Einstein’s equations with artificial boundaries

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Outline

- Introduction
- Maximal dissipative boundary conditions
- Constraint-preserving boundary conditions : A toy model example
- Results for Einstein’s equations
- Beyond maximal dissipative boundary conditions
- Conclusions

Introduction

Problem: Solve Einstein's equations on a computer.

Motivation: Binary black hole problem, gravitational waves, critical phenomena, other dynamical phenomena.

Introduction

Einstein's equations naturally split into a set of evolution equations and a set of constraints:

$$\partial_t u = X(u), \qquad C(u) = 0.$$

Free evolution: Solve the constraints at $t = 0$ only. Because of the Bianchi identities, the constraint variables $C(u)$ satisfy a set of evolution equation on their own:

$$\partial_t C = Y(C),$$

where $Y(C)$ is homogeneous in C. Usually, this allows to show that the constraints are satisfied at later times as well.

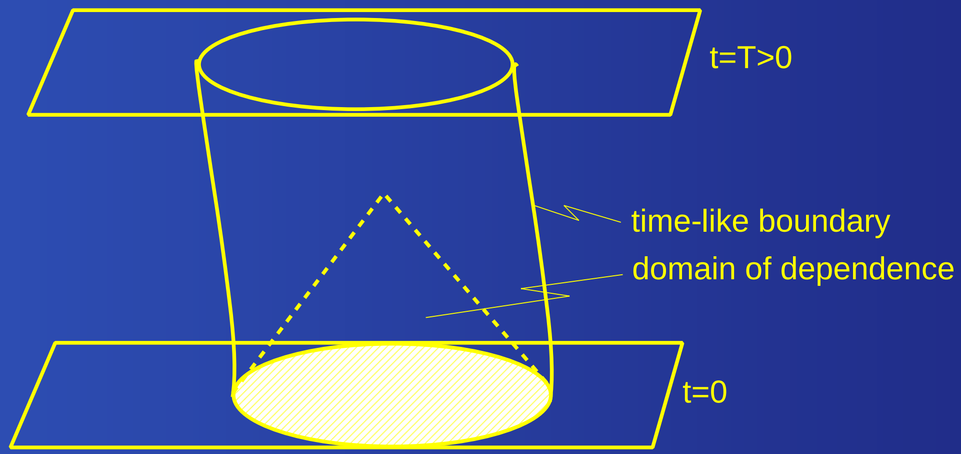
Introduction

But in view of numerical applications the situation is a little bit more complicated:

- Usually one cannot assume that $C = 0$ initially. Initial data is subject to small errors.
  How do these errors propagate in time?
  It is important to have a good understanding of the evolution equations for the constraint variables!

Introduction

- Due to finite computer memory, one usually has to deal with artificial boundaries.



The Bianchi identities only guarantee that the constraints are satisfied in the domain of dependence of the initial “region”. In order to satisfy the constraints beyond this domain, one has to be careful about the boundary conditions!

Introduction

Constraint-preserving boundary conditions (CPBC):

1998 Stewart: CPBC for the linearized field equations. *Class. Quantum Grav.* **15**, 2865 (1998).

1999 Friedrich, Nagy: Consistent boundary conditions for the nonlinear vacuum equations. Their formulation uses Weyl tensor as part of the fundamental variables. *Comm. Math. Phys.* **201**, 619 (1999).

2001 Calabrese, Lehner, Tiglio: Derivation of CPBC for the spherically symmetric case. *Phys. Rev. D* **65**, 044024 (2002).

2002 Szilagyi, Winicour: CPBC for the nonlinear vacuum equations in the harmonic gauge; homogeneous boundary conditions. gr-qc/0205044.

Introduction

2002 Calabrese, Pullin, Reula, Sarbach, Tiglio: Derivation of well posed CPBC for the generalized Einstein-Christoffel system in the weak field regime.
gr-qc/0209017, to appear in *Comm. Math. Phys*.

2003 Frittelli, Gomez: Projection of Einstein's equations along the normal to the boundary surface —-> CPBC?
gr-qc/0302032.

