# Approximation schemes for computing Gravitational Radiation

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## *Outline*

I – **General features** of approximation method, Various method found in the literature.

II – from Post-Newtonian and Post-Minkowskian, how do we get radiation reaction ?

III – THE current HOT subject in the field (computation of phase at 3pN, i.e. 1/c^6)

# *Part 1- General Features*

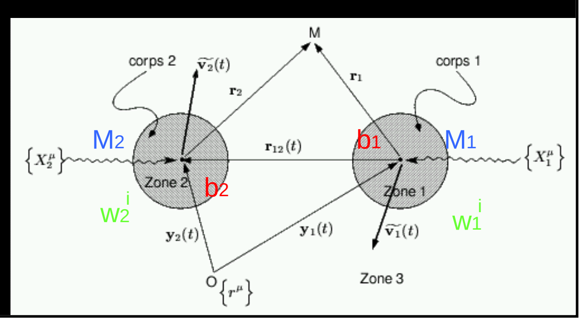
## *General features of approximation scheme.*

- System: **binary** NS, BH
- We look for **dimensionless small** parameters

$\frac{b}{r_{12}}$ e **: Geometric parameters**

$\frac{v^2}{c^2}$ $\frac{w^2}{c^2}$ **: Slow motion**

$\frac{GM}{c^2 r_{12}}$ $\frac{GM}{c^2 b}$ **: Weak field**



## What is a good analytical scheme ?

- Amenable to all orders
- Convergence
  - •Exact solution
  - •Another solution
  - •Asymptotical
- How fast does it converge ? Rapidly …

Method to **accelerate** convergence
- One body approach
- Pade approximant

## Difficulties for approximation scheme in GR.

- Non linearity
- Non local (history of the source)
- Coordinate system, reference frame (deformation of object)
- Boundary conditions

**Increasing number of terms**:

$$E_{3PN} = (m_1+m_2)c^2 + \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{Gm_1 m_2}{r_{12}} + \underbrace{\frac{E_2}{c^2}}_{8\ \text{termes}} + \underbrace{\frac{E_4}{c^4}}_{32\ \text{termes}} + \underbrace{\frac{1}{c^6}\left\{\cdots + \frac{G^4 m_1^3 m_2^2}{r_{12}^4}\left(\frac{5809}{280} - \frac{11}{3}\lambda - \frac{22}{3}\ln\left(\frac{r_{12}}{r'_1}\right)\right) + \cdots\right\}}_{130\ \text{termes}}.$$

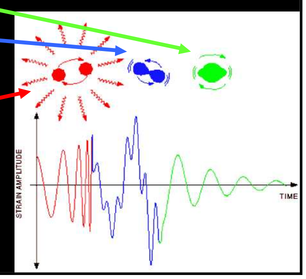
## *Many analytic approximation scheme.*

- Test particle (detector and orbit)
- Linearized gravity (GW in the far zone)

- **Kerr perturbation**

**Numerical**

- **Post-test** or « a la Futamase » (adapted to strong field)
- **Post-Minkowskian** (G)
- **Post-Newtonian** (1/c)




## *Part 2- PN and PM expansions*

## PN and PM expansions.

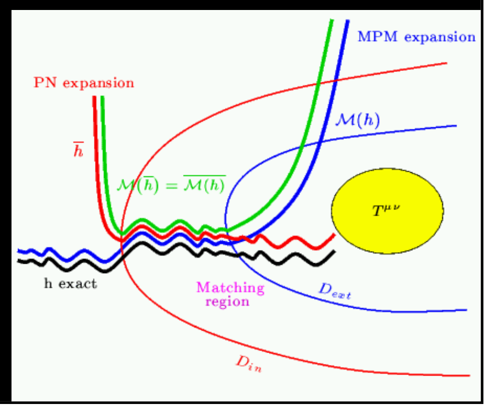
**h** is the **deviation** of the metric to the Minkowski metric.

**PN:** $$\overline{h}^{\mu\nu} = \sum_{n\geq 0} \frac{1}{c^n} \overline{h}^{\mu\nu}_n$$

**PM:** $$h^{\mu\nu} = \sum_{n\geq 0} G^n h^{\mu\nu}_{(n)}$$

- Valide in **different** domain

**Matching = Reaction**




## Mathematical validity of the post-Newtonian expansion

**Why such a question ? pN seems all right with exp.**

1- S. Chandrasekhar : 2pN equations of hydrodynamic.
2pN divergent integrals.
Kerlick solve the problem up to 2.5pN. Still a problem at 3pN

2- Imagine there is an order to which pN blows up:
To which experimental extent could we trust our theoretical results ?

