Black Holes in a Compactified Spacetime

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Talk Outline:

- Weyl metric and Schwarzschild solution in 4D
- Schwarzschild BH in compactified spacetime
- Solution, asymptotics and horizon properties

4D Weyl Black Hole

- Weyl coordinates:

$$dS^2 = L^2\,ds^2$$

$$ds^2 = -e^{2U}\,dt^2 + e^{-2U}\left[e^{2V}\left(d\rho^2 + dz^2\right) + \rho^2\,d\phi^2\right]$$

- Vacuum Einstein equations:

$$\frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{\partial^2 U}{\partial z^2} = 0$$

$$V_{,\rho} = \rho\,(U_{,\rho}^2 - U_{,z}^2), \quad V_{,z} = 2\rho\,U_{,\rho}\,U_{,z}$$

- Source for a Schwarzschild BH is a thin rod:

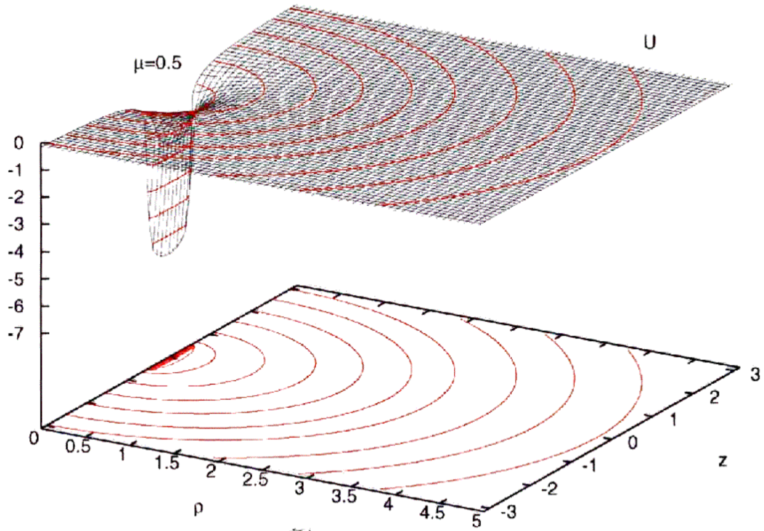
$$\Delta U = \frac{\delta(\rho)}{\rho}\,\Theta(z/\mu), \qquad \Theta(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

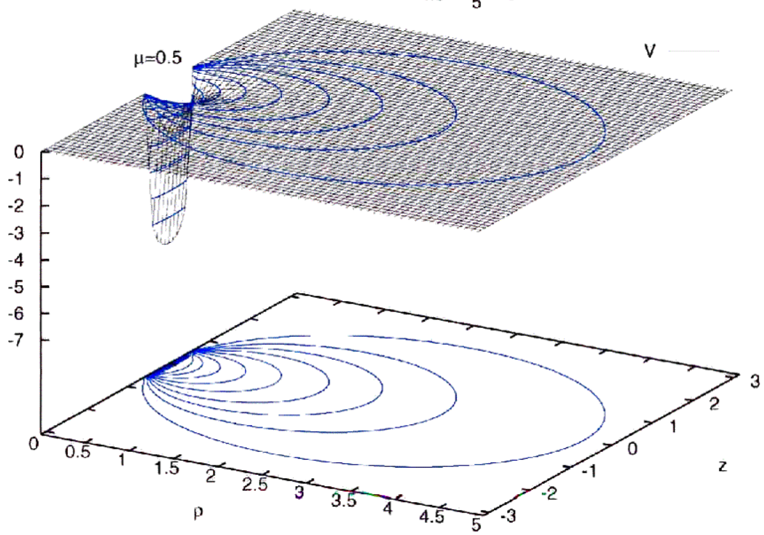
- Schwarzschild solution:

$$U_S(\rho, z) = \frac{1}{2}\ln\left(\frac{\ell - \mu}{\ell + \mu}\right), \quad V_S(\rho, z) = \frac{1}{2}\ln\left(\frac{\ell^2 - \mu^2}{\ell^2 - \eta^2}\right)$$

$$\ell = \frac{1}{2}(\ell_+ + \ell_-), \quad \eta = \frac{1}{2}(\ell_+ - \ell_-), \quad \ell_\pm = \sqrt{\rho^2 + (z \pm \mu)^2}$$