Alternative approach: Conformal field equations.

Maximal dissipative b.c.

Initial-boundary value problem:
Evolution equations as a (quasi)linear first order system

$$\partial_t u = A^j(t, x^i)\partial_j u + B(t, x^i, u), \qquad (t, x^i) \in [0, T] \times \Omega$$

with initial data $u(t = 0) = f$ and boundary data $Mu(x^i \in \partial\Omega) = g$.
VERY IMPORTANT notion in view of numerical applications:

- Well posedness estimate: $\|u(t)\| \leq C(t)(\|f\| + \|g\|)$ for all $0 \leq t \leq T$ and *all data* f, g. This implies that at each time $t > 0$ the solution depends continuously on the initial and boundary data. This is important because at the numerical level the data is contaminated with small errors.
  (One might want to control $C(t)$ as a function of time as well.)

Maximal dissipative b.c.

Remark on weakly hyperbolic systems: As an example, consider the following system with periodic b.c.

- $$\begin{aligned} \partial_t u_1 &= \partial_x u_1 + \partial_x u_2 \\ \partial_t u_2 &= \qquad\quad\; \partial_x u_2 + a u_1 \end{aligned}$$

  Fourier space: $\partial_t u = A(\omega) u$, where $A = \begin{pmatrix} i\omega & i\omega \\ a & i\omega \end{pmatrix}$ has eigenvalues $\lambda_\pm = i\omega \pm \sqrt{ai\omega}$. Take eigenfunction of A with eigenvalue λ_+ as initial data: $\|u(t)\| = \exp\left(\text{Real}(\lambda_+)t\right)\|u(0)\|$, but $\text{Real}(\lambda_+) \to \infty$ as $\omega \to \infty$ for each fixed t. (If $a = 0$ one has only a linear growth in ω).

ADM equations are weakly hyperbolic. Cannot have a stable evolution (Calabrese et al. *Phys. Rev. D* **66**, 041501 (2002)).

Maximal dissipative b.c.

For (quasi)linear first order systems

$$\partial_t u = A^j(u)\partial_j u + B(u),$$

(local) well posedness estimates can be obtained if some (simple) algebraic conditions on the principal symbol ($A(n) = A^j n_j$) are verified:

- $A(n)$ diagonalizable for all n, eigenvalues real, and eigenvectors depend smoothly on n, x and u: strongly hyperbolic. One can obtain estimates for periodic b.c. (or no b.c.)
- Special case: $A(n)$ symmetric for all n: symmetric hyperbolic. If suitable boundary conditions are specified, symmetric hyperbolic equations also imply well-posedness for the initial-boundary value problem.

Maximal dissipative b.c.

Example: $\partial_t u = A^j \partial_j u$, A^1, A^2, A^3 symmetric constant matrices.
Energy estimate: $E(t) = \int_\Omega u(t,x)^T u(t,x) d^3x$, integration by parts

$$\dot{E}(t) = 2\int_\Omega u^T A^j \partial_j u = \int_\Omega \partial_j (u^T A^j u) = \int_{\partial\Omega} u^T A^j n_j u.$$

Suppose $A^j n_j$ has only the eigenvalues $0, \pm 1$. Decompose
$u = u_+ + u_- + u_0$: $u^T A^j n_j u = u_+^2 - u_-^2$.
Maximal dissipative boundary conditions: $u_+ = L u_-$.
If L is small enough, the boundary term is nonnegative, and it follows that

$$E(t) \leq E(0).$$

In particular one has *uniqueness*.

Maximal dissipative b.c.

Generalization to

$$u_+ = Lu_- + b,$$

where b is a prescribed function at the boundary.

If the boundaries are smooth (and other technical assumptions are satisfied), the existence of a smooth solution to the initial-boundary value problem for quasilinear symmetric hyperbolic systems follows (Rauch, Secchi,...).

Maximal dissipative b.c.

