
Into Thin Air
climbing a smooth route
to null infinity

people

AEI:

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- Carsten Schnemann, Anil Zenginoglu

Ian Hinder from U. Southampton collaborates on automatized code generation

questions

answered:

- why compactification? why hyperboloidal?
- why spherical boundaries?
- why is our code called Scriwalker?
- are 92 constraints too many? can we evolve Minkowski space?

raised:

- spherical boundaries – but how? how should we solve the constraints?
- can we get rid of first order symmetric hyperbolic?

isolated systems as models of sources of gravitational radiation

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Observers situated at "astronomical" distances (e.g. gravitational wave detectors) – "looking inside along light rays" – are modeled by geometric objects at null infinity ($\approx$ "in phase" with radiation source!).

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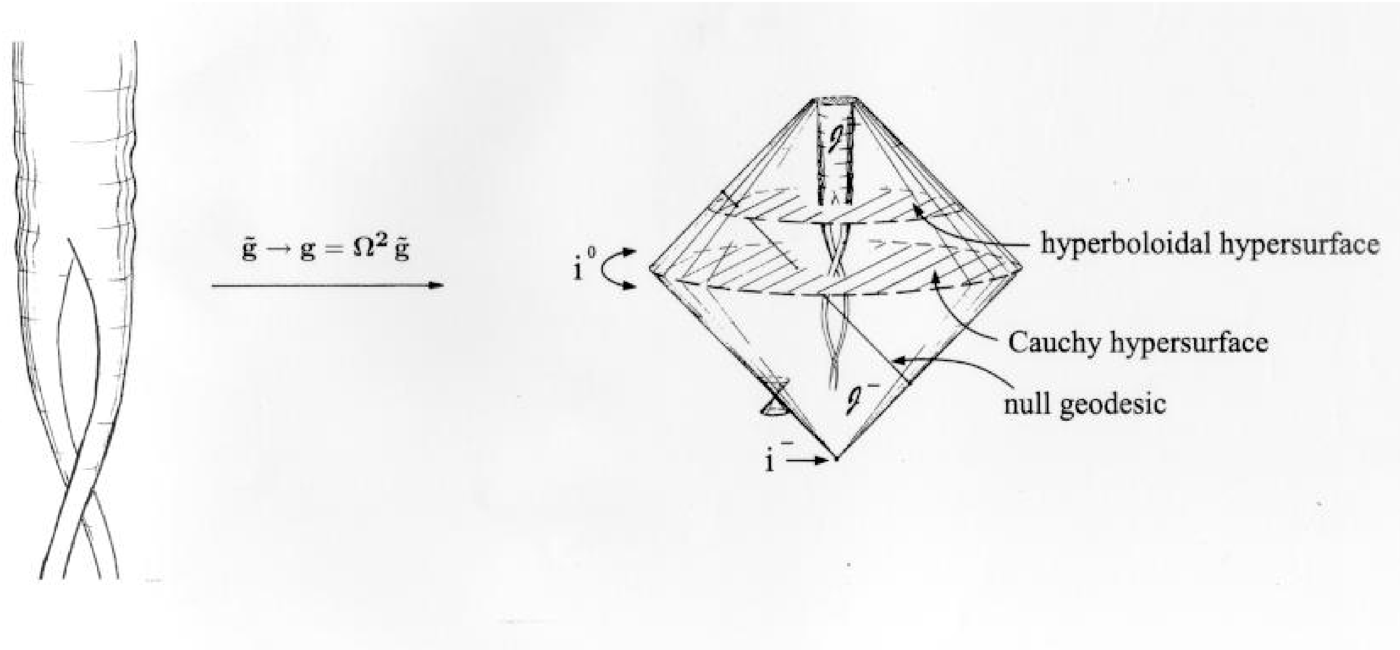
conformal compactification



Using conformal compactification to an unphysical spacetime, we can discuss AF spacetimes in terms of local differential geometry (Penrose!).

$$\tilde{g}_{ab} = \Omega^{-2} g_{ab}, \qquad \tilde{\mathcal{M}} = \{p \in \mathcal{M} \,|\, \Omega(p) > 0\}.$$

"infinity" $\to \Omega = 0$: 3-dimensional boundary of a 4-dimensional region in $\mathcal{M}$.

Remark on compactifying Einstein Equations

Can obviously not be straightforward:

Einstein's vacuum equations in terms of Ω & g_{ab}:

$$\begin{aligned}\tilde{G}_{ab}[\Omega^{-2}g] &= G_{ab}[g] - \frac{2}{\Omega}\left(\nabla_a\nabla_b\Omega - g_{ab}\nabla_c\nabla^c\Omega\right) \\ &\quad -\frac{3}{\Omega^2}g_{ab}\left(\nabla_c\Omega\right)\nabla^c\Omega.\end{aligned}$$

singular for $\Omega = 0$, multiplication by Ω^2 also does not help here $\rightarrow$ the principal part of PDEs encoded in G_{ab} would degenerate at $\Omega = 0$.

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Multiply by Ω^2:

for a vacuum spacetime $(\nabla_c\Omega)\nabla^c\Omega = 0$ @ $\mathscr{I}$ $\Rightarrow$ must consist of null surfaces!

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Under practical circumstances, e.g. computing the signal at a GW detector, $\mathscr{I}$ more realistically corresponds to an observer that is sufficiently far way from the source to treat the radiation linearly, but not so far away that cosmological effects have to be taken into account.

conformal field equations in 30 seconds

Start with splitting Riemann into trace-free (Weyl) and trace (Ricci and scalar) parts, define tracefree Ricci $\hat{R}_{ab} = R_{ab} - \frac{1}{4}\, g_{ab}\, R$ and rescaled Weyl

$$C_{abc}{}^{d} = \Omega\, d_{abc}{}^{d}.$$

$\tilde{R} = 0$ and $\tilde{R}_{ab} = 0$ imply

$$\begin{aligned} 6\,\Omega\,\nabla^{a}\nabla_{a}\Omega &= 12\,(\nabla^{a}\Omega)\,(\nabla_{a}\Omega) - \Omega^{2} R, \\ \nabla_{a}\nabla_{b}\Omega &= \frac{1}{4} g_{ab}\,\nabla^{c}\nabla_{c}\Omega - \frac{1}{2}\,\hat{R}_{ab}\,\Omega. \end{aligned} \tag{1}$$

