Discrete Differential Forms in Numerical General Relativity

- Motivation
- Discrete differential forms
- Einstein equations as differential ideal
- Discrete formulation
- Outlook

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SFB 382: Methods and Algorithms to simulate physical processes on high performance computers

Motivation

Numerical relativity is based essentially on the following procedure:

- Set up geometric differential equations
- Split into evolution equations and constraints
- Verify that constraints propagate

- Discretise evolution equations and constraints
- Solve constraints to provide initial data
- Choose gauges (coordinates, etc.)
- Evolve
- Check constraints to control the quality of solution
- Extract physical (invariant) information

Problems

- discretisation after split
- independent discretisation of evolution equations and constraints
- discrete versions are in general not compatible
- discrete constraints are not propagated by the discrete evolution
  $\rightarrow$ severe violation of constraints during simulations

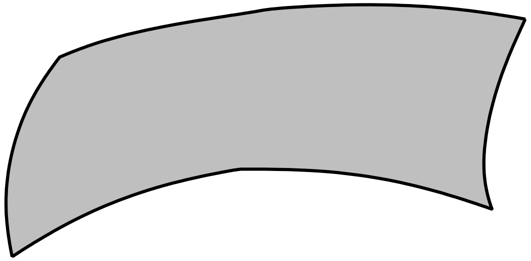
- Einstein equations are invariant under diffeomorphisms
  $\rightarrow$ simulations are coordinate dependent
- invariant information has to be determined after the simulation

- geometric character (vector, tensor) of the variables plays no role

- finite element methods are largely ignored

Discrete Differential Forms

continuous



p-dimensional submanifold S_p:

(0) point, (1) curve, (2) surface

p-form:

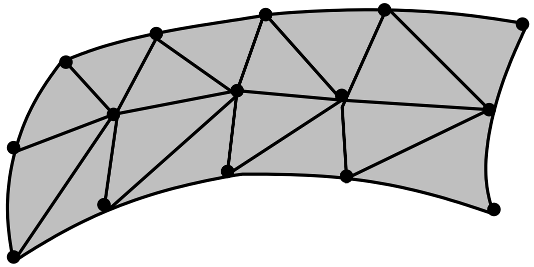
$$\omega : S_p \mapsto \int_{S_p} \omega \in \mathbb{R}$$

exterior derivative $\mathbf{d}$:

$$\int_{S_p} \mathbf{d}\omega = \int_{\partial S_p} \omega$$

Stokes' theorem

discrete



p-simplices $\mathfrak{S}_p$:

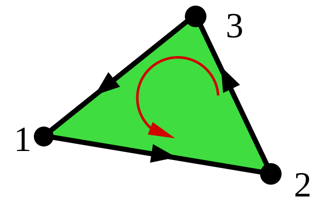
(0) node, (1) edge, (2) face

discrete p-form:

$$\omega : \mathfrak{S}_p \mapsto \omega[\mathfrak{S}_p] \in \mathbb{R}$$

Definition:

$$\mathbf{d}\omega[\mathfrak{S}_p] = \omega[\partial \mathfrak{S}_p]$$



Example:

$$\mathbf{d}\omega_{123} = \omega_{12} + \omega_{23} + \omega_{31}$$

continuous

Grassmann (wedge) product:

$$(\overset{p}{\alpha},\overset{q}{\beta}) \mapsto \overset{p+q}{\alpha\wedge\beta},$$

graded algebra

$$\alpha\wedge\beta = (-1)^{pq}\beta\wedge\alpha,$$

derivation:

$$\mathbf{d}(\alpha\wedge\beta) = \mathbf{d}\alpha\wedge\beta + (-1)^{p}\alpha\wedge\mathbf{d}\beta.$$

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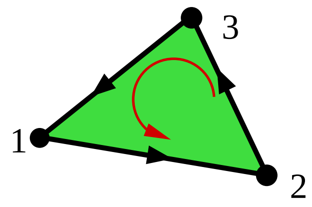
deRham cohomology

discrete

discrete Grassmann product:

$$(\overset{p}{\alpha},\overset{q}{\beta}) \mapsto \overset{p+q}{\alpha\wedge\beta}$$



Example:

$$(\overset{1}{\alpha}\wedge\overset{1}{\beta})_{123} = \frac{1}{2}\Big[\alpha_{12}\beta_{13}+\alpha_{23}\beta_{21}+\alpha_{31}\beta_{32} - \beta_{12}\alpha_{13}-\beta_{23}\alpha_{21}-\beta_{31}\alpha_{32}\Big]$$

discrete **d** is derivation

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singular cohomology

Einstein equation as differential ideal

Variables:

- (covariant) tetrad: θ^i
- $so(1,3)$ connection form: $\omega^i{}_k$

In addition:

