturbations
  - Strongly/symmetric: stable for initial value problems (no boundaries)

$$|u(T)| \le k e^{aT} |u(0)|$$

  - Strongly hyperbolic: delicate in presence of boundaries [Sarbach-Calabrese]
  - Einstein eqns, satisfied if:
    - Initial data satisfies the constraints (consistency cond'n)
    - Boundary data doesn't feed constraint violations

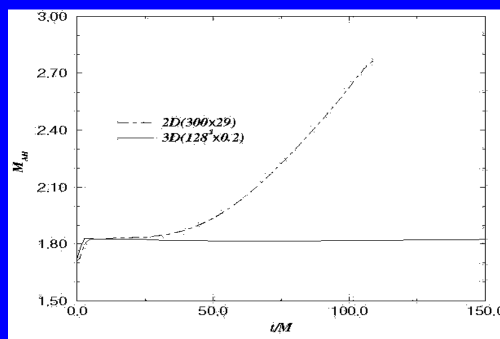


Observation: for our goals wrong BD can make ID completely irrelevant!

ISO 'best' formulation (1)

Note: evolution eqns non-unique, use constraints to 'modify' them

- One road: 'slight' modifications of ADM. 'Most popular' one: BSSN.
  - 'Structure' of eqns very much like ADM. Use constraint eqns to free the system from 'undesirable' terms.
  - Unclear hyperbolic properties in most 'flavors' found. (Though, could be made symm. hyperbolic [Sarbach-Tiglio]).



(Alcubierre et al)

•Second road: obtain explicitly symmetric/strongly hyperbolic systems.

–In most flavors, reduction to first order form used. Consequently, ~2-3 times as many variables. [Though this isn't needed, Kreiss-Ortiz (2nd/2nd); Nagy-Reula-Ortiz (1st/2nd)]

–Amenable to use rigorous results from applied math to guarantee stable implementations [most for 1st oder formulations]

Added bonus of 'hyperbolic road':

–Exploit 'expected growth' of solution [at the core of establishing stability at continuous level] to come up with 'better' options.

•Constraint behavior:

–Enlarge system, include evolution of constraints driven to constraint surface [Brodbeck, Frittelli, Huebner, Reula]

–Examine constraint growth at the onset [Lindblom-Scheel]

–Modify eqns to force constraints to behave well [Fiske]

–Choose parameters 'on the fly' according to evolution [Tiglio]

•Stability:

–Conserved quantities can be exploited to guarantee desired stability (no exponential growth!) [Calabrese,LL,Neilsen,Pullin,Sarbach,Tiglio]

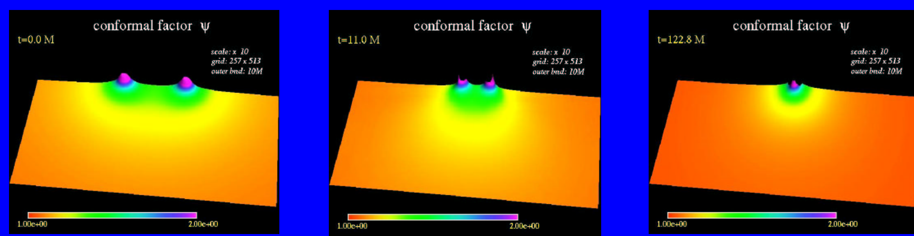
•Boundaries:

–Exploit mode propagation knowledge to (i) assess stability of evolution eqns, (ii) come up with constraint preserving boundary conditions.

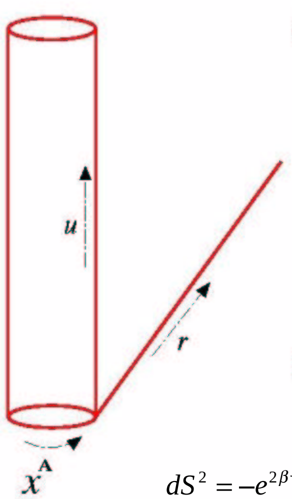
Alternative

- Adopt a *constrained evolution* strategy.
- What to solve for? [Lichnerowicz-York approach]

- Examples: GR in axysimmetry [Choptuik,Hirschman,Liebling,Pretorius]
  GR in 3D [Anderson,Hawley,Matzner]