Commute $\nabla_{c}\nabla_{b}$ in $g^{bc}\nabla_{c}\nabla_{b}\nabla_{a}\Omega$ and (1):

$$\frac{1}{4}\nabla_{a}\left(\nabla^{b}\nabla_{b}\Omega\right) = -\frac{1}{2}\,\hat{R}_{ab}\,\nabla^{b}\Omega - \frac{1}{24}\,\Omega\,\nabla_{a}R - \frac{1}{12}\,\nabla_{a}\Omega\,R,$$

No equations for g_{ab} yet! – use identity defining Weyl

$$R_{abc}{}^{d} = \Omega d_{abc}{}^{d} + \left(g_{ca}g_{b}{}^{d} - g_{cb}g_{a}{}^{d}\right)\frac{R}{12} + \left(g_{ca}\hat{R}_{b}{}^{d} - g_{cb}\hat{R}_{a}{}^{d} - g^{d}{}_{a}\hat{R}_{bc} + g^{d}{}_{b}\hat{R}_{ac}\right)/2.$$

Equations for $d_{abc}{}^{d}$ and $\hat{R}_{ab}$? – Bianchi identities $\nabla_{[a}R_{bc]d}{}^{e} = 0$ imply

$$\nabla_{b}\hat{R}_{a}{}^{b} = \frac{1}{4}\nabla_{a}R \quad \text{and} \quad \nabla_{d}C_{abc}{}^{d} = 0, \tag{2}$$

Weyl is conformally invariant,

$$\tilde{C}_{abc}{}^{d} = C_{abc}{}^{d} \quad \rightarrow \quad \tilde{\nabla}_{d}\tilde{C}_{abc}{}^{d} = \Omega\nabla_{d}\left(d_{abc}{}^{d}\right),$$

thus

$$\nabla_{d}d_{abc}{}^{d} = 0.$$

Bianchi identity combined with the definition of Weyl implies

$$\begin{aligned}\nabla_a \hat{R}_{bc} - \nabla_b \hat{R}_{ac} &= -\frac{1}{12}\left((\nabla_a R)\, g_{bc} - (\nabla_b R)\, g_{ac}\right) \\ &\quad -2\,(\nabla_d \Omega)\, d_{abc}{}^d .\end{aligned}$$

For any solution $(g_{ab}, \hat{R}_{ab}, d_{abc}{}^d, \Omega)$, R is the Ricci scalar, $\hat{R}_{ab}$ the tracefree Ricci tensor, and $\Omega\, d_{abc}{}^d$ the Weyl tensor of g_{ab}.

3+1 split $\longrightarrow$ 57 Variables:
h_{ab}, k_{ab},
$\gamma^a{}_{bc}$,
${}^{(0,1)}\hat{R}_a$, ${}^{(1,1)}\hat{R}_{ab}$,
E_{ab}, B_{ab},
Ω, Ω_0, Ω_a, $\nabla^a \nabla_a \Omega$

BUT: there is a lot of freedom, as long as Ω and E_{ab}, B_{ab} remain evolution variables!

3+1 – business as usual

signature $(-,+,+,+)$:

$$g_{ab} = h_{ab} - n_a n_b = \Omega^2(\tilde{h}_{ab} - \tilde{n}_a \tilde{n}_b),$$

$$\tilde{n}_a = \Omega\, n_a$$

extrinsic curvature:

$$\tilde{k}_{ab} = \frac{1}{2}\mathcal{L}_{\tilde{n}}\tilde{h}_{ab}, \qquad k_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab}$$

$k_{ab} = \Omega\,(\tilde{k}_{ab} + \Omega_0\, \tilde{h}_{ab})$, where $\Omega_0 = n^a \nabla_a \Omega$.

$\hat{R}_{ab}$ and $d_{abc}{}^d$ are decomposed as

$$\begin{aligned} {}^{(0,1)}\hat{R}_a &= n^b h_a{}^c \hat{R}_{bc}, \quad {}^{(1,1)}\hat{R}_{ab} = h_a{}^c h_b{}^d \hat{R}_{bd}, \\ E_{ab} &= d_{efcd} h^e{}_a n^f h^c{}_b n^d, \quad B_{ab} = d^*_{efcd} h^e{}_a n^f h^c{}_b n^d. \end{aligned}$$

hyperboloidal hypersurfaces

Components of $\tilde{h}_{ab}$ and $\tilde{k}_{ab}$ diverge in compactified coordinates – coordinate independent trace $\tilde{k}$ can be assumed regular everywhere,

$$\Omega\, k = (\tilde{k} + 3\,\Omega_0), \qquad \tilde{k}|_{\mathscr{I}} = -3\,\Omega_0$$

$\mathscr{I}^+$ ingoing null surface: $\Omega_0 < 0$ at $\mathscr{I}^+ \Rightarrow \tilde{k} > 0$.

Regular spacelike hypersurfaces in $\mathcal{M}$: hyperboloidal hypersurfaces $\equiv$ spacelike surfaces in $\mathcal{M}$ with $\lim_{r\to\infty} \tilde{k} > 0$

These surfaces are asymptotically null with respect to $\tilde{g}_{ab}$!

In $\tilde{\mathcal{M}}$ they are analogous to standard hyperboloids $t^2 - r^2 = \tilde{k}^2$ in Minkowski.

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If you go further out in space, you also have to go to later times to follow the radiation!