To summarize:

Symmetric hyperbolic first order evolution systems

and

maximal dissipative boundary conditions

(+ technical assumptions)

yields well posed initial-boundary value formulations.

BE CAREFUL IF SYSTEM IS NOT SYMMETRIC HYPERBOLIC

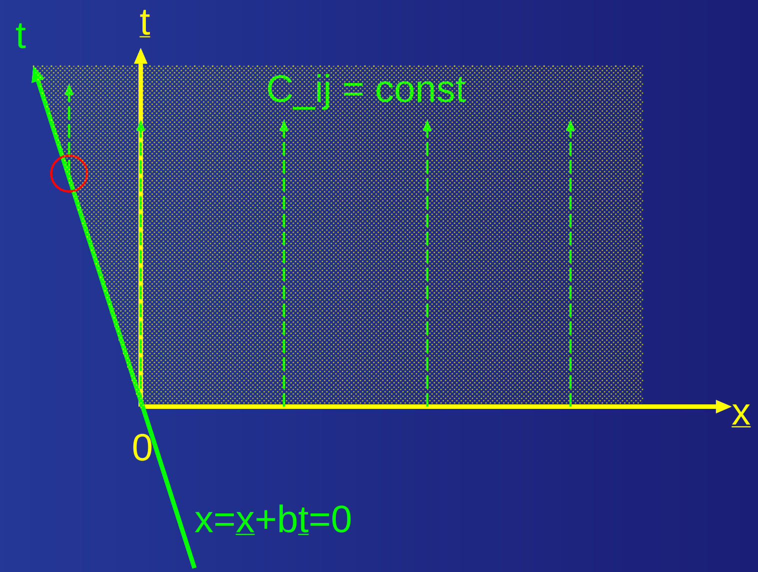
(see end of the talk)

CPBC: A toy model example

In order to illustrate how to construct constraint-preserving boundary conditions, we consider the flat wave equation, written in first order form, but with a nontrivial shift:

$$
\begin{aligned}
(\partial_t + b\partial_x)\Pi &= \partial^i d_i\,, \\
(\partial_t + b\partial_x)d_i &= \partial_i \Pi\,,
\end{aligned}
$$

where $0 < b < 1$. Solve this for $t > 0$, $x > 0$.



This system is only equivalent to the wave equation if the constraints $0 = C_{ij} = \partial_i d_j - \partial_j d_i$ are satisfied. The C_{ij} propagate according to $(\partial_t + b\partial_x)C_{ij} = 0$, so $C_{ij} = const$ along the lines $x - bt = const$.

CPBC: A toy model example

Therefore, we have to set $C_{ij} = 0$ not only on the initial slice, but also on the boundary surface $x = 0$. So we need to impose $0 = 2C_{xy} = \partial_x d_y - \partial_y d_x$.

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Eliminate those terms by using the evolution equations:

$0 = \partial_x d_y - \partial_y d_x = -b^{-1}\partial_t d_y + b^{-1}\partial_y \Pi - \partial_y d_x$.

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Therefore, we have the following equations which are intrinsic to the boundary $x = 0$:

$$
\begin{aligned}
\partial_t d_y &= \partial_y(\Pi - b\, d_x), \\
\partial_t d_z &= \partial_z(\Pi - b\, d_x).
\end{aligned}
$$

These conditions automatically imply $C_{yz} = 0$.

CPBC: A toy model example

$$\partial_t d_y = \partial_y(\Pi - b\, d_x),$$
$$\partial_t d_z = \partial_z(\Pi - b\, d_x).$$

However, these conditions do not look like maximal dissipative boundary conditions. In terms of the characteristic variables, $\Pi - b\, d_x = (1+b)u_{in} + (1-b)u_{out}$, where d_y,d_z,u_{in} are ingoing variables.

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But we can set $\Pi - b\, d_x = \psi$, where ψ is a *prescribed* function at the boundary and then integrate the above equations to obtain d_y, d_z.

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But typically, we would like to set $u_{in} = 0$ (“radiative” boundary condition). In this case, one obtains a differential equation for d_y, d_z, u_{out} that is intrinsic to the boundary. This requires to go beyond maximal dissipative boundary conditions!

