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Bagnold scaling, Density plateau, and Kinetic Theory Analysis of Dense Granular Flow

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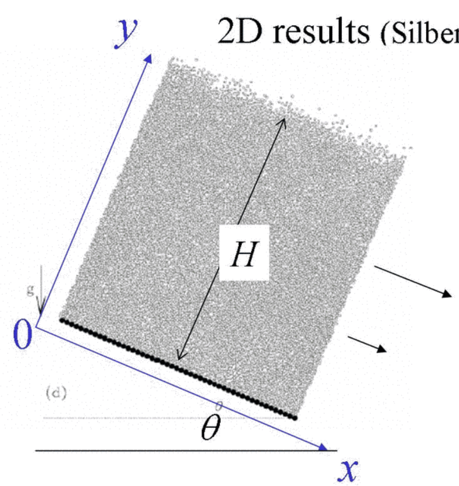
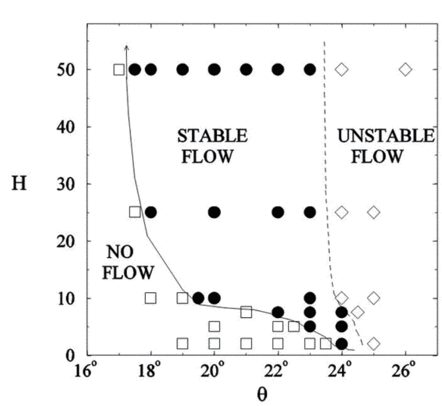
Acknowledgement: RIKEN and JSPS.

Steady dense flow down a slope

From L. Silbert et al. PRE 64, 051302 (2001)

2D results (Silbert et al. studied 3D system also)

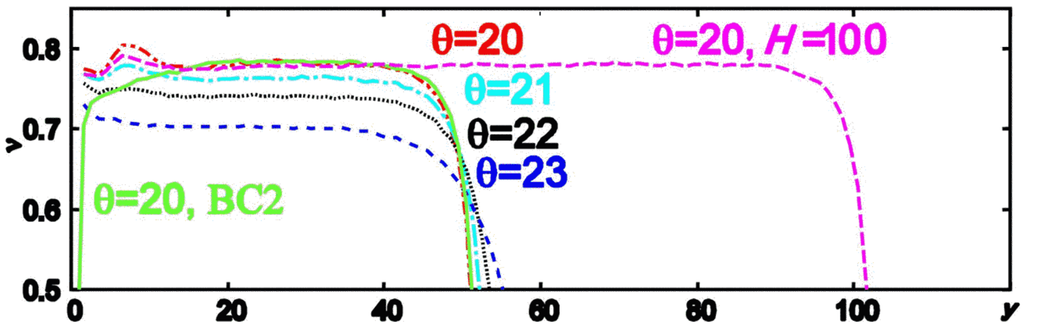




cf. O. Pouliquen, Phys. Fluids (1999): experimental phase diagram

Density Profile of Dense Flow

Silbert et al. PRE **64** 051302 (2001) found that steady flows in 2- and 3- dimension shows *constant density profile* in the bulk.



y dependence of the packing fraction ν (From our 2-D simulations)

Density plateau: the value is

- determined by inclination angle θ
- independent of total flow height H
- independent of bottom roughness

Density Profile of Dense Flow

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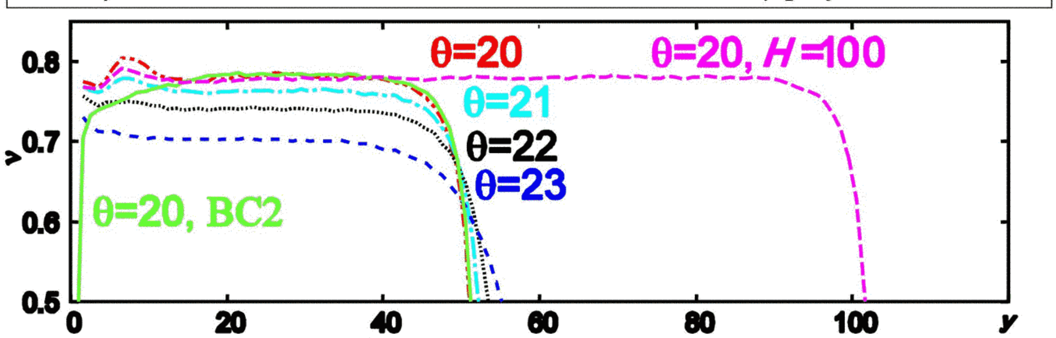


y dependence of the packing fraction ν (From our 2-D simulations)