$$
\begin{aligned}
\mathcal{C}_{\mathbf{h}\;abc} &= {}^{(3)}\nabla_a h_{bc} \\
\mathcal{C}_{\mathbf{k}\;abc} &= -\;{}^{(3)}\nabla_a k_{bc} + {}^{(3)}\nabla_b k_{ac} + \frac{1}{2} h_{ca}\,{}^{(0,1)}\hat{R}_b - \frac{1}{2} h_{cb}\,{}^{(0,1)}\hat{R}_a - {}^{(3)}\epsilon_{ab}{}^{d}\Omega B_{dc} \\
\mathcal{C}_{\gamma\;abc}{}^{d} &= -\;{}^{(3)}\nabla_a \gamma^{d}{}_{bc} + {}^{(3)}\nabla_b \gamma^{d}{}_{ac} + \gamma^{d}{}_{ae}\gamma^{e}{}_{bc} - \gamma^{d}{}_{be}\gamma^{e}{}_{ac} \\
&\quad - k_a{}^{d} k_{bc} + k_{ac} k_b{}^{d} + \frac{1}{12} h_a{}^{d} h_{bc} R - \frac{1}{12} h_{ac} h_b{}^{d} R \\
&\quad - \frac{1}{2} h_b{}^{d}\,{}^{(1,1)}\hat{R}_{ac} + \frac{1}{2} h_a{}^{d}\,{}^{(1,1)}\hat{R}_{bc} - \frac{1}{2} h_{ac}\,{}^{(1,1)}\hat{R}_b{}^{d} + \frac{1}{2} h_a{}^{d}\,{}^{(1,1)}\hat{R}_{bc} \\
&\quad - h_{ac}\Omega E_b{}^{d} + h_{bc}\Omega E_a{}^{d} + h_b{}^{d}\Omega E_{ac} - h_a{}^{d}\Omega E_{bc} \\
\mathcal{C}_{\mathbf{E}\;a} &= -\;{}^{(3)}\nabla_b E_a{}^{b} - {}^{(3)}\epsilon_{abc} k^{bd} B_d{}^{c} \\
\mathcal{C}_{\mathbf{B}\;a} &= -\;{}^{(3)}\nabla_b B_a{}^{b} + {}^{(3)}\epsilon_{abc} k^{bd} E_d{}^{c} \\
\mathcal{C}_{{}^{(\mathbf{0},\mathbf{1})}\hat{\mathbf{R}}\;ab} &= {}^{(3)}\nabla_a\,{}^{(0,1)}\hat{R}_b - {}^{(3)}\nabla_b\,{}^{(0,1)}\hat{R}_a + k_b{}^{c}\,{}^{(1,1)}\hat{R}_{ca} - k_a{}^{c}\,{}^{(1,1)}\hat{R}_{cb} + 2\,{}^{(3)}\epsilon_{ab}{}^{c}\Omega_d B_c{}^{d} \\
\mathcal{C}_{{}^{(\mathbf{1},\mathbf{1})}\hat{\mathbf{R}}\;abc} &= {}^{(3)}\nabla_a\,{}^{(1,1)}\hat{R}_{bc} - {}^{(3)}\nabla_b\,{}^{(1,1)}\hat{R}_{ac} - \frac{1}{12} h_{ac}\,{}^{(3)}\nabla_b R + \frac{1}{12} h_{bc}\,{}^{(3)}\nabla_a R + {}^{(0,1)}\hat{R}_a k_{bc} \\
&\quad - {}^{(0,1)}\hat{R}_b k_{ac} + 2\,{}^{(3)}\epsilon_{ab}{}^{d}\Omega_0 B_{dc} - 2\Omega_a E_{bc} + 2\Omega_b E_{ac} + 2h_{ca}\Omega_d E_b{}^{d} - 2h_{cb}\Omega_d E_a{}^{d} \\
\mathcal{C}_{\mathbf{\Omega}\;a} &= -\;{}^{(3)}\nabla_a \Omega + \Omega_a, \qquad \mathcal{C}_{\mathbf{\Omega_0}\;a} = -\;{}^{(3)}\nabla_a \Omega_0 + k_a{}^{b}\Omega_b - \frac{1}{2}\Omega\,{}^{(0,1)}\hat{R}_a \\
\mathcal{C}_{\mathbf{\Omega_a}\;ab} &= -\;{}^{(3)}\nabla_a \Omega_b + h_{ab}\omega + k_{ab}\Omega_0 - \frac{1}{2}\Omega\,{}^{(1,1)}\hat{R}_{ab} \\
\mathcal{C}_{\omega\;a} &= -\;{}^{(3)}\nabla_a \omega - \frac{1}{24}\Omega\,{}^{(3)}\nabla_a R - \frac{1}{12}\Omega_a R + \frac{1}{2}\Omega_0\,{}^{(0,1)}\hat{R}_a - \frac{1}{2}\Omega^{b}\,{}^{(1,1)}\hat{R}_{ba}
\end{aligned}
$$

are we trapped by too many equations?

analyze the situation . . .

first steps toward simplification

constraints:

- split into independent components – has only been done recently!

evolution equations:

- look for potential feedback terms

general:

- look at the case $\Omega = 1$ – this already leads to interesting new features as compared to standard GR formulations for NR – the inclusion of curvature variables!

- look at cases with symmetry

$$
\begin{aligned}
\mathcal{L}_n h_{ab} &= 2k_{ab}, \qquad \mathcal{L}_n k_{ab} = {}^{(3)}\nabla_c \gamma^c{}_{ab} + \gamma^d{}_{bc}\gamma^c{}_{ad} + a_a a_b + k_c{}^c k_{ab} - \gamma^c{}_{ab} a_c \\
&\quad + h_a{}^c h_b{}^d \partial_d \partial_c q - \frac{R}{12} h_{ab} - {}^{(1,1)}\hat{R}_c{}^c h_{ab} - 2\Omega E_{ab} \\
h_{ad}\,\mathcal{L}_n \gamma^d{}_{bc} &= + {}^{(3)}\nabla_a k_{bc} - a_a k_{bc} + a_c k_{ab} + a_b k_{ac} + h_{da} h_b{}^e h_c{}^f \frac{1}{N} \partial_f \partial_e N^d + \ldots \\
\mathcal{L}_n E_{ab} &= +\frac{1}{2}\,{}^{(3)}\epsilon_a{}^{cd}\,{}^{(3)}\nabla_d B_{cb} + \frac{1}{2}\,{}^{(3)}\epsilon_b{}^{cd}\,{}^{(3)}\nabla_d B_{ca} + a^c\,{}^{(3)}\epsilon_{cb}{}^d B_{da} + a^c\,{}^{(3)}\epsilon_{ca}{}^d B_{db} \\
&\quad - h_{ab} k^{cd} E_{cd} + \frac{5}{2} k_b{}^c E_{ca} + \frac{5}{2} k_a{}^c E_{cb} - 2k_c{}^c E_{ab} \\
\mathcal{L}_n B_{ab} &= -\frac{1}{2}\,{}^{(3)}\epsilon_a{}^{cd}\,{}^{(3)}\nabla_d E_{cb} - \frac{1}{2}\,{}^{(3)}\epsilon_b{}^{cd}\,{}^{(3)}\nabla_d E_{ca} + a^c\,{}^{(3)}\epsilon_{bc}{}^d E_{da} + a^c\,{}^{(3)}\epsilon_{ac}{}^d E_{db} \\
&\quad - h_{ab} k^{cd} B_{cd} + \frac{5}{2} k_b{}^c B_{ca} + \frac{5}{2} k_a{}^c B_{cb} - 2k_c{}^c B_{ab} \\
\mathcal{L}_n\,{}^{(0,1)}\hat{R}_a &= {}^{(3)}\nabla_b\,{}^{(1,1)}\hat{R}_a{}^b - \frac{1}{4}\,{}^{(3)}\nabla_a R - k_b{}^b\,{}^{(0,1)}\hat{R}_a + a_b\,{}^{(1,1)}\hat{R}_a{}^b + a_a\,{}^{(1,1)}\hat{R}_b{}^b \\
h_{bc}\,\mathcal{L}_n\,{}^{(1,1)}\hat{R}_a{}^c &= {}^{(3)}\nabla_a\,{}^{(0,1)}\hat{R}_b - \frac{1}{12} h_{ab} \mathcal{L}_n R + \ldots \\
\mathcal{L}_n \Omega &= \Omega_0, \qquad \mathcal{L}_n \Omega_0 - \omega + a^a \Omega_a - \frac{\Omega}{2}\,{}^{(1,1)}\hat{R}_a{}^a \\
\mathcal{L}_n \Omega_a &= a_a \Omega_0 + k_a{}^b \Omega_b - \frac{\Omega}{2}\,{}^{(0,1)}\hat{R}_a, \quad \mathcal{L}_n \omega = -\frac{\Omega}{24} \mathcal{L}_n R - \frac{R}{12} \Omega_0 - \frac{\Omega^a}{2}\,{}^{(0,1)}\hat{R}_a + \frac{\Omega_0}{2}\,{}^{(1,1)}\hat{R}_a{}^a
\end{aligned}
$$

