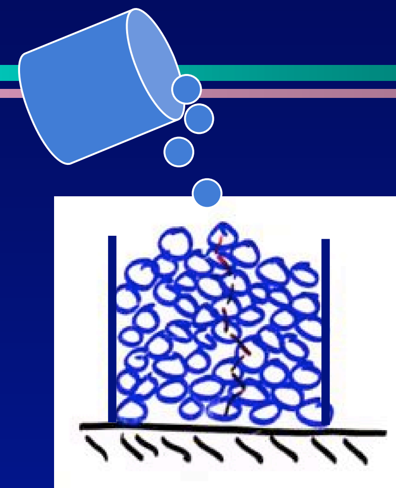
Topology of Stress Transmission in Isostatic Assemblies

David Wu
Colorado School of Mines
KITP Granular Physics Program
June 1, 2005

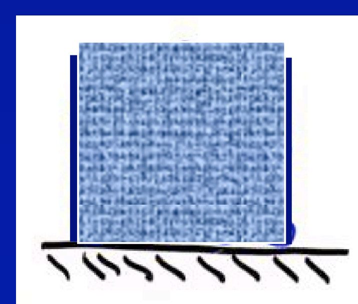
Holy Grail of Granular Statics

- Specify Preparation
  - Grain sizes, shapes, interactions
  - Prep history



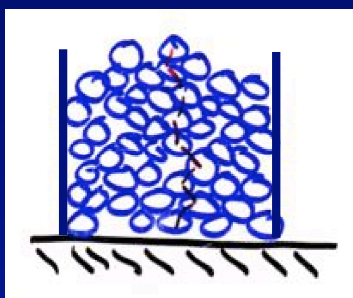
Statistical (micro)mechanics?

- Predict Macroscopic Properties
  - Elastic (?) Continuum (?)
  - Yield (stress distribution)

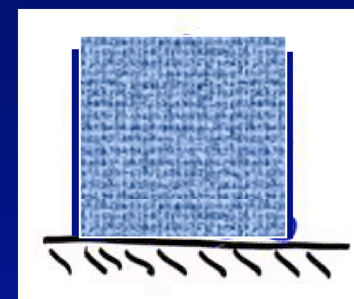


Key Issues in Granular Statics

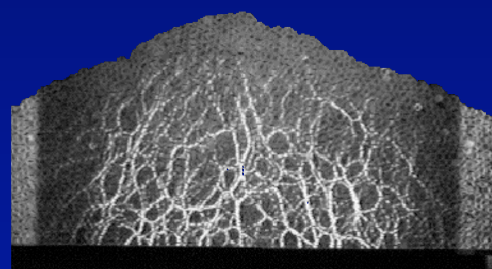
Contact network



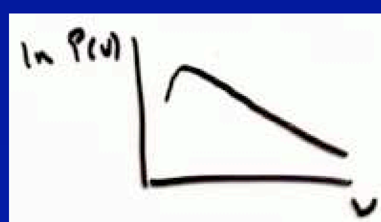
Constitutive behavior



Force chains



Where is grain contact information?

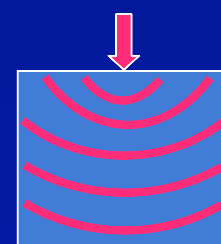


$P(f) \sim \exp(f/f_{avg})$

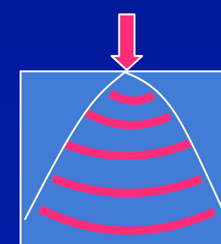
Phenomenological

Q- & vector models

Applied Load



Elliptic: elastic

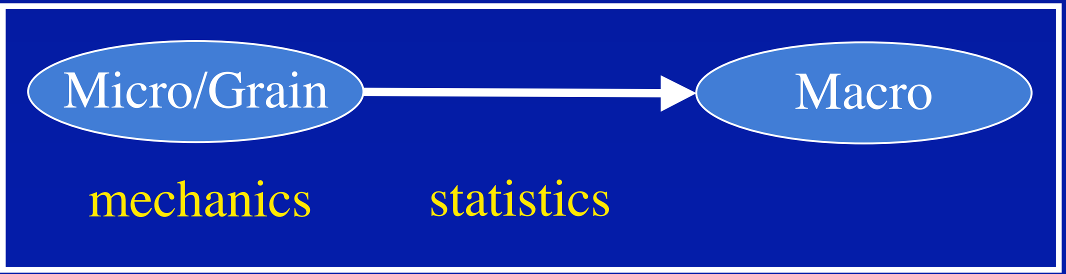


Hyperbolic: wavelike

Statistical Mechanics

- Mechanics: Static Stresses - T. Kangsadan (MS '00)
  - (Isostatic) Piles of frictionless hard disks
  - Force chain hierarchy (directed topology)
  - Implications

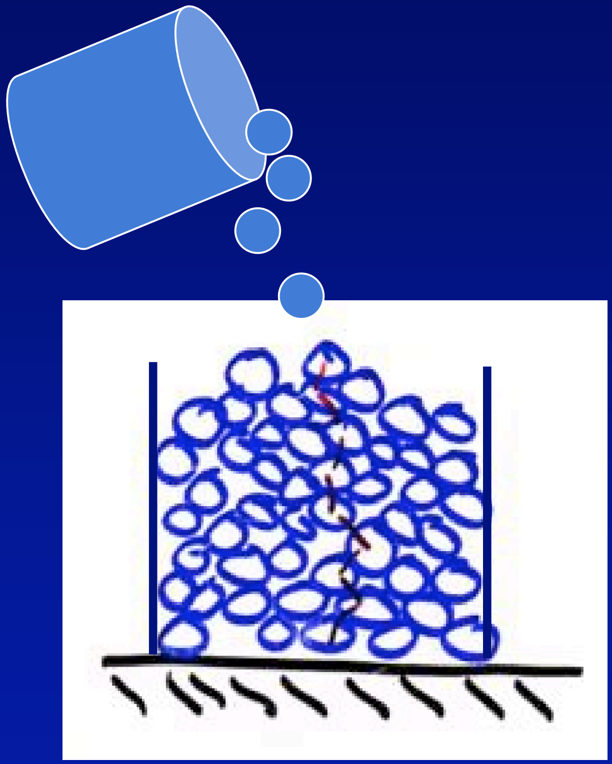
- Statistics: Averages - J. Lechman (PhD '04)
  - Simple model of pile formation
  - Force scattering
  - Fluctuations
  - Correlations