## *Reduced Einstein equations in harmonic coordinates.*

•Metric deviation : our field $\mathfrak{g}^{\mu\nu} = \sqrt{g}\, g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{16\pi G}{c^4} T^{\mu\nu}$$

$\partial_\mu h^{\mu\nu} = 0$ Harmonic gauge condition

$$\overbrace{\left(\sum_{p\geq 2} \underbrace{\partial h \partial h\, h \cdots h}_{p\ terms}\right)^{\mu\nu}}^{\Lambda^{\mu\nu}}$$

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4} \, | -g \, | \, T^{\mu\nu} + \underbrace{\Lambda^{\mu\nu}}_{\sim LL\ p-tensor} = \frac{16\pi G}{c^4} \tau^{\mu\nu}$$

Reduced Einstein equation

## *Physical hypothesis.*

1- gravitational source is spatially compact, $C^\infty$
and it is not point-like (hydrodynamical fluid)

2- Stationary in the past $\forall t < -\mathcal{T},\ \frac{\partial}{\partial t}\left[h^{\mu\nu}(\mathbf{x},\, t)\right] = 0$

3- Asymptotically Minkowski $\lim_{r\to\infty} h^{\mu\nu}_{ext}(\mathbf{x}, t) = 0$

4- pN is valide if $\frac{1}{c}|\partial_t\, A(\mathbf{x},\, t)| \ll |\partial_k\, A(\mathbf{x},\, t)|$

$1/cT \ll 1/r$ → $r \ll \lambda$

•If « slow motion », λ is big.
•Source inside the near zone

pN is valide near the source

# Post-Newtonian equation (in the vicinity of the source).

$$\overline{h}^{\mu\nu} = \sum_{n\geq 0} \frac{1}{c^n} \underset{n}{\overline{h}^{\mu\nu}}$$

•pN expansion of the field (bar means pN)

$$\Delta \underset{n}{\overline{h}^{\mu\nu}} = 16\pi \mathrm{G}\, \underset{n-4}{\overline{\tau}^{\mu\nu}} + \partial_t^2 \underset{n-2}{\overline{h}^{\mu\nu}}$$

•From the relaxed Einstein equation

$$\underset{n}{\overline{h}^{\mu\nu}} = 16\pi \mathrm{G}\, \widetilde{\Delta^{-1}} \left[\underset{n-4}{\overline{\tau}^{\mu\nu}}\right] + \partial_t^2 \widetilde{\Delta^{-1}} \underset{n-2}{\overline{h}^{\mu\nu}} + \sum_{l\geq 0} \underset{n}{A}{}^{\mu\nu}_{L}(t)\, \widehat{x}_L$$

Most general homogeneous terms

Generalised poisson integral

$$\underset{n}{\overline{h}^{\mu\nu}} = \sum_{k=0}^{[n/2]} (\partial_t^{2k} \widetilde{\Delta^{-k-1}})(\underset{n-4-2k}{\overline{\tau}^{\mu\nu}}) + \sum_{k=0}^{[n/2]-1} \underset{n-2k}{\overset{(2k)}{A^{\mu\nu}_L}} \widetilde{\Delta^{-k}}(\hat{x}_L)$$

•Using a recurrence

$$\overline{h}^{\mu\nu} = \frac{16\pi G}{c^4} \mathcal{I}^{-1}(\overline{\tau}^{\mu\nu}) + \sum_{k,l\geq 0} \overset{(2k)}{A_L} \widetilde{\Delta^{-k}}(\widehat{x}_L)$$

•The mathematical structure.

•$A_L(t)$ are unkown for the moment.