Schwarzschild Solution





Distorted Black Holes

Geroch & Hartle (1981)

- A distorted black hole is described by a static axisymmetric Weyl metric with a regular Killing horizon. One can write the solution (U, V) for a distorted black hole as

$$U = U_S + \hat{U}\,, \quad V = V_S + \hat{V}\,,$$

where (U_S, V_S) is the Schwarzschild solution with mass $M = \mu L$

- Both V and V_S vanish at the axis $\rho = 0$ outside the horizon, hence the function $\hat{V}$ has the same property
- The function $\hat{U}$ obeys the homogeneous 3D Laplace equation
- One of the equations for $\hat{V}$ is of the form

$$\hat{V}_{,z} = 2\rho\left(U_{S,\rho}\hat{U}_{,z} + U_{S,z}\hat{U}_{,\rho} + \hat{U}_{,\rho}\hat{U}_{,z}\right)$$

- Near the horizon $\hat{U}$ is regular, while $U_{S,\rho} = O(\rho^{-1})$ and $U_{S,z} = O(1)$. Thus near the horizon $\hat{V}_{,z} \sim 2\hat{U}_{,z}$

- Integrating this relation along the horizon from $z=-\mu$ to $z=\mu$ and using the relations $\hat{V}(0,-\mu)=\hat{V}(0,\mu)=0$, we obtain that $\hat{U}$ has the same value u at both ends of the line segment H

- By integrating the same equation along the segment H from the end point to an arbitrary point of H one obtains for $-\mu \le z \le \mu$

$$\hat{V}(0,z)=2\left[\hat{U}(0,z)-u\right]$$

- Geroch and Hartle demonstrated that if $\hat{U}$ is a regular smooth solution in any small open neighborhood of H (including H itself) which takes the same values, u, on the both ends of the segment H, then the solution is regular at the horizon and describes a distorted black hole

- We use the coordinate transformation

$$\rho = e^{u}\sqrt{r(r-2\mu_0)}\,\sin\theta\,,$$

$$z=e^{u}\,(r-\mu_0)\,\cos\theta\,,$$

and we define $\mu_0=\mu\, e^{-u}$

- The metric of a distorted black hole takes the form

$$ds^2=-e^{-2\hat{U}}\left(1-\frac{2\mu_0}{r}\right)dt^2$$

$$+e^{2(\hat{V}-\hat{U}+u)}\left(1-\frac{2\mu_0}{r}\right)^{-1}dr^2$$

$$+e^{2(\hat{V}-\hat{U}+u)}\,r^2\left(d\theta^2+e^{-2\hat{V}}\sin^2\theta\,d\phi^2\right).$$

- The event horizon is described by the equation $r=2\mu_0$, and the 2-dimensional metric on its surface is

$$d\gamma^2=4\mu_0^2\left[e^{2(\hat{U}-u)}\,d\theta^2+e^{-2(\hat{U}-u)}\sin^2\theta\,d\phi^2\right].$$

It is a sphere S^2 deformed in an axisymmetric manner

- The horizon surface has area

$$A=16\pi\mu_0^2L^2\,.$$

- The surface gravity κ is constant over the horizon surface:

$$\kappa=\frac{e^{u}}{4\mu_0}L^{-1}\,.$$

Compactified Schwarzschild Black Hole

Israel & Khan (1964); Myers (1987); Korotkin & Nicolai (1994)

- Periodic solution! Use Fourier decomposition:

$$U(\rho, z) = U_0(\rho) + \sum_{k=1}^{\infty} U_k(\rho)\, \cos(kz)$$

- Source term for a compactified BH:

$$\Theta(z/\mu) = a_0 + \sum_{k=1}^{\infty} a_k\, \cos(kz)$$

$$a_0 = \frac{\mu}{\pi}\,, \qquad a_k = \frac{2}{\pi k}\sin(k\mu)$$

- Radial mode equation:

$$\frac{d^2U_k}{d\rho^2} + \frac{1}{\rho}\frac{dU_k}{d\rho} - k^2\, U_k = a_k\, \frac{\delta(\rho)}{\rho}$$

- Radial mode functions:

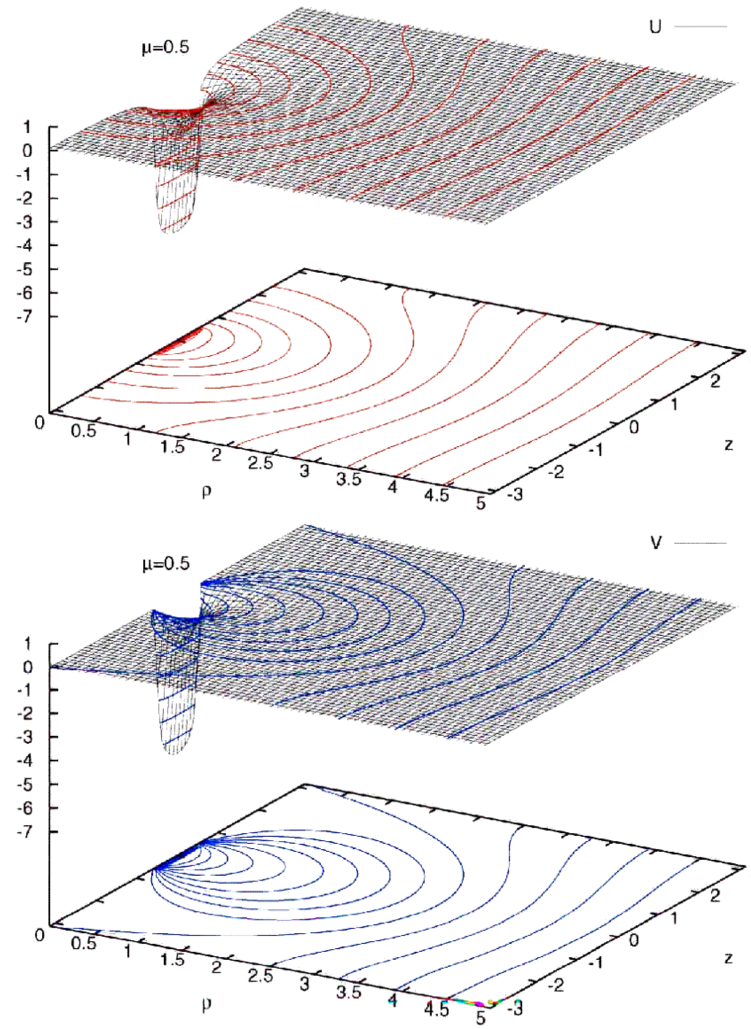
$$U_0(\rho) = a_0 \ln(\rho), \quad U_k(\rho) = -a_k K_0(k\rho)$$

- Compactified Schwarzschild solution:

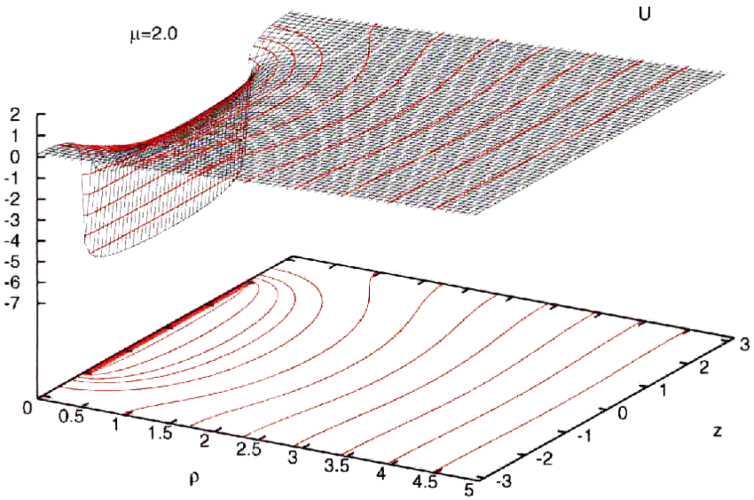
$$U(\rho, z) = \frac{\mu}{\pi}\ln\rho - 2\sum_{k=1}^{\infty}\frac{\sin(k\mu)}{\pi k}\cos(kz)\, K_0(k\rho)$$