I. Mechanics
How does it all stack up?

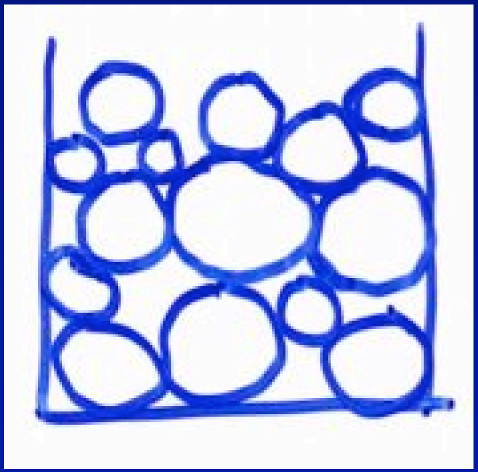
Microscopic Mechanical Analysis

- Basic model
  - Hard, frictionless particles
  - Choice of Ensemble:
    - Under some ambient load (gravity, confinement)
    - Explicit boundary
  - Isostatic packing
- Examine force response function



Review of Isostatic Packings

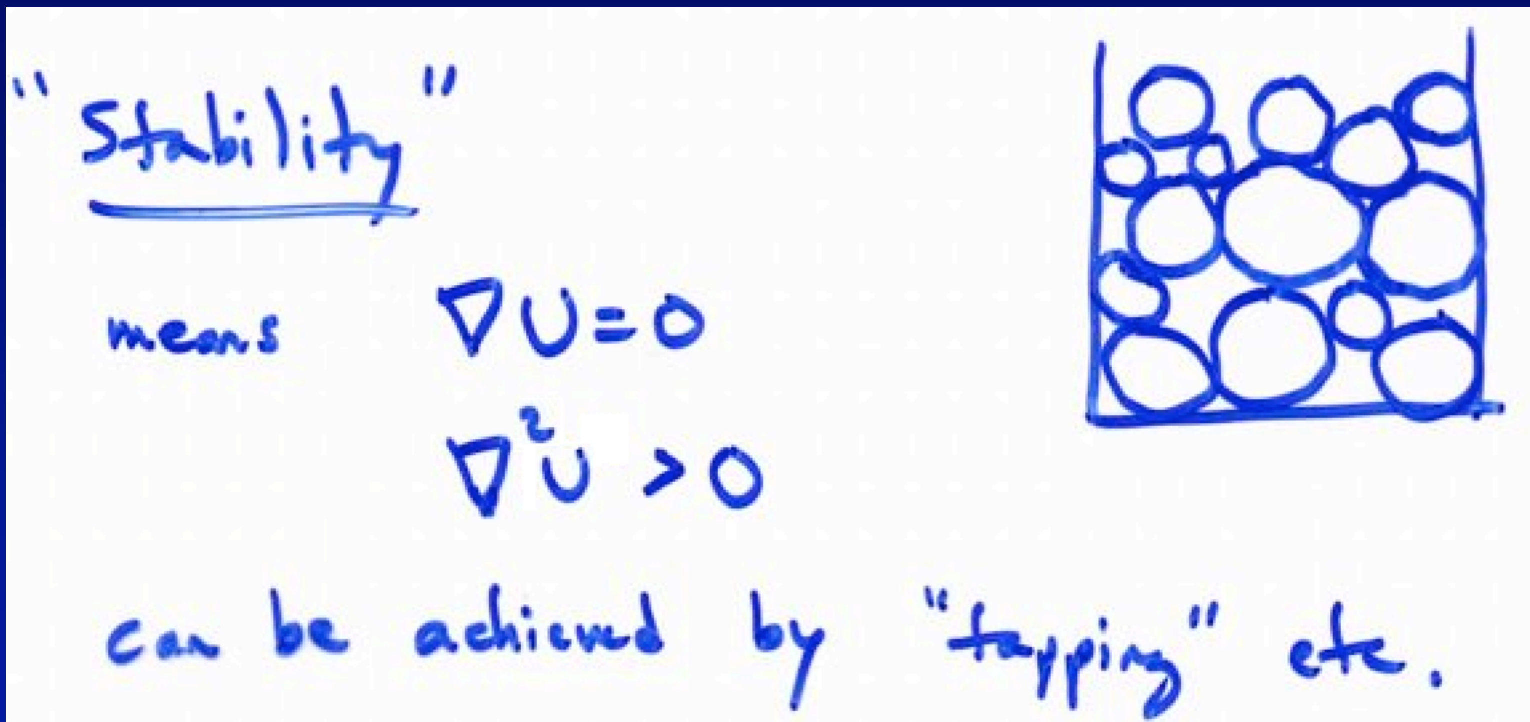
- Isostatic means
  - minimum number of contacts to provide rigidity
  - any applied stress is uniquely resolved
- Maxwell-Cremona count is satisfied
  - # of constraints = # of degrees of freedom
- For random sized disks, probability of accidental extra contacts is zero.
  - Isostatic graphs with generic lengths are infinitesimally rigid