Caracteristic formulation

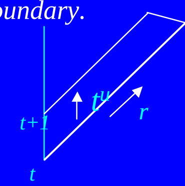


$$ds^2 = -\left(e^{2\beta}V/r - r^2 h_{AB}U^A U^B\right)du^2 - 2e^{2\beta}dudr$$
$$-2r^2 h_{AB}U^B du dx^A + r^2 h_{AB}dx^A dx^B$$

$$\beta_{,r} = F_\beta[h_{AB}];$$
$$U^A_{,rr} = F^A_U[\beta, h_{AB}];$$
$$V_{,r} = F_V[U^A, \beta, h_{AB}]$$
$$2\,(r h_{AB})_{,ur} - \frac{V}{r}\,(r h_{AB})_{,rr} = F_{AB}[V, U^A, \beta, h_{AB}]$$

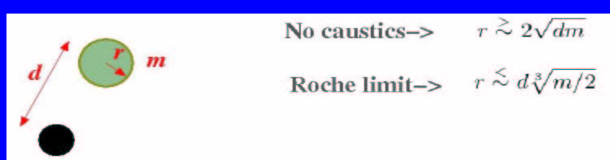
$$dS^2 = -e^{2\beta}V/r\; du^2 + h_{AB}(dx^A - r^2 U^A du)(dx^B - r^2 U^B du)$$

- Spacetime can be compactified, outer boundary is a *causal boundary*.
  - No boundary conditions needed there, no inconsistencies fed in.



- Structure of eqns: (2) first order in time + (4) ‘contraints’
- Coordinates: more restricted, free only at a given worldtube
- Well posed known only for the case inner boundary also null.
- *Main difficulty:* Caustics/crossovers render coordinates singular!
  - Way out: (I) Treat them ‘analytically’ [Friedrich-Stewart]
    (II) Use multiple patches & independent formulations on each [LL].
- Characteristic codes ‘finished’ and in place:
  - 3D [Bishop-Gomez-LL-Maharaj-Winicour; Bartnik-Norton]
  - 2D [D’Inverno et. al]