- space-time 'metric': $\eta_{ik} = (+,-,-,-)$

Cartan's structure equation:

$$\mathbf{d}\theta^i + \omega^i{}_k \wedge \theta^k = 0,$$
$$\mathbf{d}\omega^i{}_k + \omega^i{}_l \wedge \omega^l{}_k = \Omega^i{}_k$$

Bianchi identity:

$$\mathbf{d}\Omega^i{}_k + \omega^i{}_l \wedge \Omega^l{}_k - \omega^l{}_k \wedge \Omega^i{}_l = 0.$$

Nester-Witten form: $$L_i = \frac{1}{2}\varepsilon_{ijkl}\omega^{jk}\wedge\theta^l$$

identity: $$\mathbf{d}L_i = \underbrace{S_i}_{\sim\omega^2} + \underbrace{E_i}_{\sim G_{ab}}$$

Sparling: $$\mathbf{d}S_i = 0 \iff G_{ab} = 0.$$

exterior system for the variables θ^i, $\omega^i{}_k$:

$$\boxed{\begin{aligned} \mathbf{d}\theta^i + \omega^i{}_k \wedge \theta^k &= 0, \\ \mathbf{d}L_i - S_i &= 0. \end{aligned}}$$

(2-form)

(3-form)

gauge freedom: Lorentz rotations of the tetrad

Applications of the exterior system

- Einstein’s energy balance
- Landau-Lifshitz and Einstein pseudo-tensor
- Bondi mass loss, light focussing
- Positive mass theorem, Penrose inequality

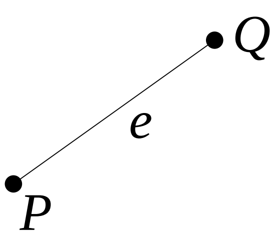
Discrete formulation

1. Choose the topology of the time slices
2. Triangulate with 4-simplices
3. Replace continuous by discrete forms

Discrete variables:

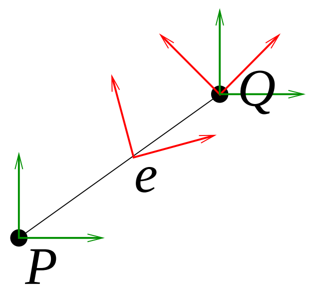
values of θ^i and $\omega^i{}_k$ on **edges**: $\theta^i[e]$, $\omega^i{}_k[e]$

Geometric meaning:



$$l[e]^2 = \eta_{ik}\theta^i[e]\theta^k[e]$$

squared length

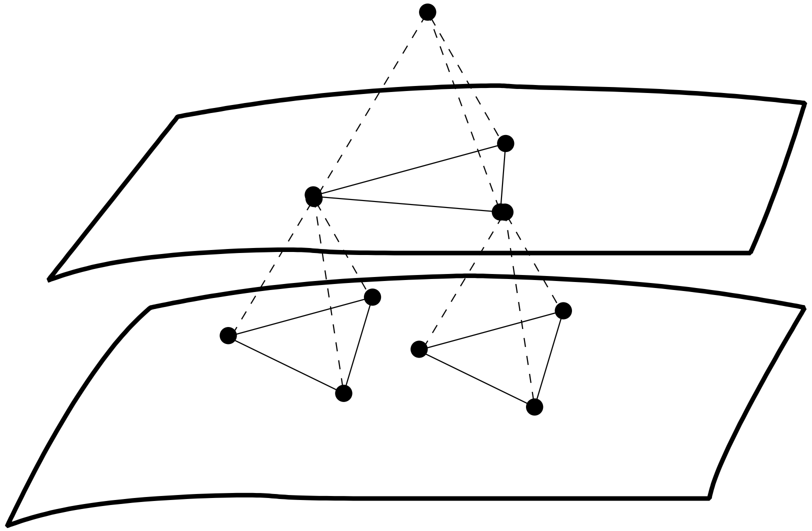


$$R^i{}_k(e) = \exp(\omega^i{}_k[e])$$

holonomy

- Choose gauge: Lorentz-rotations of the tetrad
- Evaluate the discrete forms on 2- resp. 3-simplices
  $\rightarrow$ algebraic, non-linear system for $\{\theta^i[e], \omega^i{}_k[e]\}$
- Split into evolution equations and constraints
  determined by the causal character of the simplices
- Bianchi identity is satisfied also for discrete formulation
  $\rightarrow$ essential for consistency of the equations
- squared length of an edge e, holonomy along an edge e:
  coordinate independent description of space-time

Stepping in time



- Triangulation consists of tetrahedra
- Each tetrahedron determines a unique point in the future
  'dual' triangulation in the next time slice, staggering
- edges will connect null separated points
- CFL condition built in
- can be used to fix the Lorentz gauge

Outlook

- Implementation in simple cases ($1+1$-systems)
  - spherical symmetry
  - pp-waves
- Investigation of the properties of the equations
  - propagation
  - hyperbolic character
- tetrad gauge?
- boundary conditions?
- methods of solution?