$$2\, N_{particles} = N_{contacts}$$

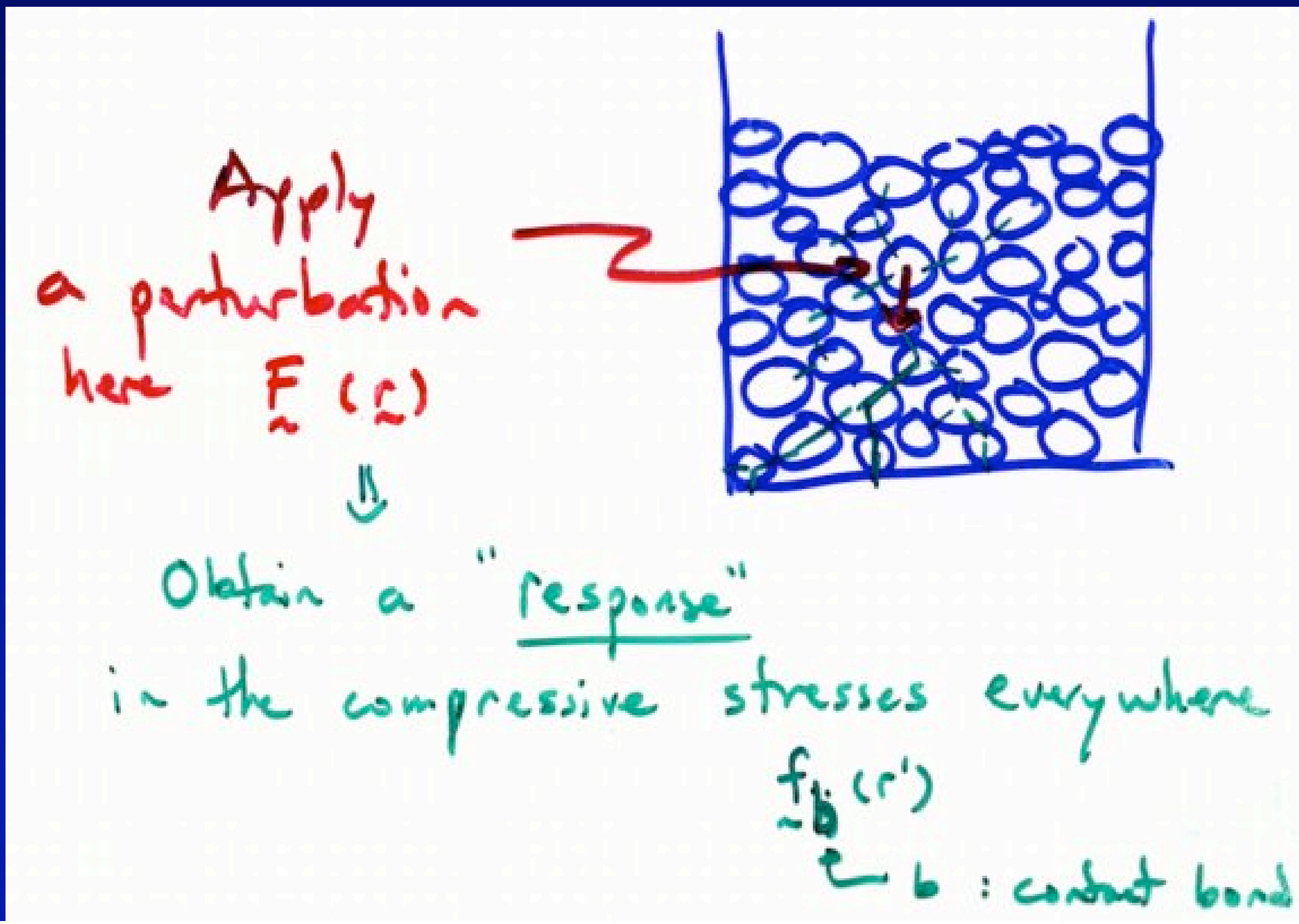
For more on rigidity theory: R. Connelly, W. Whiteley

Stability of Packs



- => Response function is well-defined
- => Response is linear for small perturbations (under load)
- Accommodates softness and adhesion