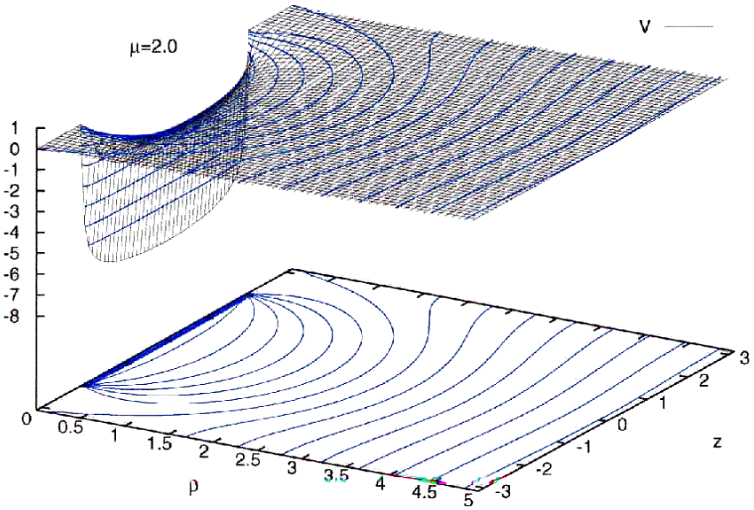
Black Holes in a Compactified Spacetime

Compactified Schwarzschild Solution (I)



Compactified Schwarzschild Solution (II)





Black Holes in a Compactified Spacetime

Far-Away Asymptotic

- Solution far away:

$$U(\rho,z)|_{\rho\to\infty} \approx \frac{\mu}{\pi}\ln\rho, \quad V(\rho,z)|_{\rho\to\infty} \approx \frac{\mu^2}{\pi^2}\ln\rho + v(\mu)$$

For $\mu \ll 1$, $v \sim \mu^2$.

- Asymptotic metric is Kasner: [Rindler if $\mu = \pi$]

$$ds^2 = -\rho^{2\frac{\mu}{\pi}}\,dt^2 + e^{2v}\rho^{-2\frac{\mu}{\pi}(1-\frac{\mu}{\pi})}\,(d\rho^2 + dz^2) + \rho^{-2\frac{\mu}{\pi}}\,\rho^2\,d\phi^2$$

$\Rightarrow$ For small μ, 2μ is an angle deficit at $\rho=\infty$ and

- Asymptotic (Komar) mass: $m = \frac{1}{4\pi}\int \xi^{\mu;\nu}_{(t)}\,d\sigma_{\mu\nu} = \mu$

Near-Horizon Asymptotic

- Solution near horizon: $\hat{U}(z) = \lim_{\rho\to 0}[U(\rho,z) - U_S(\rho,z)]$

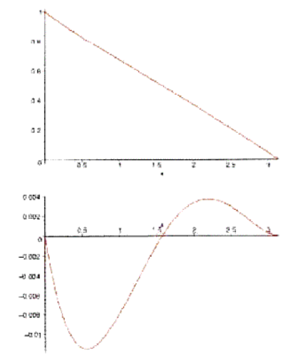
$$\hat{U}(z) = \frac{\mu}{\pi}\ln(4\pi) + \frac{1}{2}\ln\left[f(\tfrac{\mu+z}{2})f(\tfrac{\mu-z}{2})\right], \quad |z| \le \mu$$

- Function $f(x)$:

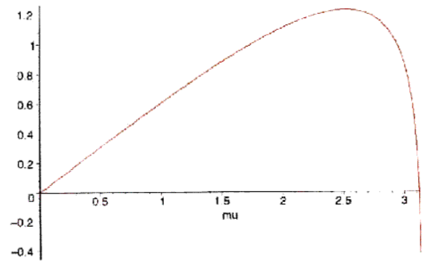
$$f(x) = \frac{x}{\pi^2}\sin x\,\Gamma^2(\tfrac{x}{\pi})$$

$$f(0) = 1, \quad f(\tfrac{\pi}{2}) = \tfrac{1}{2}, \quad f(\pi) = 0$$

$$f(x) \approx 1 - \frac{x}{\pi}$$



Redshift Factor



- Redshift factor:

$$u = \frac{\mu}{\pi}\ln(4\pi) + \frac{1}{2}\ln f(\mu)$$

$$u \approx \frac{\mu}{\pi}\ln(4\pi) + \frac{1}{2}\ln\left(1 - \frac{\mu}{\pi}\right)$$