How can we understand the bulk rheology with this density plateau?

Qualitatively: Dimensional analysis

Quantitatively: Kinetic theory analysis

Dimensional analysis gives density plateau: Bagnold scaling

Shear flow of rigid gains is characterized by

- particle mass m
- diameter σ
- number density n
- shear rate $\dot{\gamma}$

$\dot{\gamma}^{-1}$ is the only time scale in the system.

Shear Stress

$$S = m\sigma^{2-d} A(\nu)\dot{\gamma}^2$$

Bagnold Scaling

R.A. Bagnold (1954)

d : dimension,
packing fraction $\nu \approx \sigma^d n$

Extended Bagnold scaling explains the density plateau

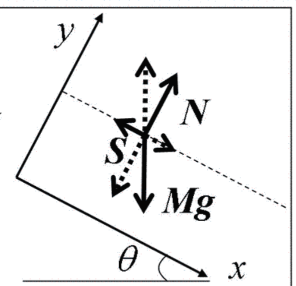
Bagnold scaling from dimensional analysis

$$S = m\sigma^{2-d} A(\nu)\dot{\gamma}^2$$

Extend to Pressure

$$N = m\sigma^{2-d} B(\nu)\dot{\gamma}^2$$

Force balance in steady flow



At any y :

$$S/N = \tan\theta$$

In the bulk where Bagnold scaling holds:

$$\tan\theta = \frac{A(\nu)}{B(\nu)}$$

Density is determined by θ!

- indep. of y (constant profile).
- indep. of total height H.
- indep. of boundary conditions.

Extended Bagnold scaling explains the density plateau

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$$N = m\sigma^{2-d} \ldots$$

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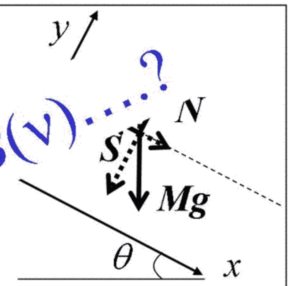

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Functional forms of A(ν) and B(ν)….?

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Theory for Granular Flow that gives Constitutive Relations

Dimensional analysis gives $\tan\theta = A(\nu)/B(\nu)$,
but the functional forms for A and B are unknown.

Any theory that gives constitutive relations?

Kinetic Theory of (slightly) inelastic hard spheres

- "Microscopic" Theory
- Quantitative agreement for relatively low-density flow with small dissipation

▶ How good (or how it fails) in Dense Slope Flow?

Compare the theory with the simulation data to see if the theory give a starting point to describe the dense flow.

Hydrodynamic Model for Granular flow

"Hydrodynamics variables"

number density : $n(\mathbf{r},t)$

mean velocity : $\mathbf{v}(\mathbf{r},t)=\langle \mathbf{c}\rangle$

granular temperature : $T(\mathbf{r},t)=\frac{m}{d}\left\langle (\mathbf{c}-\mathbf{v})^2\right\rangle$

Eq. of continuity: $D_t n = -n\nabla\cdot\mathbf{v}$

Eq. of motion: $mnD_t\mathbf{v} = mn\,\mathbf{f} + \nabla\cdot\overleftrightarrow{\Sigma}$

Eq. of energy: $\frac{d}{2}n\,D_t T = -\nabla\cdot\mathbf{q} + \overleftrightarrow{\Sigma}\cdot\nabla\mathbf{v} - \Gamma$

$(D_t = \partial/\partial t + \mathbf{v}\cdot\nabla)$

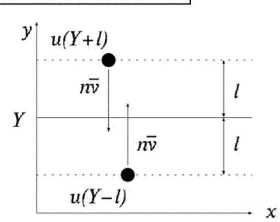
dissipation of fluctuation energy due to inelasticity, friction, etc.