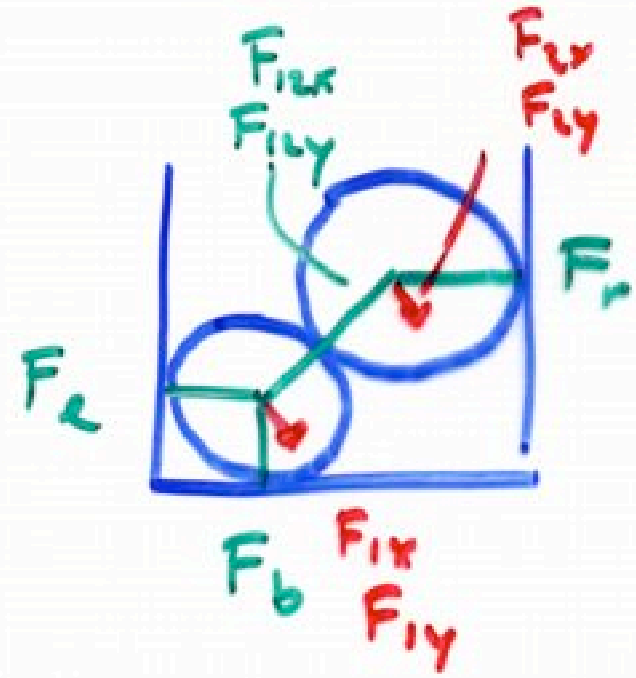
Force Response Functions



Linear Set of Constraint Eq^{n}s

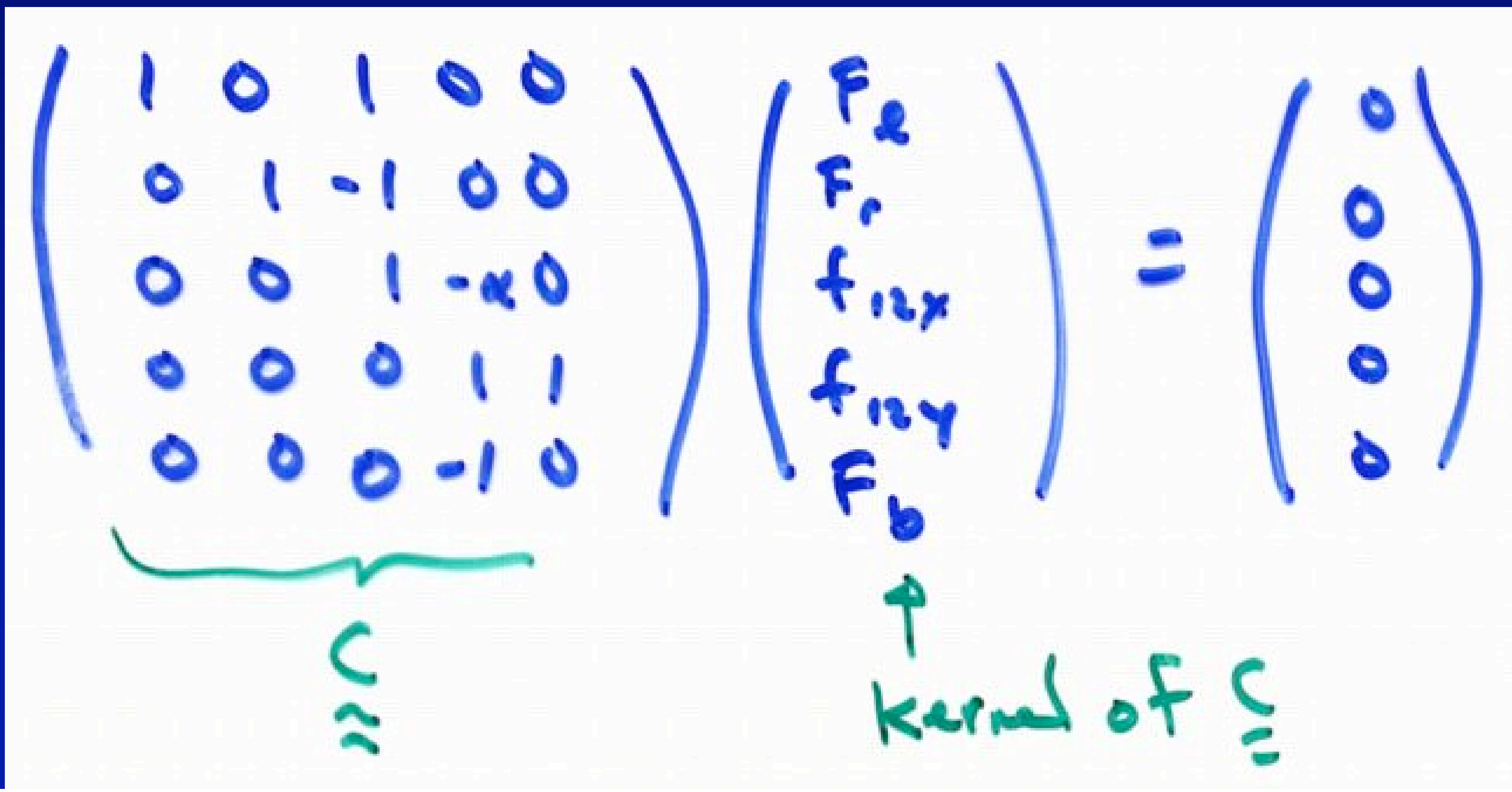
E.g.

$$\begin{cases} f_{2x} + f_{12x} + F_l = 0 \\ f_{1x} + f_{12x} + F_r = 0 \\ f_{2y} + f_{12y} + F_b = 0 \\ f_{1y} - f_{12y} = 0 \\ f_{12x} = \alpha f_{12y} \end{cases}$$