Results for Einstein's equations

Einstein-Christoffel system: Evolution equations are system of six coupled wave equations written in first order form:

$$\begin{aligned} \partial_t K_{ij} &= \partial^k f_{kij} \, , \\ \partial_t f_{kij} &= \partial_k K_{ij} \, . \end{aligned}$$

plus a bunch of constraints, $0 = C(K_{ij}, f_{kij}; \eta)$, where η is a parameter. Constraint variables C satisfy a first order evolution system on their own. This system has been shown to be symmetric hyperbolic if $0 < \eta < 2$ and strongly hyperbolic otherwise.

Results for Einstein’s equations

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- Analyze the characteristic fields of the evolution system for the constraints. Impose homogeneous maximal dissipative b.c. for this system. If $0 < \eta < 2$ the general theory guarantees that the constraints are zero if so initially. Furthermore, for each fixed t, small initial errors give rise to small errors at time t.

$$C_{in} = LC_{out}$$

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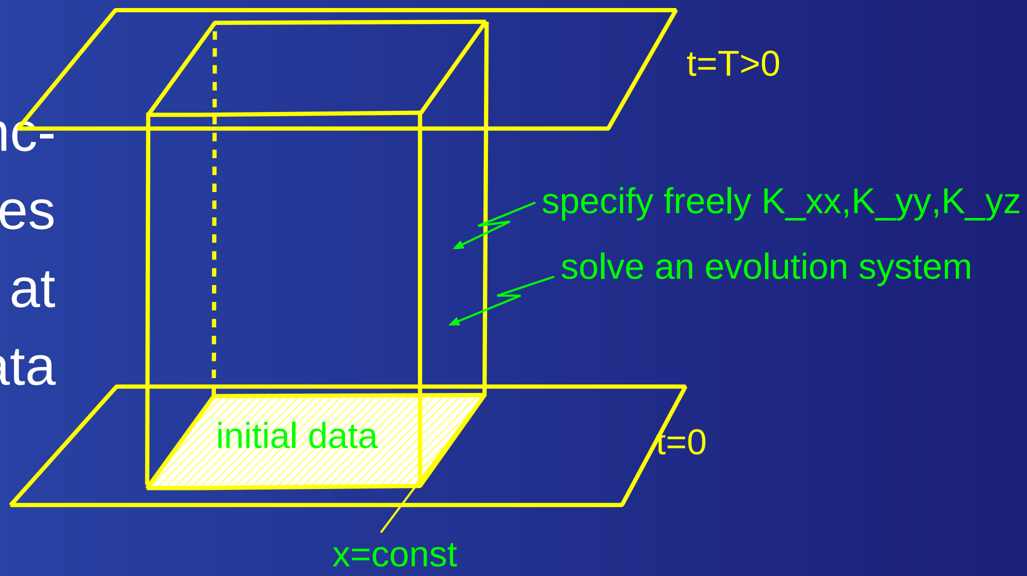
- Translate these conditions into conditions for the main variables. Eliminate normal derivatives by using the main evolution eqns.
- For a suitable coupling between the in- and outgoing fields the resulting system gives rise to a closed symmetric hyperbolic evolution system at the boundary. Once this system is solved, one obtains data for maximal dissipative boundary conditions for the main system.

Results for Einstein's equations

One ends up with three source functions that are free: They can be used to specify Dirichlet or Neumann-like conditions on the normal-normal and transverse-traceless components of the metric components (the former corresponding to a gauge freedom, the latter to a physical choice).

Once these free source functions are specified, one solves the closed evolution system at the boundary and obtains data for the main evolution system.



Compatibility conditions at edges!

Results for Einstein's equations

So the boundary conditions we have found are CPBC and yield a well-posed initial-boundary value formulation. However, it is far from clear that the b.c. are suitable for physical purposes since likely, they are going to introduce reflections at the boundaries.