# Generalised Poisson integral.

Analytical continuation :

$$\widetilde{\Delta^{-1}} S\,(\boldsymbol{z}) = -\frac{1}{4\pi} \underset{B=0}{\mathrm{FP}} \int_{\mathbb{R}^3} \frac{d^3\boldsymbol{x}}{|\,\boldsymbol{x}-\boldsymbol{z}\,|} \left(\frac{r}{r_0}\right)^{B} S(\boldsymbol{x})$$

$$\mathcal{S} = \left\{ S \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}) \mid S(\boldsymbol{x}, t) = \sum_{p,q} r^p \ln^q r\, s_{p,q}(\boldsymbol{n}) + \mathcal{O}(r^{-N}) \right\}$$

Basically:

$$\begin{aligned} \widetilde{\Delta^{-1}} S\,(\boldsymbol{z}) \;=\; & -\frac{1}{4\pi} \int_{r<\Lambda} \frac{d^3\boldsymbol{x}}{|\,\boldsymbol{x}-\boldsymbol{z}\,|} S(\boldsymbol{x}) \\ & - \sum_{\substack{p\leq p_0 \\ p\neq -3}} \frac{\Lambda^{3+p}}{3+p} \Big( \sum_{l=0}^{p_0-p} \frac{(2l+1)!!}{l!} \hat{z}_L \int d\Omega\, \hat{n}_L\, s_{p+l}(\boldsymbol{n}, t) \Big) \\ & - \ln\Big(\frac{\Lambda}{r_0}\Big) \Big( \sum_{l=0}^{p_0+3} \frac{(2l+1)!!}{l!} \hat{z}_L \int d\Omega\, \hat{n}_L\, s_{l-3}(\boldsymbol{n}, t) \Big) \end{aligned}$$

Usual Poisson operator with a cut-off

Substract homogeneous terms

## Post-Minkowskian expansion (outside the source).

(Work due to Blanchet and Damour)

$$h^{\mu\nu} = \sum_{n\geq 0} G^n h^{\mu\nu}_{(n)}$$

•pM expansion valide outside the source.

$$\Box h^{\mu\nu}_{(n)} = \Lambda^{\mu\nu}_{(n)}\,,\qquad \partial_\mu h^{\mu\nu}_{(n)} = 0\,,$$

No mater, source of gravitation

Multipolar post-Minkowskian expansion.

$$\Box h^{\mu\nu}_{(1)} = 0\,,\qquad \partial_\mu h^{\mu\nu}_{(1)} = 0\,.$$

$$\mathcal{M}(h^{\mu\nu}_{(n)}) = \widetilde{\Box^{-1}_{\mathrm{Ret}}}(\mathcal{M}(\Lambda^{\mu\nu}_{(n)})) + q^{\mu\nu}_{(n)}$$

$$h^{\mu\nu}_{(1)} = \sum_{l\geq 0} \widehat{\partial}_{\scriptscriptstyle L}\left(\frac{F^{\mu\nu}_{\scriptscriptstyle L}(t-r)}{r}\right)$$

$$q^{\mu\nu}_{(n)} = \sum_{l\geq 0} \widehat{\partial}_{\scriptscriptstyle L}\left(\frac{\mathcal{F}^{\mu\nu}_{(n)\scriptscriptstyle L}(t-r)}{r}\right)$$

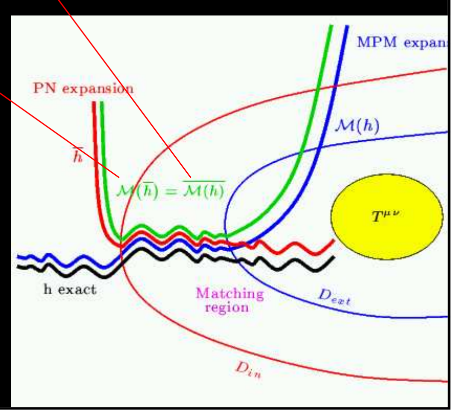
No-incoming radiation

Most general harmonic fonction

## Matching equation.

Expansion to 0.

Expansion to infinity.




• In the matching region we have numerically:

$$\overline{h}^{\mu\nu} = h^{\mu\nu}_{exact} = \mathcal{M}(h^{\mu\nu})$$

• Expansions don't have the same math struct.