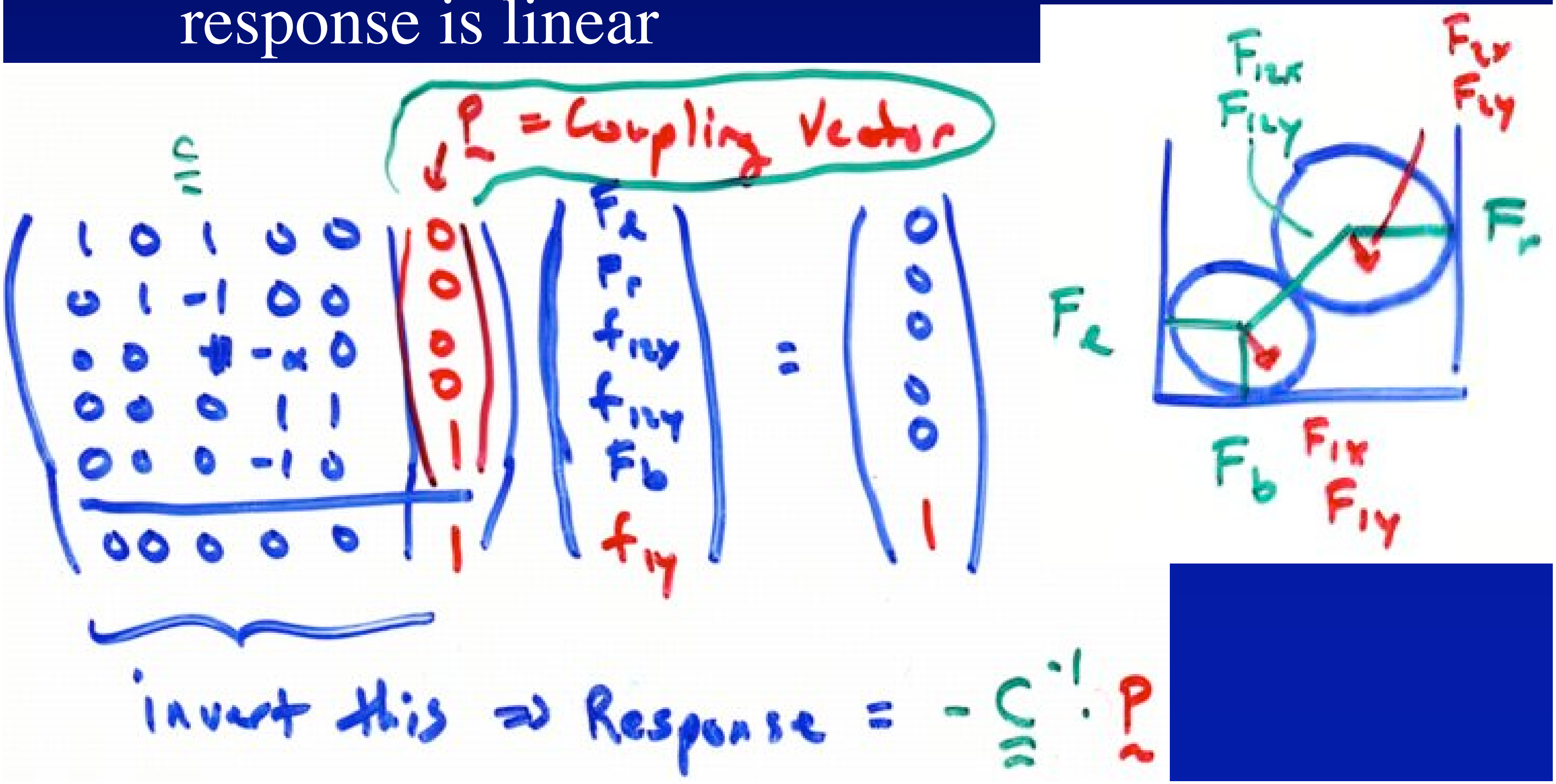
Rigidity Matrix, C

- Encodes contact geometry
- For no external forces:



Response to a Perturbation

- Response to $f_{1y} \neq 0$: take $f_{1y} = 1$, since response is linear



Response Function for Rigid Packs

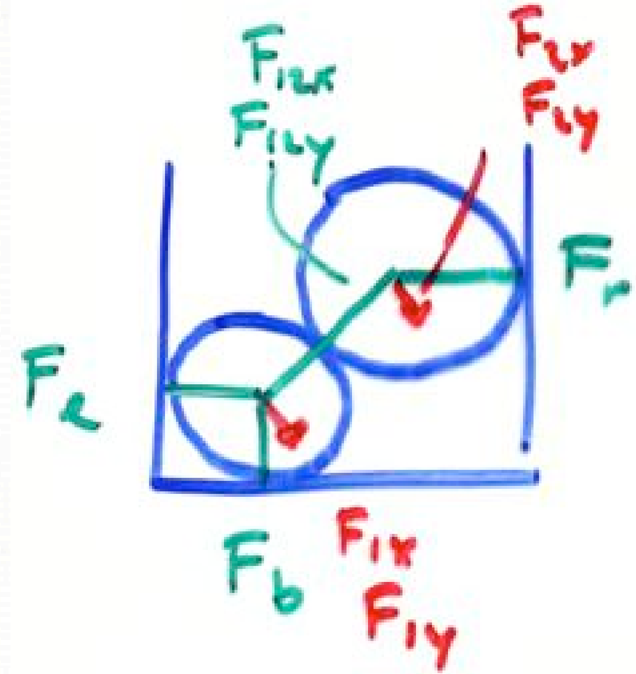
But $\underset{\sim}{P}$ is a unit vector usually

$$\Rightarrow \text{Response} = -\underset{\approx}{C}^{-1}$$

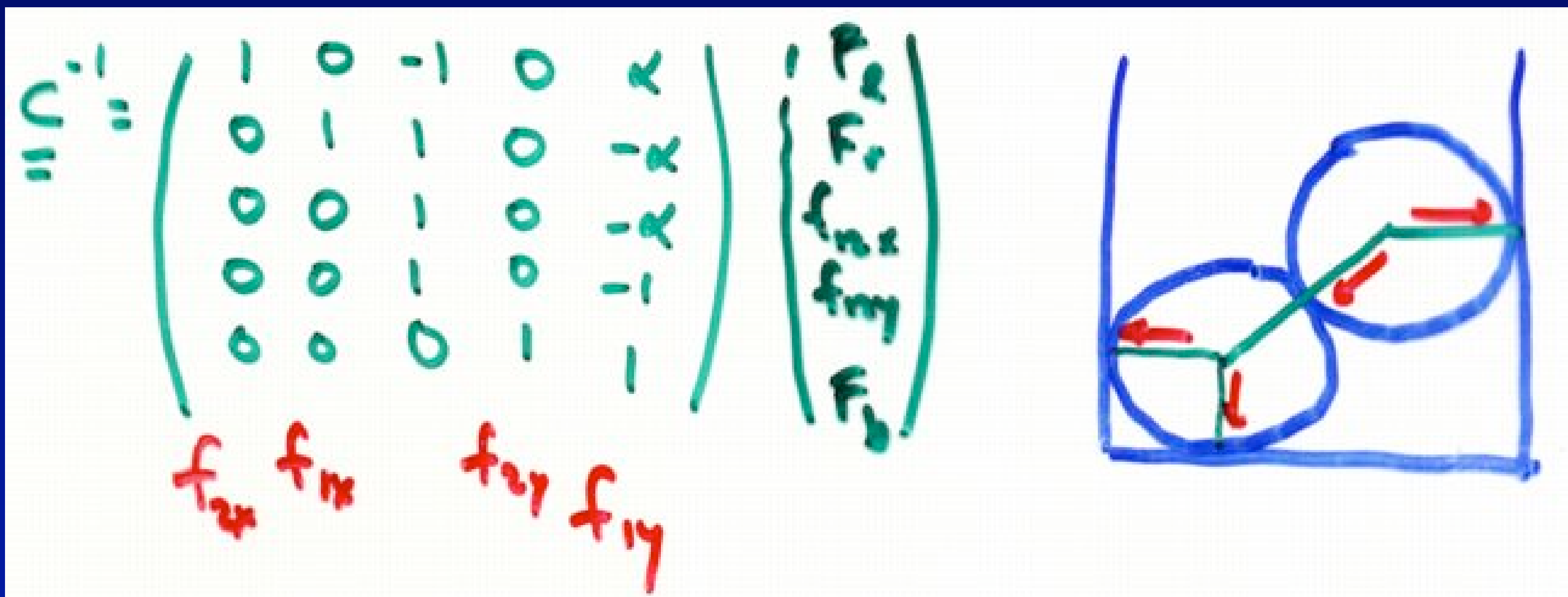
e.g.