Example: Black hole – ‘neutron’ star system

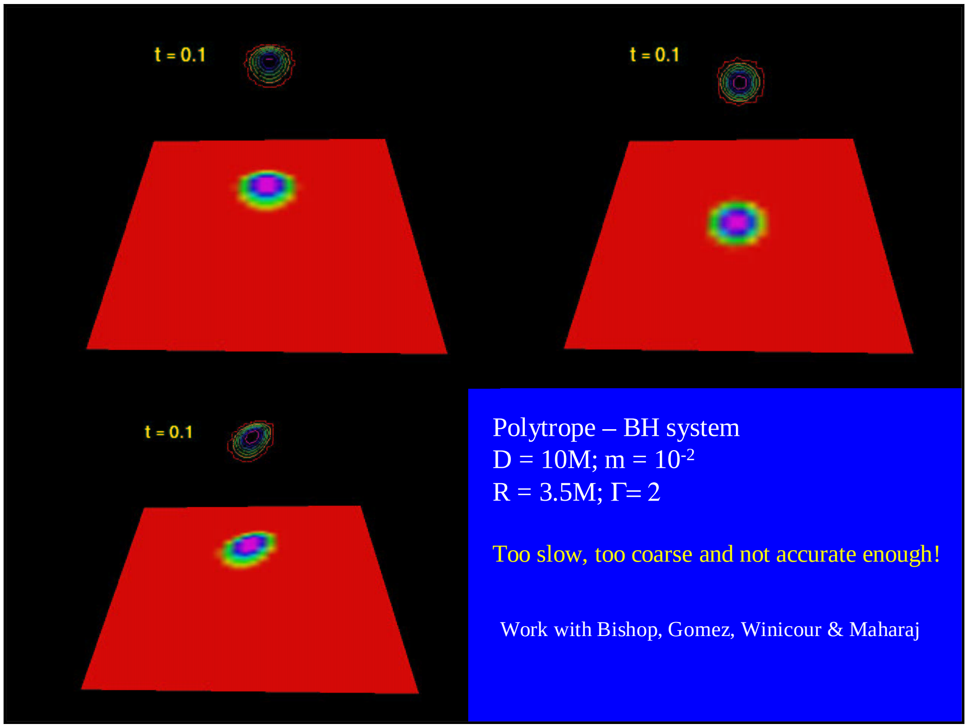


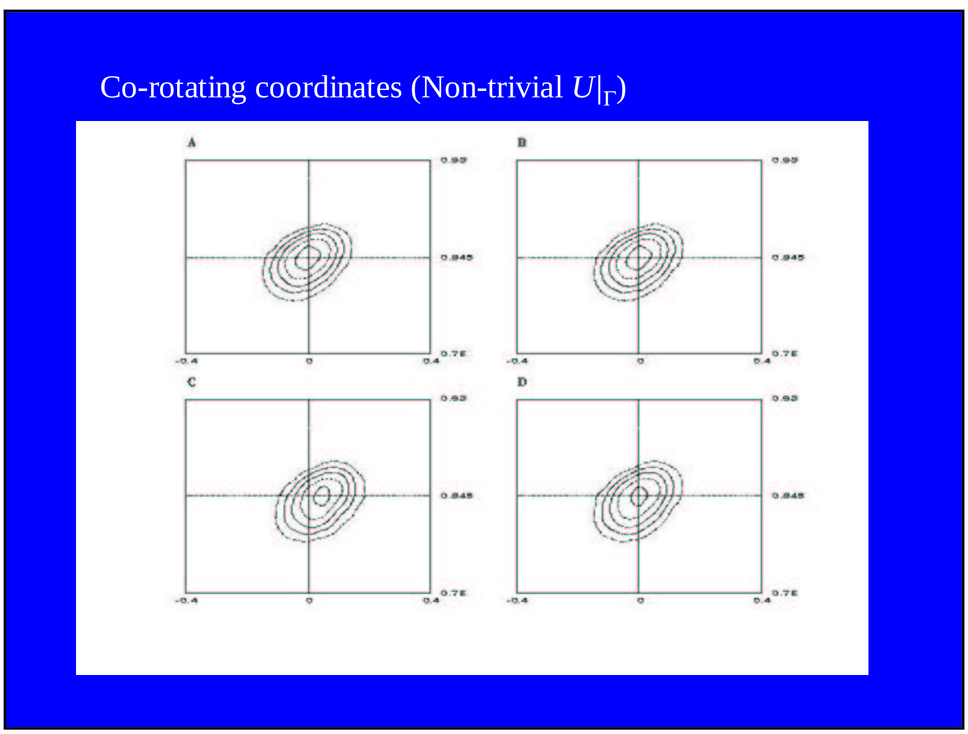
- Bdry data –> induced from Schwarzschild
- Initial data. Unconstrained in this approach. Almost any data is consistent!. Physically relevant?… In this case, obtained from a ‘Newtonian’ correspondence

•Write eqns in conservation form (need ‘new’ variables).

•$P:=(\rho,\varepsilon,u) \rightarrow N:=[det(g)]^{1/2} (T^{oo}, T^{0i}, J^0)$

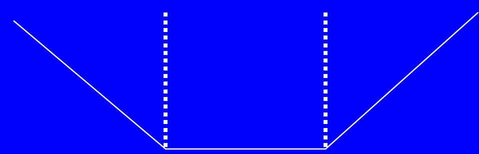
•$N_{,t} + F^j_{,x} = S$ (*)


t = 0.1
t = 0.1
t = 0.1
Polytrope – BH system
D = 10M; m = 10^-2
R = 3.5M; Γ= 2
Too slow, too coarse and not accurate enough!
Work with Bishop, Gomez, Winicour & Maharaj


Co-rotating coordinates (Non-trivial U|Γ)
A
B
C
D

Outlook

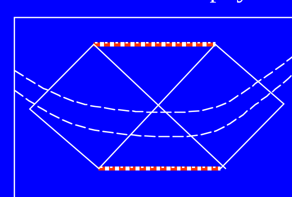
- In its simplest form, suited for:
  - Perturbations off single black hole spacetimes
  - Study of asymptotic structure of spacetimes
  - Matching to a Cauchy formulation to 'remove' hard boundaries.



- Open questions
  - Stability (rigorous) to timelike-null boundary problem?
    - 'experimental' evidence strong [ie. problems not seen]
    - Translate steps involved in assessing stability of IVBP of Cauchy approach.
  - Physically interesting boundary conditions.