## Consequences of the matching.

$$\mathcal{M}(\overline{h}^{\mu\nu}) = \frac{16\pi G}{c^4}\,\mathcal{I}^{-1}(\mathcal{M}(\overline{\tau}^{\mu\nu})) + \sum_{k,l\geq 0} \overset{(2k)}{A^{\mu\nu}_L}(t)\;\Delta^{-k}(\hat{x}_L) - \frac{4G}{c^4}\sum_{k,n,l\geq 0}\frac{(-1)^l}{l!(2n)!}\,\widehat{\partial}_{L}\left(r^{2n-1}\right)\,\underset{\mathsf{B}=0}{\mathrm{FP}}\int d^3\mathbf{x}\, r^{\mathsf{B}}\Delta^{-k}(\hat{x}_L)\;\partial_t^{2n+2k}(\overline{\tau}^{\mu\nu})$$

$$\overline{\mathcal{M}(h^{\mu\nu})} = \mathcal{I}^{-1}(\overline{\mathcal{M}(\Lambda^{\mu\nu})}) \Longleftrightarrow \frac{16\pi G}{c^4}\,\mathcal{I}^{-1}(\overline{\mathcal{M}(\tau^{\mu\nu})}) + \frac{4G}{c^4}\sum_{l\geq 0}\sum_{k\geq l}(-1)^l\,\frac{2^l}{(2k+1)!}\;\Delta^{-k}(\hat{x}_L)\left\{\overset{(2l+2k+1)}{\overline{\mathcal{R}}^{\mu\nu}_L}(t) + \overset{(2l+2k+1)}{\overline{\mathcal{F}}^{\mu\nu}_L}(t)\right\} - \frac{4G}{c^4}\sum_{n\geq 0}\sum_{l\geq 0}\frac{(-1)^l}{l!(2n)!}\;\widehat{\partial}_{L}\left(r^{2n-1}\right)^{(2n)}\overline{\mathcal{F}}^{\mu\nu}_L(t)\,.$$

## pN can be iterated to all orders in (1/c)!

$$\overline{h}^{\mu\nu} = \frac{16\pi G}{c^4}\mathcal{I}^{-1}(\overline{\tau}^{\mu\nu}) + \sum_{k,l\geq 0}\overset{(2k)}{A_L}\widetilde{\Delta^{-k}}(\widehat{x}_L)$$

$$A^{\mu\nu}_L = \frac{1}{c^{2l+1}}\frac{4G}{c^4}\frac{2^l(-1)^l}{(2l+1)!}\left(\overset{(2l+1)}{\overline{\mathcal{R}}^{\mu\nu}_L}(t) + \overset{(2l+1)}{\overline{\mathcal{F}}^{\mu\nu}_L}(t)\right)$$

$$\overline{\mathcal{F}}^{\mu\nu}_L(t) = \frac{1}{c^{2k}}\sum_{k\geq 0}\underset{\mathsf{B}=0}{\mathrm{FP}}\int \mathrm{d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^{\mathsf{B}}\Delta^{-k}(\hat{x}_L)\;\partial_t^{2k}(\overline{\tau}^{\mu\nu})$$

$$\overline{\mathcal{R}}^{\mu\nu}_L(t) = \underset{\mathsf{B}=0}{\mathrm{FP}}\int d^3\mathbf{y}\,\hat{y}_L\,|\,\mathbf{y}\,|^{\mathsf{B}}\int_1^{\infty} dz\,\gamma_l(z)\mathcal{M}(\tau^{\mu\nu})(\mathbf{y}, t - z\frac{|\,\mathbf{y}\,|}{c})$$

In the Burke-Thorne gauge:

$$\begin{cases} (h^{00}_R + h^{kk}_R) = \frac{4G}{5c^7}x^i x^j\left[\overset{(5)}{\mathsf{M}_{ij}}(t) + \frac{4GM}{c^3}\int_0^{+\infty}\mathrm{d}u\,\overset{(7)}{\mathsf{M}_{ij}}(t-u)\left[\ln\left(\frac{u}{2}\right) + \frac{11}{12}\right]\right] \\ h^{0i}_R = o(c^{-6}) + o(c^{-9}) \\ h^{ij}_R = o(c^{-5}) + o(c^{-8}) \end{cases}$$

# *Part 3- The Hot Subject*

## *Dynamic of a binary system up to 3pN.*

•Blanchet and Faye: 3pN for point-like stars (harmonic).

$\lambda$ at 3pN ?

Regularisation method (because of $\delta$): hadamard.

•Damour, Schaefer, Jaranowski: same thing (ADM).

$\omega_{static}$ at 3pN ?

Same regularisation method.

•DSJ used dimensional regularisation:

$\omega_{static} = 0$.