$$\underset{\approx}{C}^{-1} = \begin{pmatrix} 1 & 0 & -1 & 0 & \alpha \\ 0 & 1 & 1 & 0 & -\alpha \\ 0 & 0 & 1 & 0 & -\alpha \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Response to: f_{2x} f_{1x} f_{2y} f_{1y}



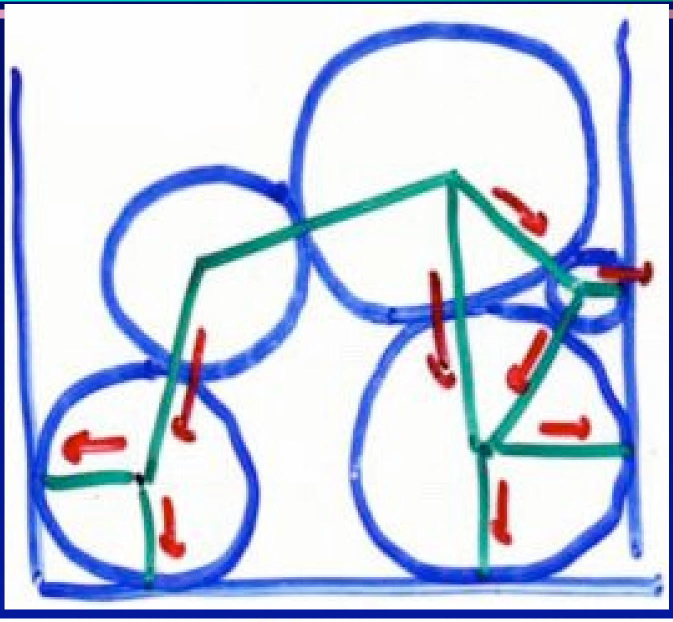
Force Response Hierarchy

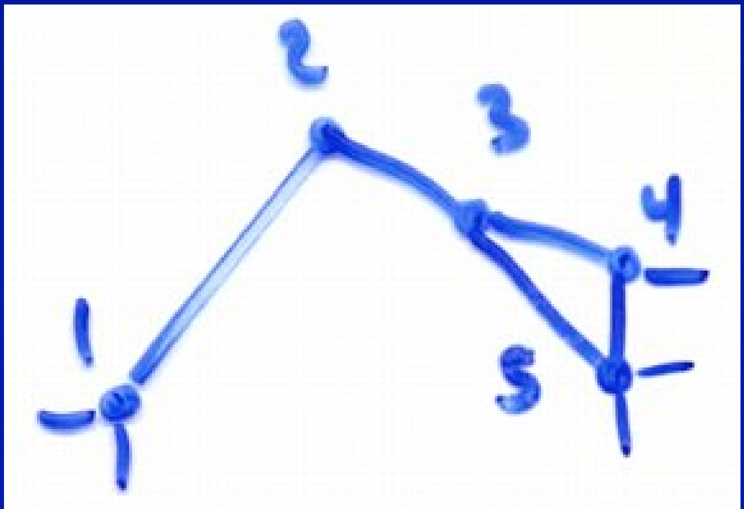


- Why is there a directionality?
  - Note: Not related to gravity!
  - Purely topological

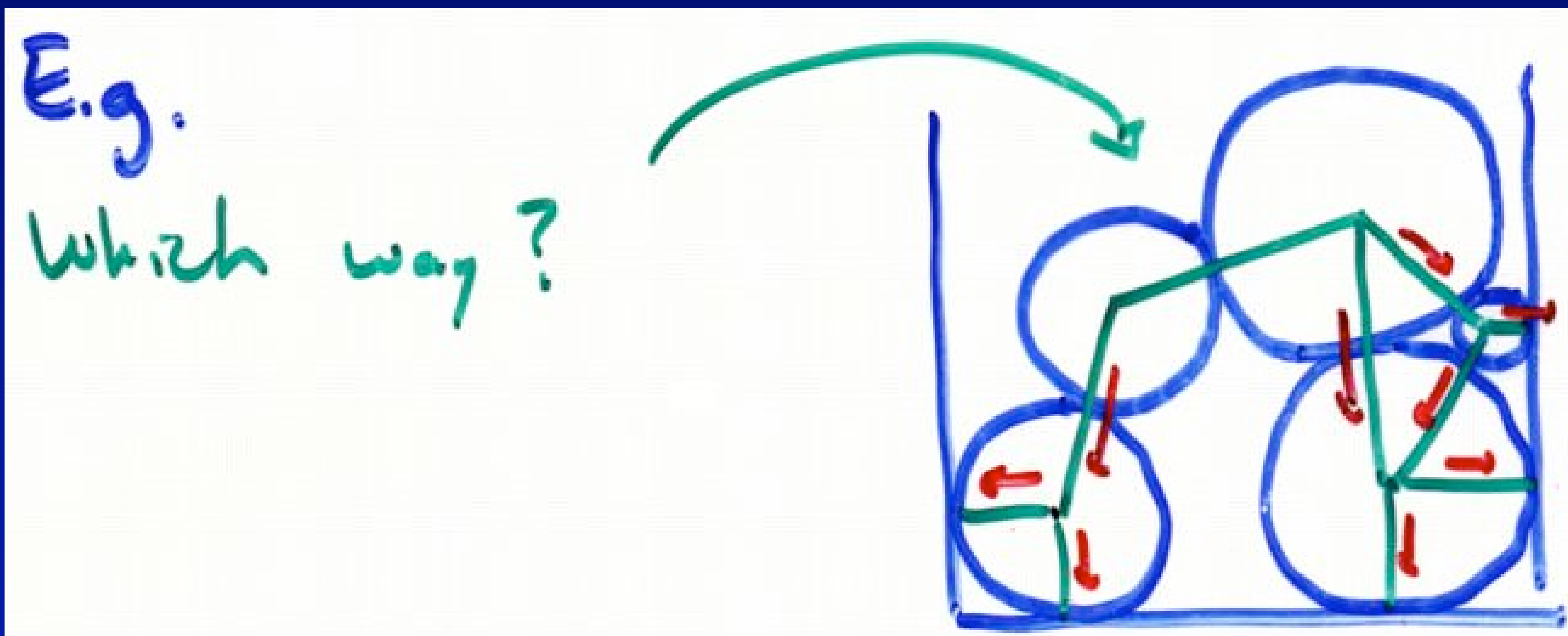
Sequential Hierarchy of Rigidity

- Rigidity starts at the boundary
- A particle (group) needs 2 (3) non-colinear bonds to be stabilized
- Once stabilized, its response function is "resolved" into that of the supporting particles
- The particle (group) effectively becomes part of the rigid boundary.