The task

Create an approach to numerical relativity which is at least as flexible as the traditional Cauchy approach, yet as free from ambiguities arising from approximating the global nature of the problem as the characteristic approach.

Apart from a general computational framework, detailed algorithms and software modules are needed for the

1. construction of initial data on hyperboloidal hypersurfaces,
2. treatment of grid boundaries,
3. time evolution, including choice and implementation of gauge conditions,
4. computation of gravitational wave information and additional analysis of physical properties of numerically constructed spacetimes.

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Phase I: before we start our ascent, look back on history and experiment with existing codes. . .

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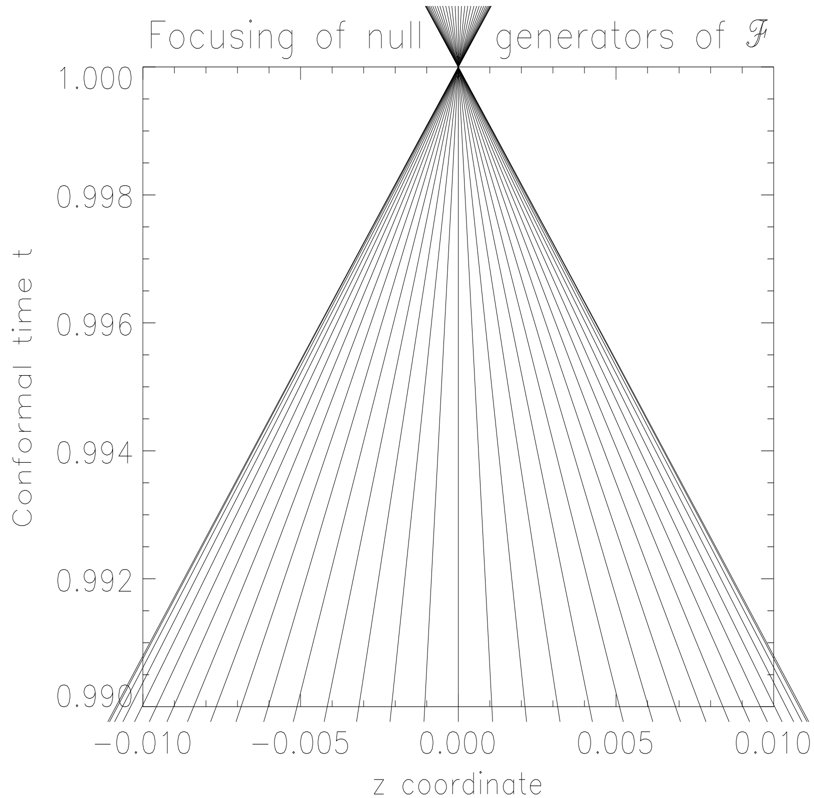
$\sim$ 1.5 yrs. have been spent studying earlier work of Hübner, Frauendiener, Weaver, Siebel ($\rightarrow$ SH, gr-qc/0204057 [LNP 617], gr-qc/0204043 [LNP 604])

To infinity and beyond . . .



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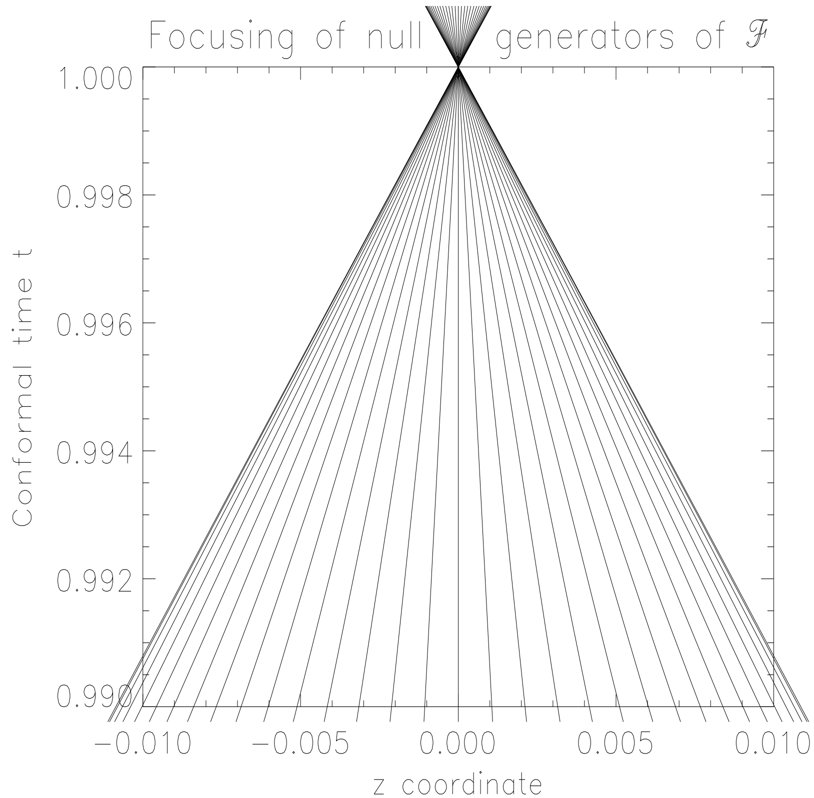

Weak data evolve into regular i^+ – resolved as one grid cell!