**Does the Nature depend on regularisation method ?**

**Of course not !**

$$E_{3PN} = (m_1+m_2)c^2 + \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{Gm_1m_2}{r_{12}} + \underbrace{\frac{E_2}{c^2}}_{8\ \text{termes}} + \underbrace{\frac{E_4}{c^4}}_{32\ \text{termes}} + \underbrace{\frac{1}{c^6}\left\{\cdots + \frac{G^4 m_1^3 m_2^2}{r_{12}^4}\left(\frac{5809}{280} - \frac{11}{3}\lambda - \frac{22}{3}\ln\left(\frac{r_{12}}{r'_1}\right)\right) + \cdots\right\}}_{130\ \text{termes}}.$$

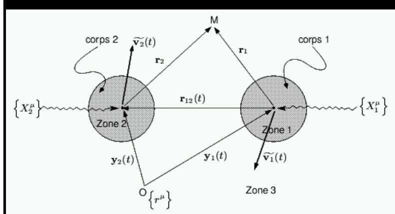
## *Effacing principle.*

- The external problem should not depend on the internal structure of the compact bodies up to 5pN !
- Assume: we take internal structure into account !

$$\rho_{*\alpha}(r^i) = \frac{m_\alpha \Psi'_\alpha(x_\alpha)}{4\pi b_\alpha^3 x_\alpha^2}$$

- We have 2 expansions: pN and « post-point-like » in power of $b_1$ and $b_2$.

Radius: $b_\alpha \sim \frac{\mathrm{G} m_\alpha}{c^2}$




$$\frac{\mathrm{d}^2 y_1^i}{\mathrm{d}t^2} = A_1^i\Big[M_1,\ M_2,\ \mathbf{r}_{12},\ \tilde{\mathbf{v}}_1,\ \tilde{\mathbf{v}}_2,\ \mathrm{G},\ \frac{1}{c},\ \text{structure interne}\Big]$$

$$\frac{\mathrm{d}^2 y_2^i}{\mathrm{d}t^2} = A_2^i\Big[M_1,\ M_2,\ \mathbf{r}_{12},\ \tilde{\mathbf{v}}_1,\ \tilde{\mathbf{v}}_2,\ \mathrm{G},\ \frac{1}{c},\ \text{structure interne}\Big]$$

## *First try: only a profile of density.*

$$E_{3PN} = \frac{1}{c^6}\Big\{\frac{35 m_1 v_1^8}{128} \ldots$$

$$+\frac{\mathrm{G}^3 m_1^2 m_2^2}{r_{12}^3}\Big(\frac{547}{12}(n_{12}v_1)^2 - \frac{3115}{48}(n_{12}v_1)(n_{12}v_2) - \frac{123}{64}(n_{12}v_1)(n_{12}v_{12})\pi^2 - \frac{575}{18}v_1^2 + \frac{41}{64}\pi^2(v_1v_{12}) + \frac{4429}{144}(v_1v_2)\Big)$$

$$+\frac{\mathrm{G}^3 m_1^3 m_2}{r_{12}^3}\Big(\mathbf{\Lambda}_4(n_{12}v_1)^2 + \mathbf{\Lambda}_5(n_{12}v_1)(n_{12}v_2) + \mathbf{\Lambda}_6(n_{12}v_2)^2 + \mathbf{\Lambda}_7 v_1^2 - \mathbf{\Lambda}_8(v_1v_2) + \mathbf{\Lambda}_9 v_2^2 + \mathbf{\Lambda}_{10}(n_{12}v_1)(n_{12}v_{12})\ln\left(\frac{r_{12}}{b_1}\right) + \mathbf{\Lambda}_{11}(v_1v_{12})\ln\left(\frac{r_{12}}{b_1}\right)\Big)\Big\}$$

$$+1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^7}\right).$$

$$+\frac{\mathrm{G}^2 m_1^2 m_2}{r_{12}^2}\Big(-\frac{49}{8}(n_{12}v_1)^4 + \frac{75}{8}(n_{12}v_1)^3(n_{12}v_2) - \frac{187}{8}(n_{12}v_1)^2(n_{12}v_2)^2 + \frac{247}{24}(n_{12}v_1)(n_{12}v_2)^3 + \frac{49}{8}(n_{12}v_1)^2 v_1^2 + \frac{81}{8}(n_{12}v_1)(n_{12}v_2)v_1^2 - \frac{21}{4}(n_{12}v_2)^2 v_1^2 + \frac{11}{2}v_1^4 - \frac{15}{2}(n_{12}v_1)^2(v_1v_2) - \frac{3}{2}(n_{12}v_1)(n_{12}v_2)(v_1v_2) + \frac{21}{4}(n_{12}v_2)^2(v_1v_2) - 27v_1^2(v_1v_2) + \frac{55}{2}(v_1v_2)^2 + \frac{49}{4}(n_{12}v_1)^2 v_2^2 - \frac{27}{2}(n_{12}v_1)(n_{12}v_2)v_2^2 + \frac{3}{4}(n_{12}v_2)^2 v_2^2 \ldots \frac{55}{} \ldots \frac{135}{} \ldots\Big)$$