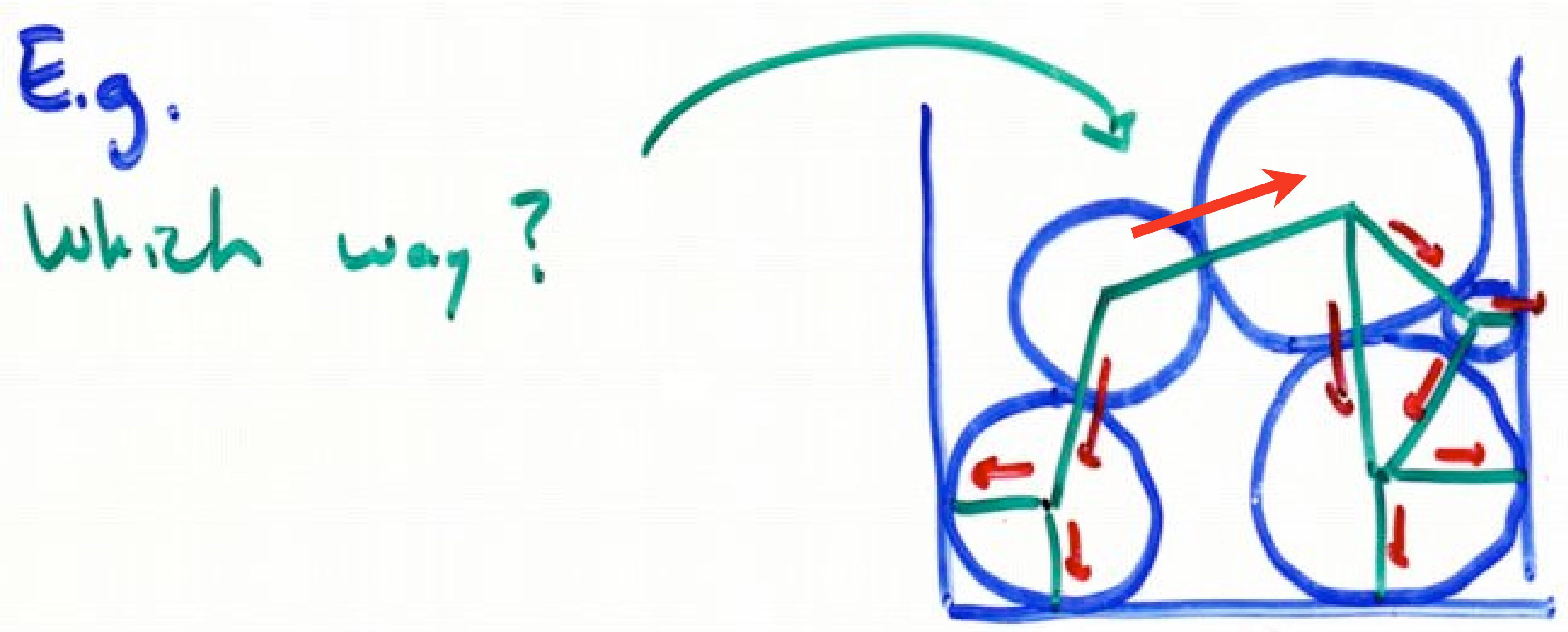




Quiz

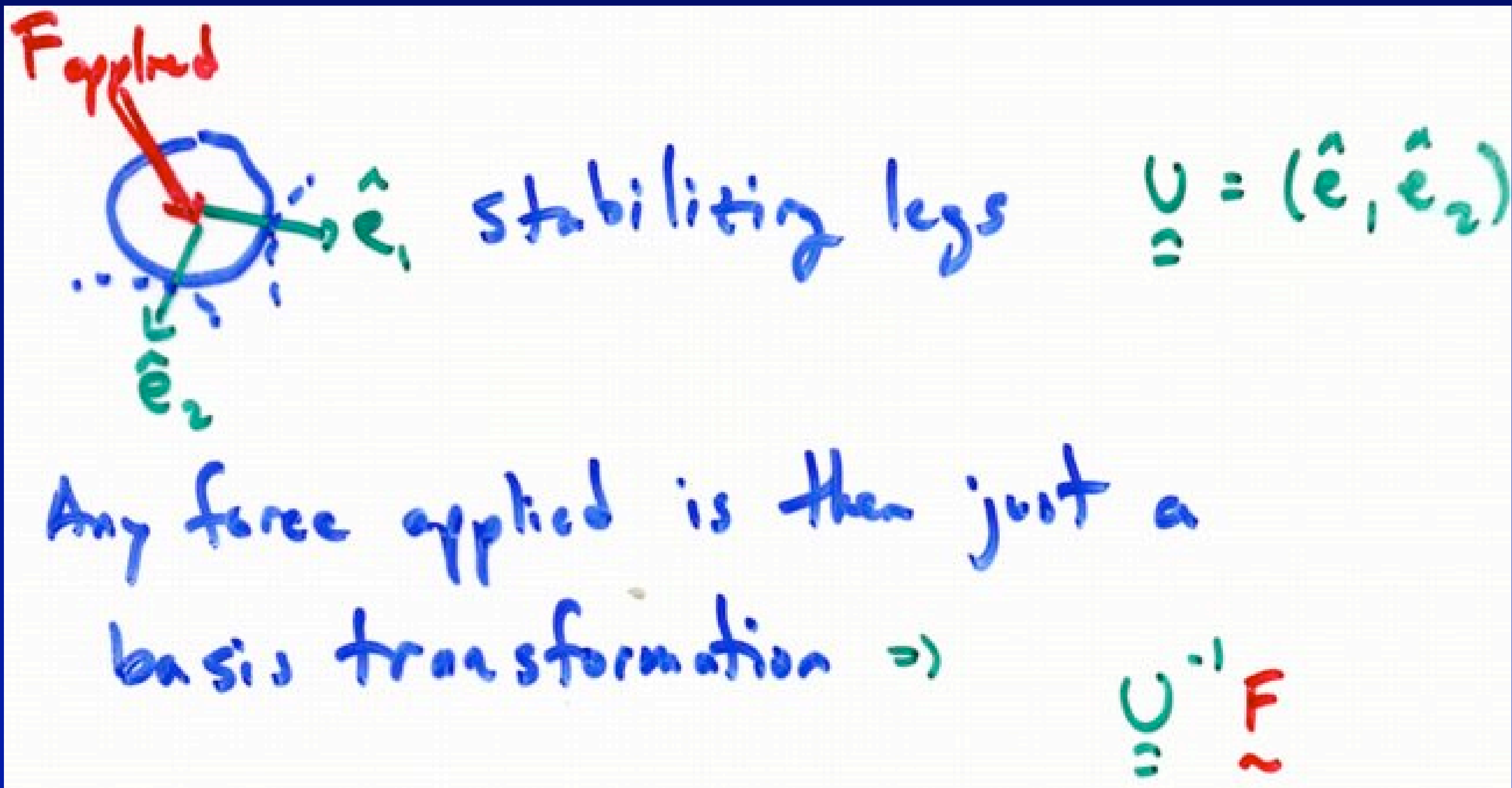