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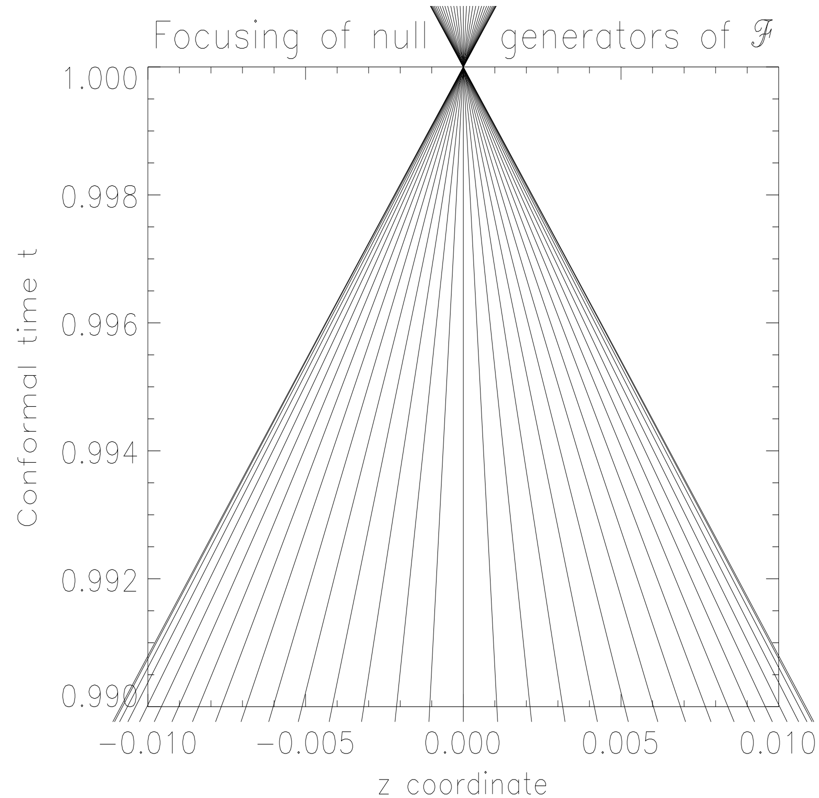

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BUT: $\Delta t = 1$ can be a very long time, especially near the end . . .

The complete future of (the physical part of) the initial slice can thus be reconstructed in a finite number of computational time steps!

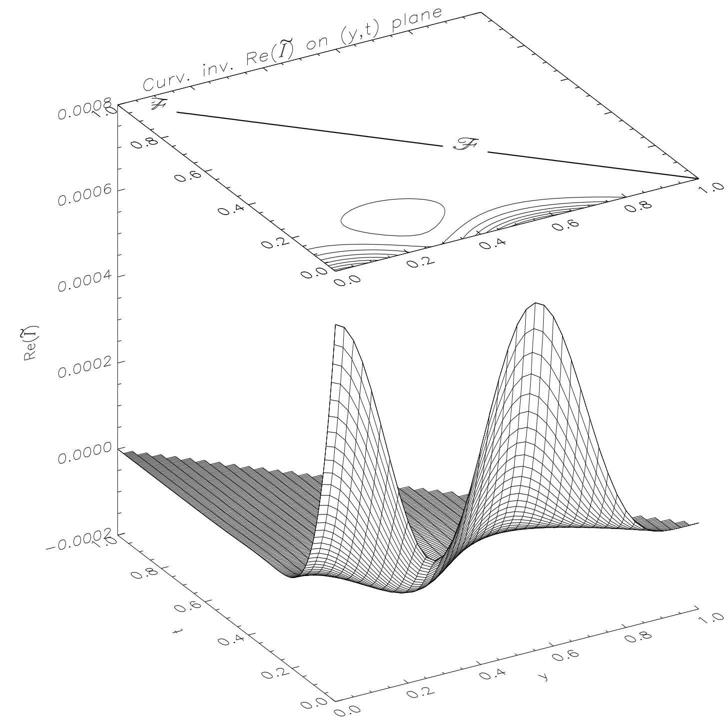


Figure 1: $\tilde{I} = \Omega^6 I$.

already nontrivial: Minkowski evolutions

3 standard ways of compactifying Minkowski:

1. Pseudostatic A (Minkowski → Minkowski)

$$ds^2 = -dt^2 + d\Sigma^2_{R^3} = \Omega^2(-dT^2 + dR^2 + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

$$\Omega = \left(R^2 - T^2\right)^{-1} = \left(r^2 - t^2\right), \tag{3}$$

where

$$r = \frac{R}{R^2 - T^2}, \qquad t = \frac{T}{R^2 - T^2}.$$

2. Pseudostatic B (textbook) map into part of Einstein static universe ($R_g = 6$),

$$ds^2 = -dt^2 + d\Sigma^2_{S^3} = \Omega^2\left(-dT^2 + dR^2 + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right), \tag{4}$$

$$\Omega^2 = 4\,(1+(T-R)^2)^{-1}\,(1+(T+R)^2)^{-1} = 4\cos^2\frac{t-\rho}{2}\cos^2\frac{t+\rho}{2}.$$

Here the coordinate transformations are

$$\rho = \arctan(T+R) - \arctan(T-R), \tag{5}$$

$$t = \arctan(T+R) + \arctan(T-R). \tag{6}$$

3. Static

$$ds^2 = -\Omega^2 dt^2 - 2rdrdt + dr^2 + r^2 d\Omega^2$$

$$\Omega = \frac{1-r^2}{2}, \quad R = 12\frac{(1-r^2)(3+r^2)}{(1+r^2)^3}, \quad trK = 3.$$

stable?