- The redshift factor u has maximum u_*

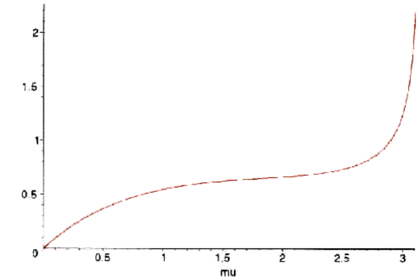
$$u_* = \ln(4\pi) - \frac{1}{2}\left[1 + \ln 2 + \ln(\ln(4\pi))\right] \approx 1.22$$

at

$$\mu_* = \pi(1 - 1/(2\ln(4\pi))) \approx 2.52\,.$$

- For $\mu > \mu_*$ the function u rapidly falls down, becoming negative and logarithmically divergent at $\mu = \pi$

Properties of the Horizon



- The irreducible mass μ_0 and the surface gravity κ in the same approximation are

$$\mu_0 = \mu\,\exp(-u) \approx \mu\,(4\pi)^{-\frac{\mu}{\pi}}\left(1 - \frac{\mu}{\pi}\right)^{-\frac{1}{2}}$$

$$\kappa = \frac{e^{-2u}}{4\mu} \approx \frac{1}{4\mu}\,(2\pi)^{2\frac{\mu}{\pi}}\left(1 - \frac{\mu}{\pi}\right)$$

- For $\mu \to \pi$, they behave as $\mu_0 \to \infty$ and $\kappa \to 0$

- Another invariant is the proper distance l between the 'north pole', $z = \mu$, and 'south pole', $z = -\mu$

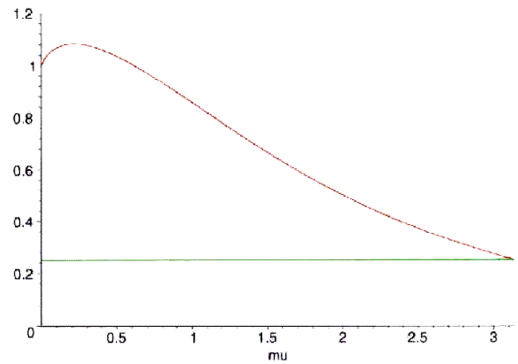
$$l(\mu) = 2\int_{\mu}^{\pi} dz\, e^{-U(0,z)}$$

Properties of the Horizon

$$l \approx 2\sqrt{\pi^2 - \mu^2}\, E(\varphi, k) \; + 2\mu\sqrt{\tfrac{\pi+\mu}{\pi-\mu}}\, F(\varphi, k) \; - (\pi - \mu)\,,$$

$$\varphi = \sqrt{1 - \mu/\pi}\,, \quad k = \frac{1}{\sqrt{1 - (\mu/\pi)^2}}$$

- In particular one has $l(0) = 2\pi\,, \quad l(\pi) = \pi/2$
- In the limit $\mu \to \pi$, when the coordinate distance Δz between the poles tends to 0, the proper distance between them remains finite. This happens because in the same limit the surface gravity tends to 0



$l/(2\pi)$ as a function of μ

Properties of Event Horizon

- The metric $d\sigma^2$ (of total area 4π) is

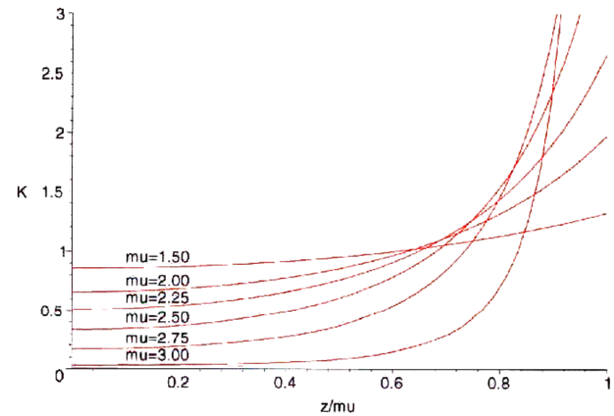
$$d\sigma^2 = F(z)\, dz^2 + \frac{d\phi^2}{\mu^2\, F(z)}$$

$$F(z) \approx \frac{1}{\mu^2 - z^2} + \frac{1}{4\pi^2(1 - \mu/\pi)}$$

- The Gaussian curvature is

$$K \approx \frac{16\pi^2(\pi - \mu)^2[(2\pi - \mu)^2 + 3z^2]}{[(2\pi - \mu)^2 - z^2]^3}$$