Conformal approach to EE

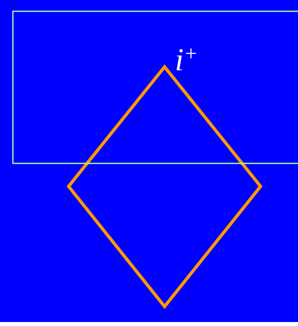
- [Friedrich 80's]. Key: construct a conformal spacetime and obtain the physical one a posteriori.



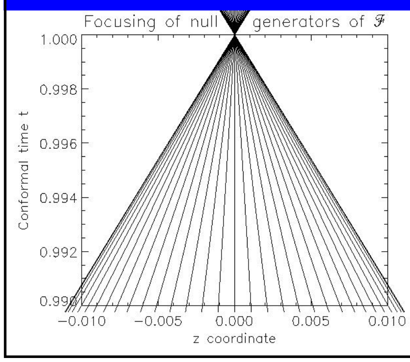
$$(\tilde{M},\tilde{g}_{ab}) \quad s.t.\ M_{phys} \subset \tilde{M};\tilde{g} = \Omega^2 g_{phys}$$

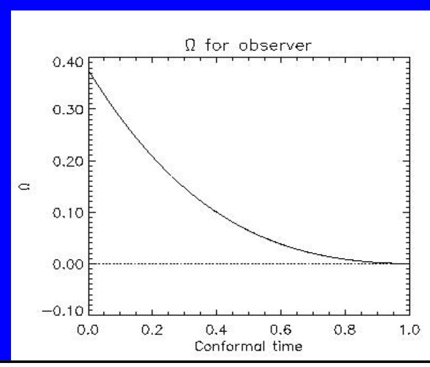
- Peculiarities: (Cauchy approach)
  - Eqns more 'involved', include eqns for (conformal) Weyl tensor
  - 5 'gauge' functions. Lapse, shift and 'extra' one $(\bar{g}_{ab},\bar{\Omega}) = (\Theta^2 \tilde{g}_{ab},\Theta\Omega)$
  - Outer boundary: 'hidden' by future null infinity (ie. disconnected from physical spacetime)
  - Initial data:
    - Conformal constraints quite complicated.
    - As far as 'physical spacetime', only need to solve for a subset of variables [Friedrich, Anderson, Chrusciel]. Procedure solve in physical, tailor/extrapolate outside…[Frauendiener]

Example



- 'Linear waves' on flat spacetime
  [Husa-Weaver]
- Evolution tracked; whole spacetime obtained; regular i^+
- Reproduced Friedrich's result
  (~related to Christodolou-Klainerman's)





Advantages

- No need for physically relevant boundary conditions, stable ones will do
- Flexible to consider other approaches within it; ie. Characteristic approach in principle equally doable.
- If asymptotic structure is to be explored, ideally suited for it!

Disadvantages(?)/little explored areas

- Initial data issue: Why expect good evolutions if constraints violated in the unphysical part?
- Coordinate conditions, what's a convenient way to fix the extra freedom?
- Common to 'standard' Cauchy approach:
  - If using a free evolution, can one choose an 'ideal' re-formulation of the eqns?

Other questions

- Accuracy?
  - Most expected radiated energy from ‘most violent’ systems ~5%
  - ‘systematic’ errors must be well below this!
  - ➔ back of envelope calculation (uniform grid):
    - 4 orders of magnitude off at least for 1 day turn around (Teraflop machine)
    - 100’s terabytes…
- Need adaptive mesh capabilities and/or extra infrastructure (domain decomposition, multiple patches, alternatives to finite differences: [spectral methods, finite elements]).

Final comments

- In GR one size doesn’t fit all, same holds for numerical relativity.
  - 3+1: general, flexible gauge, timelike outer boundary, most popular. Consistent initial data requires elliptic eqns be solved [Lichnerowicz-York]. (Constrained evolutions feasible).
  - characteristic: more restrictive, rigid gauge, null outer boundary (at future null infinity), when it can be applied works ‘scarily’ well. Can be used to study most ‘asymptotic questions’. Consistent initial data is trivial.
  - conformal: general, flexible gauge, outer boundary hidden by null infinity, can accommodate for the 2 previous, ID and gauge more involved. Can be used to study all ‘asymptotic questions’. Consistent initial data, similar to standard 3+1 in physical spacetime. (Constrained evolutions feasible.)

Very open questions

- How to give physically relevant initial/boundary data?
- For boundary data, how to give it so that is stable?