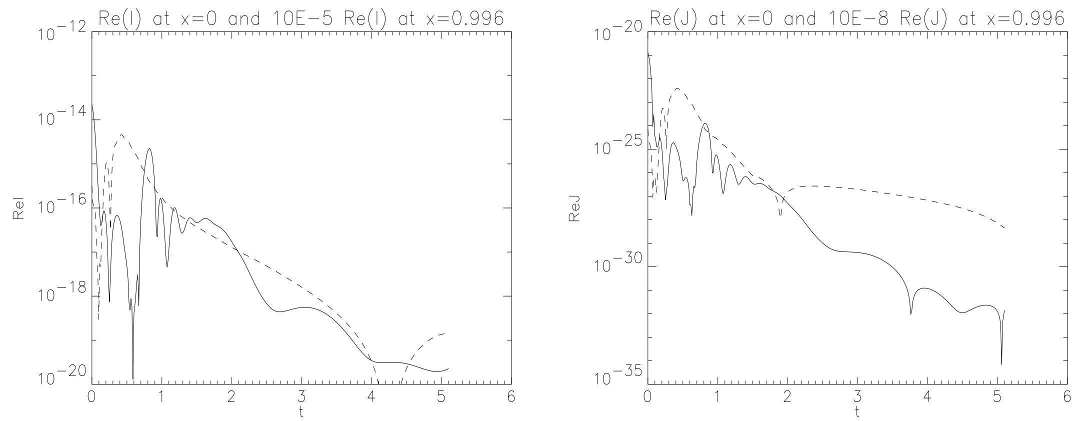
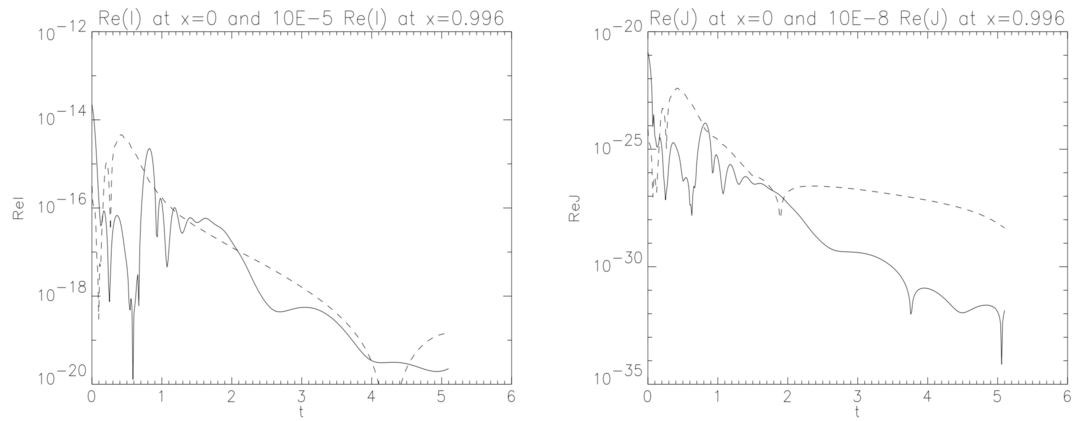


Figure 2: gridpoint at center, grid point at $x = 0.996$ (dashed).



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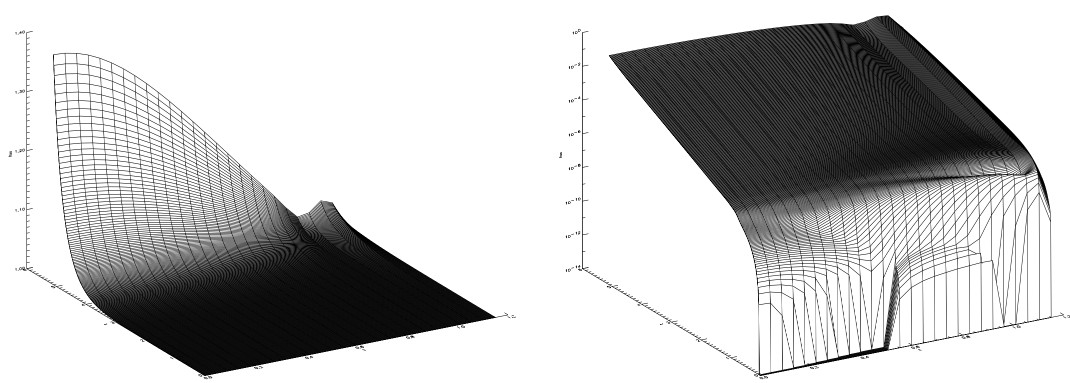

Figure 3: h_{xx} for $x \geq 0$ vs. t with linear and logarithmic scaling.

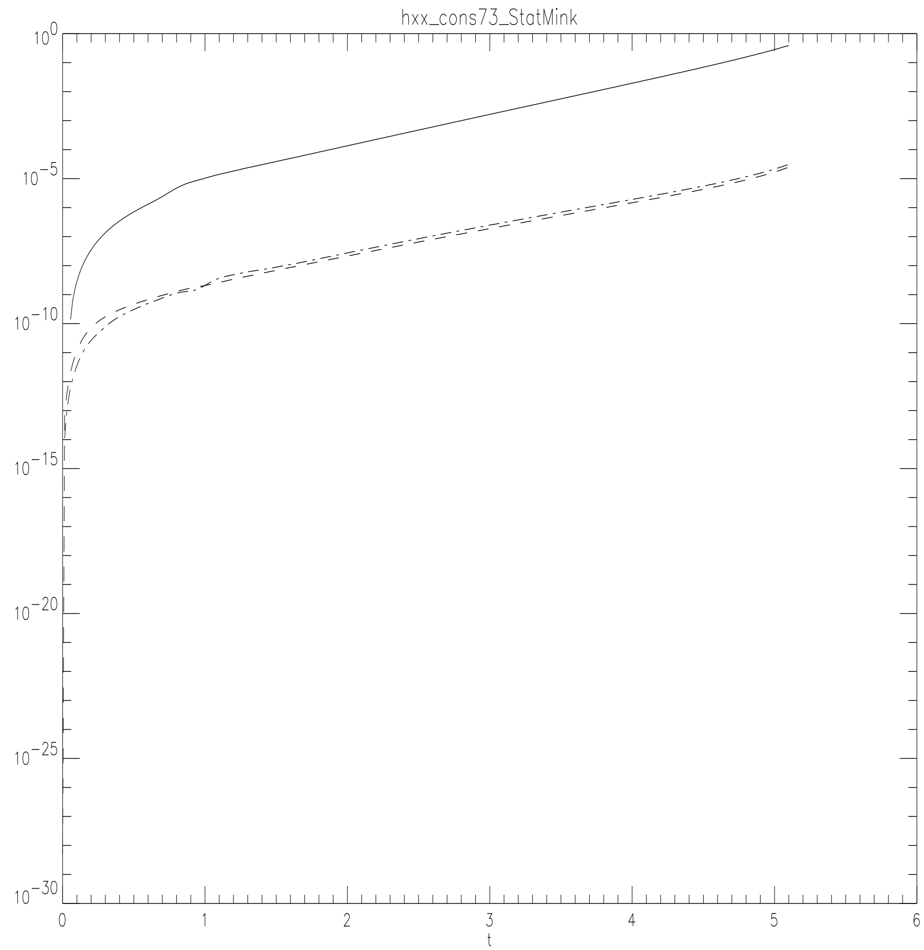

Figure 4: h_{xx} (unbroken) and constraints ${}^{(3)}\nabla_x h_{xx}$ & ${}^{(3)}\nabla_x \Omega = \Omega_x$.