More general constraint-preserving boundary conditions where one can control part of the ingoing characteristic fields can be constructed. But do they yield a well posed system?

Beyond maximal dissipative b.c.

Detecting ill-posed modes using Laplace-Fourier techniques: Consider linear symmetric hyperbolic system with constant coefficients

$$\partial_t u = A^x \partial_x u + A^y \partial_y u + A^z \partial_z u, \quad t > 0, x > 0,$$

with boundary conditions of the form

$$M(\partial_t, \partial_y, \partial_z) u = g(t, y, z).$$

Look for solutions of the form $u(t, x, y, z) = e^{st + i(w_y y + w_z z)} f(x)$, where $Re(s) > 0$, w_y, w_z real.

Test: If $g = 0$ there should be no such solutions. Otherwise the system is ill posed: Because if there is such a solution for some s, $Re(s) > 0$, then there is also a solution for αs, $\alpha > 0$ and for each fixed t

$$|u_\alpha(t, x, y, z)|/|u_\alpha(0, x, y, z)| = e^{\alpha Re(s) t} \longrightarrow \infty.$$

Beyond maximal dissipative b.c.

Introducing the ansatz $u(t,x,y,z) = e^{st+i(w_y y + w_z z)} f(x)$ into the evolution and boundary equations gives

$$sf = A\partial_x f + i(A^y w_y + A^z w_z)f, \qquad L(s, iw_y, iw_z)f = 0.$$

Solution has the form $f(x) = Pe^{M_- x}\sigma_-$, $Re(M_-) < 0$ with $LP\sigma_- = 0$. Therefore, one has to verify the determinant condition

$$\det(L(s, iw_y, iw_z)P) \neq 0, \qquad Re(s) > 0.$$

Beyond maximal dissipative b.c.

Applications:

- Wave equation in first order form with shift: Sommerfeld-like conditions: passes test.
- More general CPBC for the linearized Einstein-Christoffel system: Sommerfeld-like conditions: passes test.

Calabrese, Sarbach, gr-qc/0303040, to appear in *J. Math. Phys*

Beyond maximal dissipative b.c.

Applications:

- More general CPBC for the linearized Einstein-Christoffel system: Consider values for η outside the interval $(0, 2)$, where the evolution system for the constraint variables is not known to be symmetric hyperbolic: test fails for $\eta < 0$ or $\eta > 8/3$!

In the last case, one can show that the ill posed modes ($u \sim e^{st+iw_y y+iw_z z}$) violate the constraints in the sense that $C(u) \neq 0$. This means that the initial-boundary value problem for the constraint variables is not well posed! This is surprising because the system is strongly hyperbolic and we specify boundary conditions that set the ingoing modes to zero!

In this sense boundary conditions are not constraint-preserving!

Beyond maximal dissipative b.c.

Remark: Recent proposal by Frittelli, Gomez:
Initial data on a hypersurface with normal vector n^μ is subject to the constraints $G_{\mu\nu}n^\mu = 0$. At time-like boundaries, it is natural to analyze the equations $G_{\mu\nu}v^\mu = 0$, where v^μ is the unit outward normal to the boundary.
When introducing first spatial derivatives of the metric as extra variables (as is the case in the Einstein-Christoffel system) these equations are intrinsic to the boundary in the sense that they involve only derivatives that are tangential to the boundary.
Do these equations provide CPBC?
Do they pass the test?

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Do these equations provide CPBC?
Do they pass the test? NOT IF $\eta < 0$ or $\eta > 8/3$.
So it is really important to analyze the evolution of the constraint variables!

Conclusions

- IBVP in GR far from being solved (given a generic hyperbolic formulation of Einstein's equations). Important for NR since in 3d simulations one cannot push the boundary far away.
- Partial results in the linearized case. Generalization to nonlinear case might require more powerful techniques (Fourier- Laplace).
- Imposing the projection of Einstein's equations along the normal to the boundary surface is not enough.
- Surprising mathematical result: Strongly hyperbolic system with zero ingoing characteristic fields at the boundary can lead to an unstable system.