- The Gaussian curvature is positive in the interval $|z| < \mu$



Properties of Event Horizon

- Redshift factor:

$$u \equiv \hat{U}(\mu) = \frac{\mu}{\pi}\ln(4\pi) + \frac{1}{2}\ln f(\mu)$$

- Irreducible mass: $[A = 16\pi\mu_0^2]$

$$\mu_0 = \mu\, e^{-u} \approx \mu\,(4\pi)^{-\frac{\mu}{\pi}}\left(1-\frac{\mu}{\pi}\right)^{-\frac{1}{2}}$$

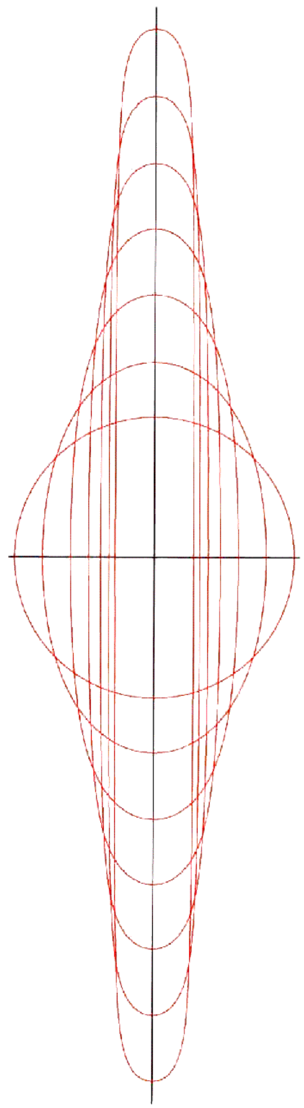
- Surface gravity:

$$\kappa = \frac{e^{-2u}}{4\mu} \approx \frac{1}{4\mu}(2\pi)^{2\frac{\mu}{\pi}}\left(1-\frac{\mu}{\pi}\right)$$

- Embedding

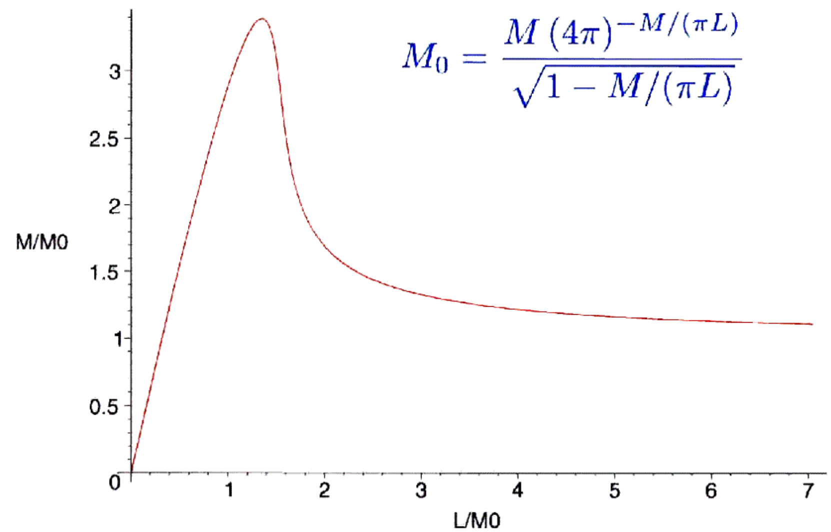
$$dl^2 = dh^2 + dr^2 + r^2\, d\phi^2$$

$$\frac{dh}{dz} = \sqrt{F - \frac{F'^2}{4F^3}}$$



Is BH in Compactified Spacetime Stable?

- Quasi-static adiabatic process: $[A \propto M_0^2$ fixed$]$



$$M_0 = \frac{M\,(4\pi)^{-M/(\pi L)}}{\sqrt{1 - M/(\pi L)}}$$

Summary

- Black hole in a compactified spacetime:
  - Exact 4D solutions found in Weyl coordinates
  - BH horizon is deformed due to self-attraction
  - Simple approximation to find size and shape
  - Nice solvable toy problem, but what about 5D?