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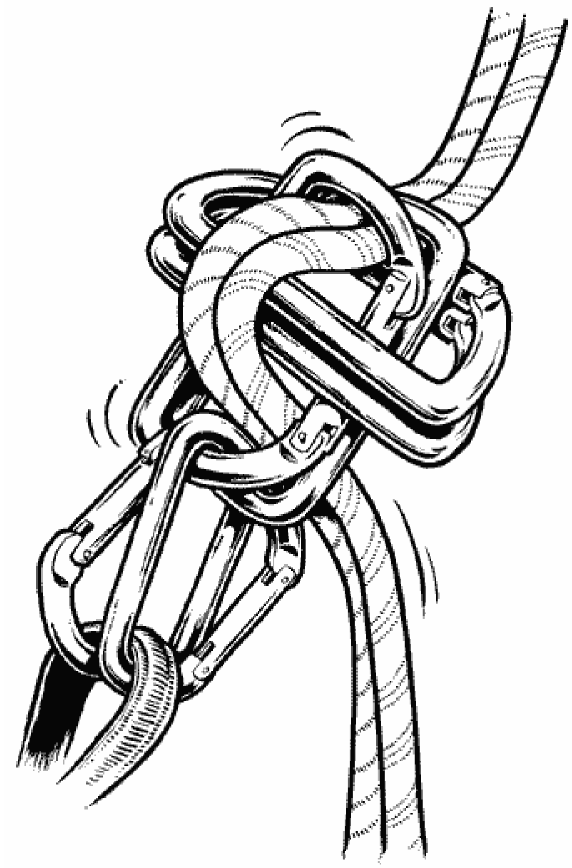
- we need a cheap, clean and flexible code $\longrightarrow$ Cactus

interlude: gear talk . . .



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stress abstract point of view – focus on algorithms!

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- templates for 3+1 splits – define spatial objects, declare names of metric etc., and corresponding rules (EM, ADM, CFE, wave eq., . . .)

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- generic package to aid in generation of Cactus Thorns (I. Hinder)

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- pattern matching for mathematical expressions is a powerful tool!

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- monitor **all** constraints

back to our project . . .

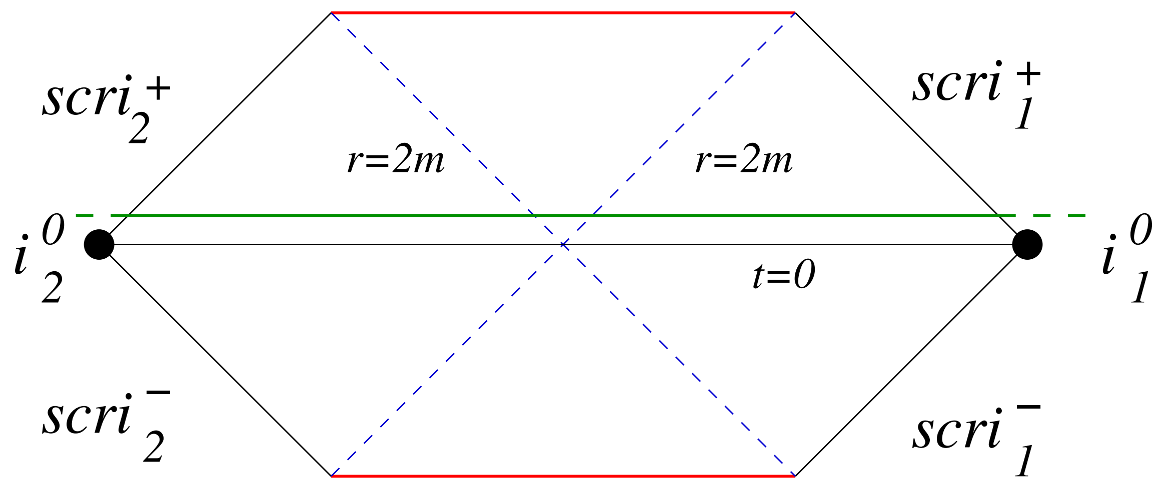
The "traditional" paradigm

Based on extended hyperboloidal intial value problem + compactification in time.

$\mathscr{J}+$ moves, typically contracts.

Constraints are violated outside $\mathscr{J}+$, "spillover" hoped to converge away.

Aim at global structure, no excision.



The “new” paradigm

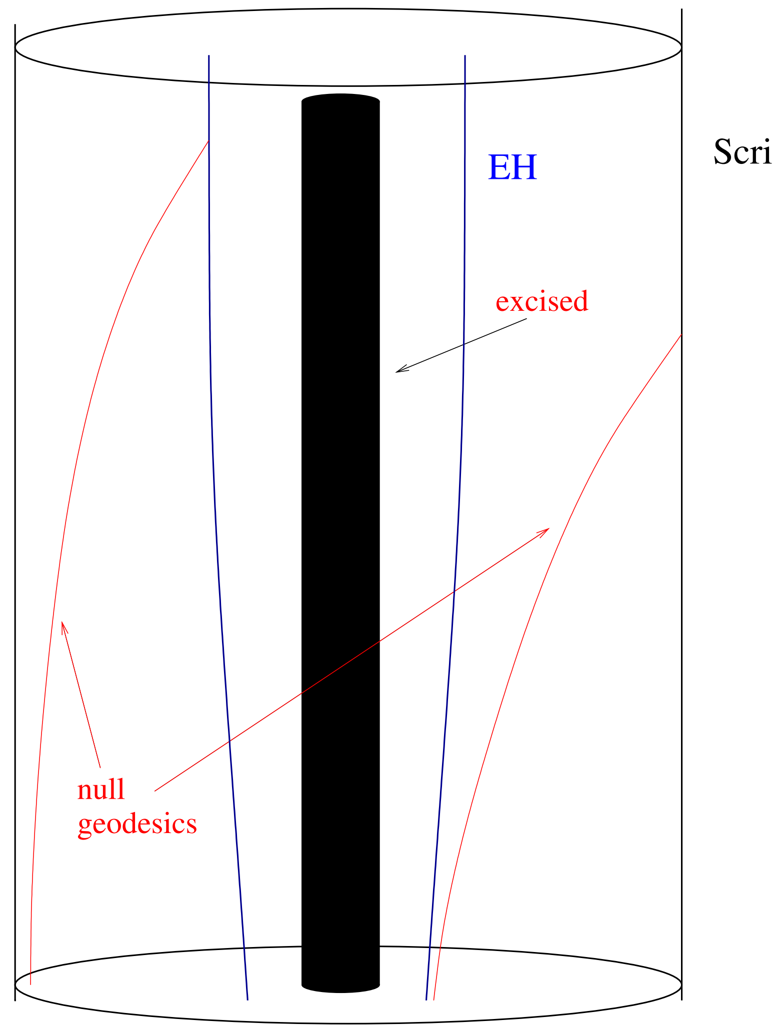
Focus on “astrophysical scenarios” – do not compactify in time.