Λ's look like:

$$\int_0^1 \mathrm{d}w\left[\int_0^w \mathrm{d}v\,\frac{\Psi_\alpha(v)}{v^4}\left[\int_0^v \mathrm{d}u\,\Psi_\alpha^2(u)\right]\right],$$

$$+\frac{7}{4}v_2^2 \ldots$$

$$\int_0^1 \mathrm{d}w\,\frac{\Psi_\alpha(w)}{w^2}\left[\int_0^w \mathrm{d}v\,\frac{\Psi_\alpha(v)}{v^2}\left[\int_0^v \mathrm{d}u\,\frac{\Psi_\alpha(u)}{u^2}\right]\right]$$

## Second try: the full problem with maximum complexity.

- We have included pressure, internal energy and internal velocity.
- Lorentz and Einstein contraction !

$$X^i = \widetilde{x}^i + \frac{1}{c^2}\left[\left(\underbrace{\frac{(\widetilde{\mathbf{v}_1\mathbf{x}})}{2}\,\widetilde{v}_1{}^i}_{\text{Lorentz}} + \underbrace{\frac{Gm_2}{r_{12}}\,\widetilde{x}^i}_{\text{Einstein}}\right) + \underbrace{\frac{Gm_2}{2r_{12}^3}(\mathbf{r_{12}}\widetilde{\mathbf{x}})\widetilde{x}^i}_{\text{terme de marée}}\right]$$

Local coordinates

Global coordinates

$$E_{\text{N}} = m_1c^2 + m_2c^2 + \frac{m_1v_1^2}{2} + \frac{m_2v_2^2}{2} - \frac{Gm_1m_2}{r_{12}} - \frac{2Gm_1^2}{3b_1} - \frac{2Gm_2^2}{3\,b_2} + \frac{Gm_1^2\int_0^1 \frac{\Psi_1(u)^2}{u^2}\,du}{3b_1} + \frac{Gm_2^2\int_0^1 \frac{\Psi_2(u)^2}{u^2}\,du}{3b_2}$$

Constant of time

Renormalisation

$$M_\alpha\Big|_{\text{1PN}} = m_\alpha - \frac{2Gm_\alpha^2}{3c^2b_\alpha} + \frac{Gm_\alpha^2\int_0^1 \frac{\Psi_\alpha(u)^2}{u^2}\,du}{3c^2b_\alpha}$$

$$E_{\text{N}} = M_1c^2 + \frac{M_1v_1^2}{2} - \frac{GM_1M_2}{2r_{12}} + (1 \leftrightarrow 2)$$