Framework of Kinetic Theory

Granular temperature T is treated as a separate variable.

ex. Shear stress: given by momentum flux

$ml(\nu)\,\dot{\gamma}\,n\sqrt{T/m}$ ($l(\nu)$: mean free path)



$$S = m^{1/2}\sigma^{1-d} f_2(\nu) T^{1/2}\dot{\gamma}$$

Dissipation per collision □ T
Collision rate □ $T^{1/2}$

Similar argument gives

Pressure: $N = \sigma^{-d} f_1(\nu)\,T$

Dissipation: $\Gamma = m^{-1/2}\sigma^{-d-1} f_3(\nu)\,T^{3/2}$

Heat Flux: $q = -m^{-1/2}\sigma^{1-d} f_4(\nu)\,T^{1/2}\,\partial_y T$

The functional form of $f_i(\nu)$ depends on the details of models and approximations.

Bagnold Scaling□

divergence of heat flux $\nabla \cdot \mathbf{q}$ is negligible in energy eq.

Kinetic Theory $S \propto T^{1/2}\dot{\gamma}, N \propto T$ ⇒ Bagnold scaling holds if $T \propto \dot{\gamma}^2$ $(S \propto \dot{\gamma}^2, N \propto \dot{\gamma}^2)$

Eq. of energy balance: $-\partial_y q + S\,\dot{\gamma} - \Gamma = 0$

If $\nabla \cdot \mathbf{q}$ *is negligible*, we have:

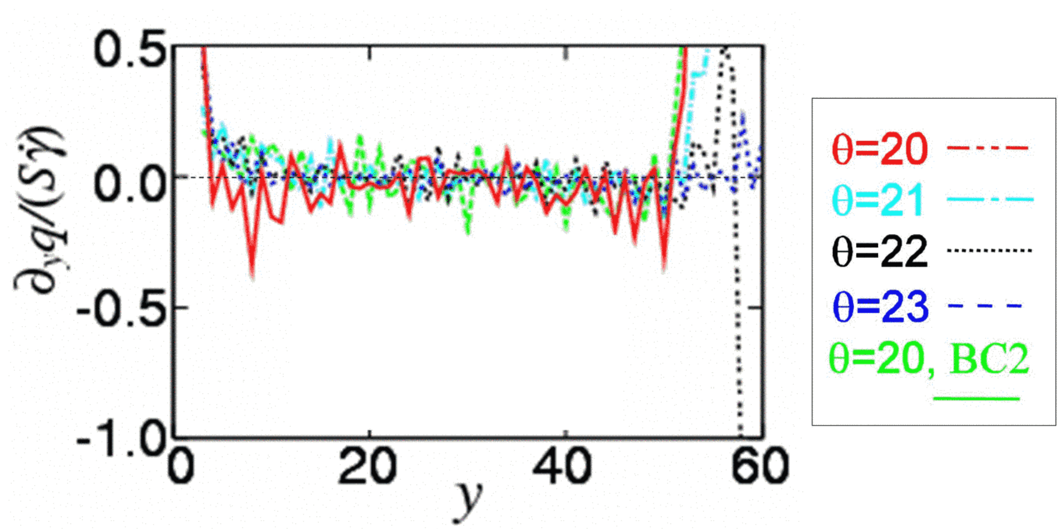
$$S\,\dot{\gamma} = \Gamma$$

⇓

$$T = [f_2(\nu)/f_3(\nu)]\, m\sigma^2\, \dot{\gamma}^2$$

Divergence of heat flux is small

compared to viscous heating and dissipation



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Eq. of energy balance: $-\partial_y q + S\dot{\gamma} - \Gamma = 0$