Avoid spacetime regions of uncontrollable constraint violation: $\mathscr{I}$ is the limit – requires spherical boundary!

$\mathscr{I}$-fixing shift must be made compatible with well–posedness.

Can we come close to a Bondi-gauge?

Coordinate gauges might mimick uncompactified case – how can we handle conformal gauge?



Where are we now in the conformal approach?

We have seen some of the upcoming problems, we have chosen a route (roughly), we have our gear in place $\longrightarrow$ We have built up base-camp!

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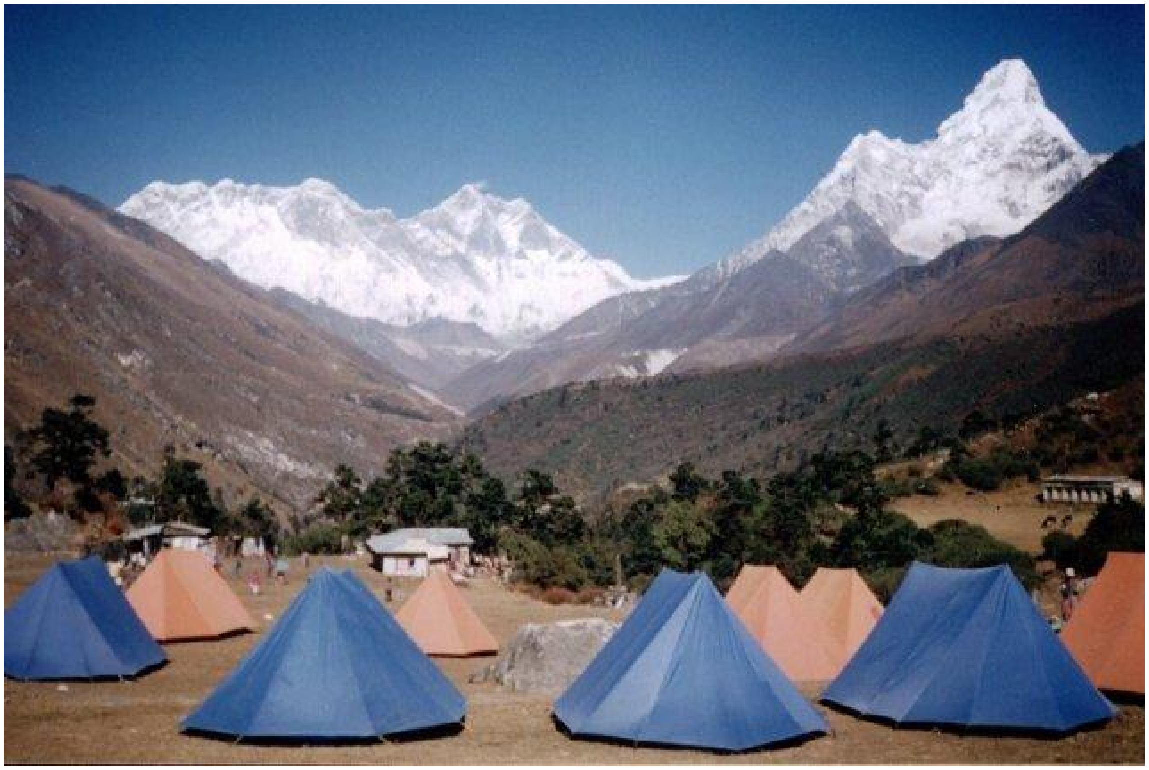

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outlook: the route

. . . organized in camps, according to standard big mountain climbing strategy, we go back and forth between camps regularly . . .

Camp I – periodic boundary conditions in 3D

For the moment: Focus on periodic boundary conditions to get a clean problem.

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- compare different formulations
- get a feeling for $\Omega \equiv 1$–case
- experiment with gauge conditions
- Mexico tests as essential health checks

  There are number of interesting tests to be performed with periodic boundaries beyond Mexico I!

Camp II – 1D with boundary

- Test gauges and formulations!
- Schwarzschild? – Static representation of Minkowski?
- Can instabilities be understood mathematically?
- Understand solution of the constraints at least in this simple case!
- Aim: stable evolution of Schwarzschild!

Camp III – 2D

- Obtain a large class of initial data!

  Assume $\tilde{k}_{ab} = \frac{1}{3}\,\tilde{k}\,\tilde{h}_{ab}$

$$\tilde{h}_{ab} = \phi^4 \left(\bar{\Omega}^{-2} h_{ab}\right) \quad \to \quad \tilde{R}(\tilde{h}) = \tilde{k}_{ab}\tilde{k}^{ab} - \tilde{k}^2,$$

  $\Rightarrow$ "elliptic" equation – principal part vanishes @ $\mathcal{S}$
  $\Rightarrow$ boundary values fixed!

$$\Omega^2 \,\triangle\, \phi + \cdots = 0.$$

- Experiment with evolution inside spherical boundary?

Camp III – toy models in 3D with boundary

Proceed the natural way: Wave equation, Maxwell, linearized Einstein

2 Elements to be tested:

1. spherical boundary

2. boundary at future null infinity

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no boundary conditions needed/allowed execpt potentially for gauge, but gauge is tricky & probably need to feed in info from constraint propagation along boundary?

Camp IV – GR with boundary at $\mathscr{I}$

Implement a generic initial data solver that *works* for hyperboloidal slices.

Which issues will arise from combining the machinery needed to deal with spherical boudnaries with the full nonlinear theory?

Camp V – improve stability

The standard tricks apply, in addition there is extra gauge freedom, e.g. trK is completely free!

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