## What about the energy at 1pN ?

$$\begin{aligned}
E_{\text{1PN}} = {} & \frac{3m_1v_1^4}{8\,c^2} + \frac{3m_2v_2^4}{8c^2} - \frac{Gm_1^2v_1^2}{3c^2b_1} - \frac{Gm_2^2v_2^2}{3c^2b_2} + \frac{G^2m_1^2m_2}{2c^2r_{12}^2} + \frac{G^2m_1m_2^2}{2c^2r_{12}^2} \\
& - \frac{Gm_1m_2(\mathbf{n_{12}v_1})(\mathbf{n_{12}v_2})}{2c^2r_{12}} + \frac{3Gm_1m_2v_1^2}{2c^2r_{12}} + \frac{3Gm_1m_2v_2^2}{2c^2r_{12}} - \frac{7Gm_1m_2(\mathbf{v_1v_2})}{2c^2r_{12}} \\
& + \frac{13G^2m_1^3}{18c^2b_1^2} + \frac{13G^2m_2^3}{18c^2b_2^2} + \frac{2G^2m_1^2m_2}{3c^2b_1r_{12}} + \frac{2G^2m_1m_2^2}{3c^2b_2r_{12}} \\
& + \frac{2G^2m_1^3\int_0^1 u\,\Psi_1(u)\,du}{9c^2b_1^2} + \frac{2G^2m_1^3\int_0^1 \frac{\int_0^u \frac{\Psi_1(v)^2}{v^2}dv\,\Psi_1(u)}{u^2}\,du}{9c^2b_1^2} + \frac{Gv_1^2m_1^2\int_0^1 \frac{\Psi_1(u)^2}{u^2}\,du}{6c^2b_1} \\
& - \frac{7G^2m_1^3\int_0^1 \frac{\Psi_1(u)^2}{u^2}\,du}{9c^2b_1^2} - \frac{G^2m_1^2m_2\int_0^1 \frac{\Psi_1(u)^2}{u^2}\,du}{3b_1r_{12}} + \frac{8G^2m_1^3\int_0^1 \frac{\int_0^u v\,\Psi_1(v)\,dv\,\Psi_1(u)^2}{u^5}\,du}{9c^2b_1^2} \\
& + \frac{2G^2m_2^3\int_0^1 u\,\Psi_2(u)\,du}{9c^2b_2^2} + \frac{2G^2m_2^3\int_0^1 \frac{\int_0^u \frac{\Psi_2(v)^2}{v^2}dv\,\Psi_2(u)}{u^2}\,du}{9c^2b_2^2} + \frac{Gm_2^2v_2^2\int_0^1 \frac{\Psi_2(u)^2}{u^2}\,du}{6c^2b_2} \\
& - \frac{7G^2m_2^3\int_0^1 \frac{\Psi_2(u)^2}{u^2}\,du}{9c^2b_2^2} - \frac{G^2m_1m_2^2\int_0^1 \frac{\Psi_2(u)^2}{u^2}\,du}{3c^2b_2r_{12}} + \frac{8G^2m_2^3\int_0^1 \frac{\int_0^u v\,\Psi_2(v)\,dv\,\Psi_2(u)^2}{u^5}\,du}{9c^2b_2^2} \\
& + \frac{G^2m_1^3\int_0^1 f_{\Pi_1}(u)\,du}{c^2b_1^2} + \frac{G^2m_2^3\int_0^1 f_{\Pi_2}(u)\,du}{c^2b_2^2}\,.
\end{aligned}$$

Does not depend on the internal structure

Depends on IS and varies in time

Depends on IS but does not vary in time

## *Consequences of renormalisation at 3pN.*

General relation between:
Baryonic mass and renormalised mass

$$m_\alpha = M_\alpha + \gamma_1 \frac{G M_\alpha^2}{c^2 b_\alpha} + \gamma_2 \frac{G^2 M_\alpha^3}{c^4 b_\alpha^2}$$

Baryonic masses

Renormalised masses

We didn't talk about these terms:

$$\begin{aligned}
\Delta_1 E_{1PN} &= \beta_1 \frac{G m^2 v^2}{c^2 r_{12}^3} b_1^2 \,, \\
\Delta_2 E_{1PN} &= \beta_2 \frac{G^2 m^3}{c^2 r_{12}^4} b_1^2 \,, \\
\Delta_3 E_{2PN} &= \beta_3 \frac{G^2 m^3 v^2}{c^4 r_{12}^3} b_1 \,, \\
\Delta_4 E_{2PN} &= \beta_4 \frac{G^3 m^4}{c^4 r_{12}^4} b_1 \,,
\end{aligned}$$

$$\begin{aligned}
\Delta_1 E_{1PN}\Big|_{\text{renormalisé}} &= \beta_1 \gamma_2 \frac{G^3 M^4 v^2}{c^6 r_{12}^3} \,, \\
\Delta_2 E_{1PN}\Big|_{\text{renormalisé}} &= \beta_2 \gamma_2 \frac{G^4 M^5}{c^6 r_{12}^4} \,, \\
\Delta_3 E_{2PN}\Big|_{\text{renormalisé}} &= \beta_3 \gamma_1 \frac{G^3 M^4 v^2}{c^6 r_{12}^3} \,, \\
\Delta_4 E_{2PN}\Big|_{\text{renormalisé}} &= \beta_4 \gamma_1 \frac{G^4 M^5}{c^6 r_{12}^4} \,.
\end{aligned}$$

## ***Conclusion***

**Don't even think about asking**
**What could happen at 4pN …**