If $\nabla\cdot\mathbf{q}$ *is negligible*, we have:

$$S\dot{\gamma} = \Gamma$$

:This realizes in the bulk!

$$T = \left[f_2(\nu)/f_3(\nu)\right] m\sigma^2\dot{\gamma}^2$$

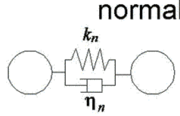
$$\tan\theta = \frac{S}{N} = \frac{m^{1/2}\sigma^{1-d} f_2(\nu) T^{1/2}\dot{\gamma}}{\sigma^{-d} f_1(\nu) T} = \frac{\sqrt{f_2(\nu) f_3(\nu)}}{f_1(\nu)}$$

Compare with data!

Our MD Simulation

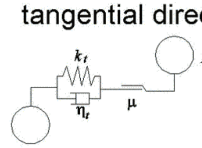
Soft-sphere model(DEM), but with **very hard** particle stiffness: in 2D with friction

normal direction



$$F_n^{ij} = k_n\Delta - \eta_n v_n$$

tangential direction



$$F_t^{ij} = \min\left(|h_t|, \mu|F_n^{ij}|\right)\mathrm{sign}(h_t)$$

$$h_t = -k_t u_t - \eta_t v_t$$

Parameters	$k_n = 2\times10^5$ $k_t = (2/7)k_n$ $\eta_n = 16.8$ $\eta_t = 0$
normal restitution coefficient:	$e_p = 0.92$ Relatively large!
Coulomb friction coefficient:	$\mu = 0.5$
tangential restitution coefficient for non-slip collision:	$\beta_0 = 1$

Units: length : σ time : $\sqrt{\sigma/g}$ mass : m

Constitutive relations from kinetic theory

Jenkins and Richman (1985): 2-dimensional hard disks

constant normal and tangential restitution coefficient e_p and β

f_1	$(4/m\pi\sigma^2)\nu(1+2r\nu g_0(\nu))$
f_2	$(\sqrt{m}/B\sigma g_0(\nu)\sqrt{\pi})\,(1+\nu g_0(\nu)(r+a))\left[1+\nu g_0(\nu)[(3r-2)r+2ar-a^2(1+\tilde{T}/\kappa T)]\right] + (4\sqrt{m}\nu^2 g_0(\nu)r/\sigma\pi^{3/2})(1+a/2r)$
f_3	$(4\nu^2 g_0(\nu)r/\sqrt{m}\sigma^3\pi^{3/2})\,[8(1-e_p) + 4\kappa(1+\beta)(1+e_p)^{-1}(1+\kappa)^{-2}[2+\kappa(1-\beta)-(1+\beta)\tilde{T}/T]]$

$\kappa = 4I/(m\sigma^2)$, $a = \kappa(1+\beta)/[2(1+\kappa)]$, $\tilde{T} = I\left\langle(\omega - \langle\omega\rangle)^2\right\rangle$

$r = (1+e_p)/2$, and $B = \left[-(1-\tilde{T}/T)a^2 + (5-8r)a + 2(5-3r)\right]/2$.

$g_0(\nu)$ is the radial distribution function.

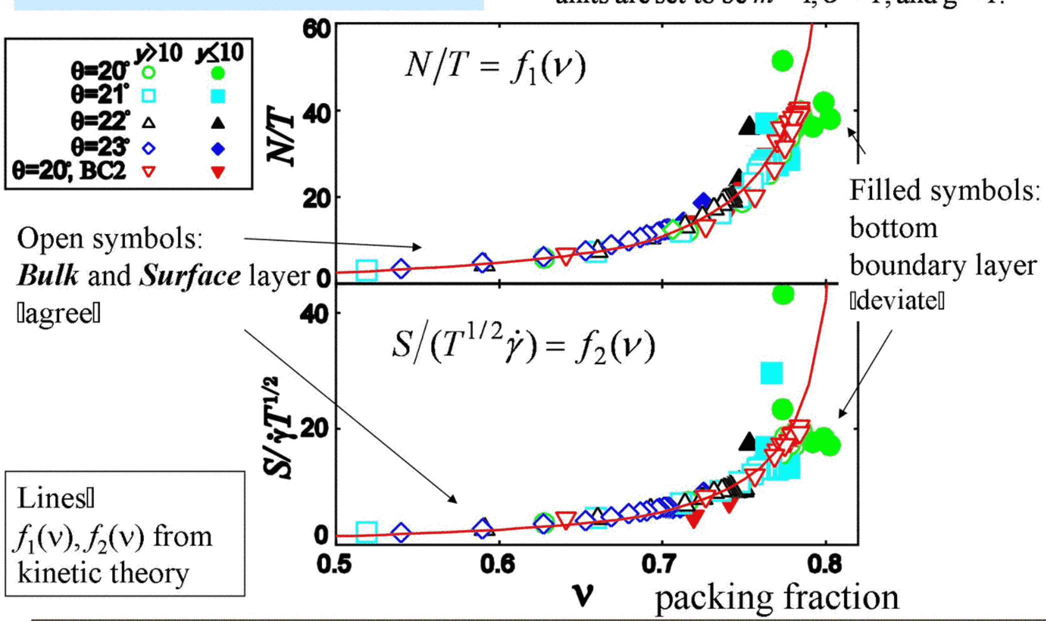
Tangential interaction is different

– Ambiguity in β and the Rotational temperature

- set $\beta = \beta_0$ and try several values of rotational temperature

Pressure and Shear stress

dimensionless :
units are set to be $m = 1$, $\sigma = 1$, and $g = 1$.

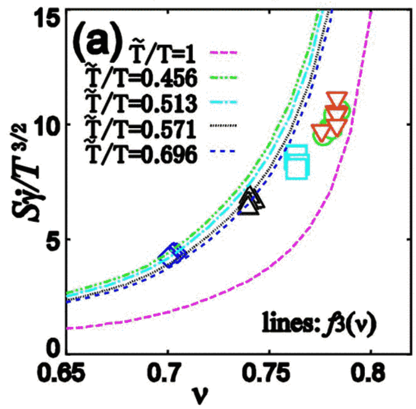


The stresses in the bulk and the surface layer agree with the kinetic theory quantitatively, even in the high density region.

Dissipation term and bulk density

Density dependence of the dissipation term ($S\dot{\gamma} = \Gamma$ is used)

$\Gamma / T^{3/2} = f_3(\nu)$



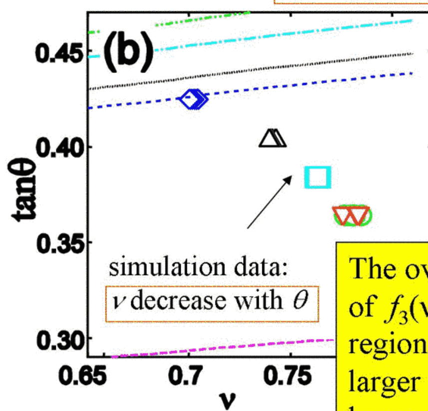
$f_3(\nu)$ increases faster than simulation data: **Theory somehow overestimate** the dissipation in the denser region!

$\tan\theta$ vs. ν (bulk)

kinetic theory (lines):

$$\frac{\sqrt{f_2(\nu) f_3(\nu)}}{f_1(\nu)} = \tan\theta$$

ν increase with θ



The overestimation of $f_3(\nu)$ in denser region results in larger $\tan\theta$ for larger density in the kinetic theory.

Summary of our results

- **We have analyzed dense granular flow down a slope by using the kinetic theory.**
  - The normal and shear **stresses** obtained from the constitutive relations of kinetic theory **agree** with our simulation data, even though the density is high.
  - The Kinetic theory **overestimate** the density dependence of **the energy dissipation term in denser region**.
  - Due to this discrepancy in the energy dissipation term, the kinetic theory gives increasing density upon increasing inclination angle, which is opposite from the simulation result.

N. Mitarai and H. Nakanishi, Phys. Rev. Lett. Vol. 94, 128001 (2005)