Answer

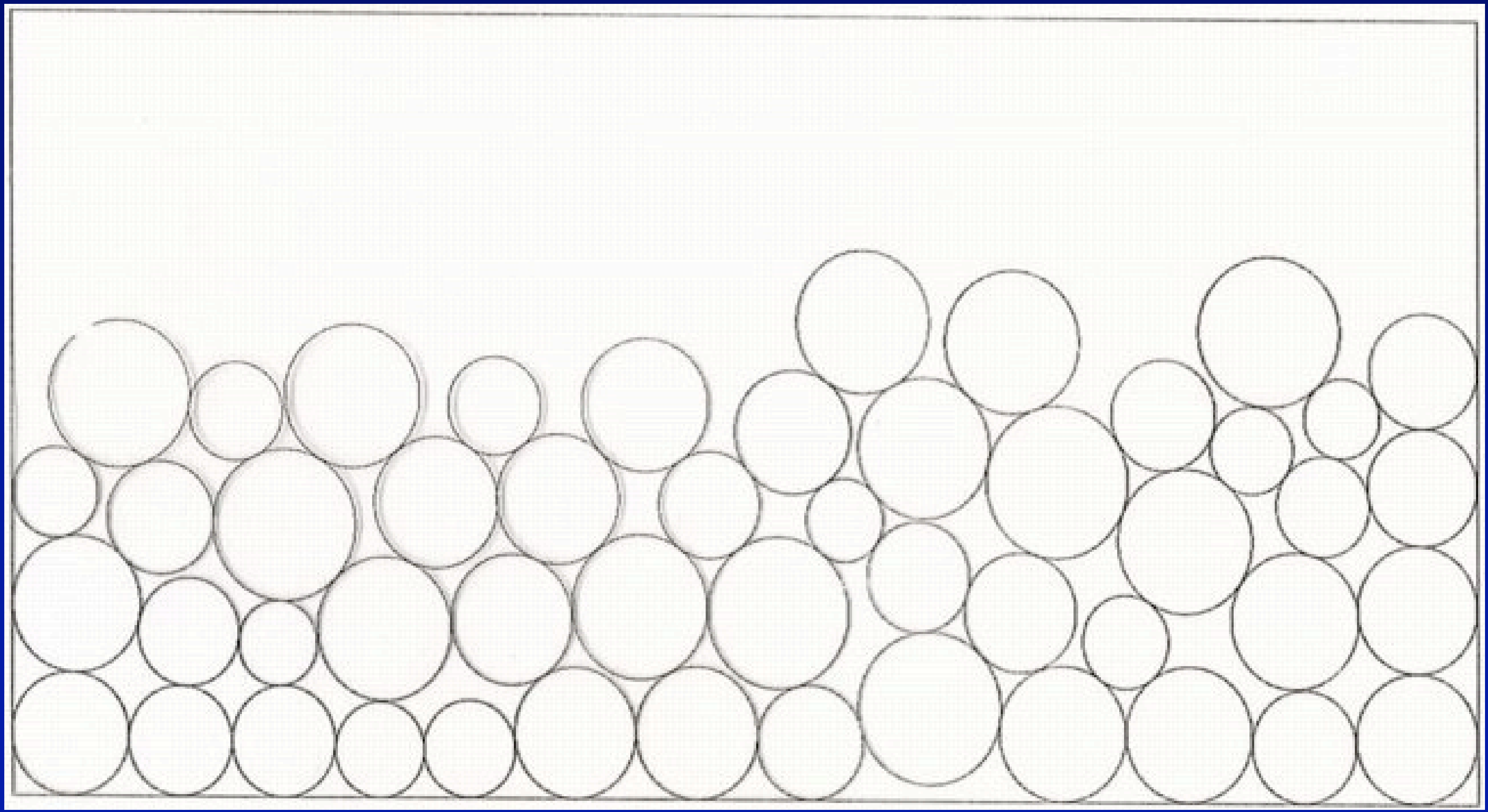


Every particle has exactly 2 outgoing bonds!

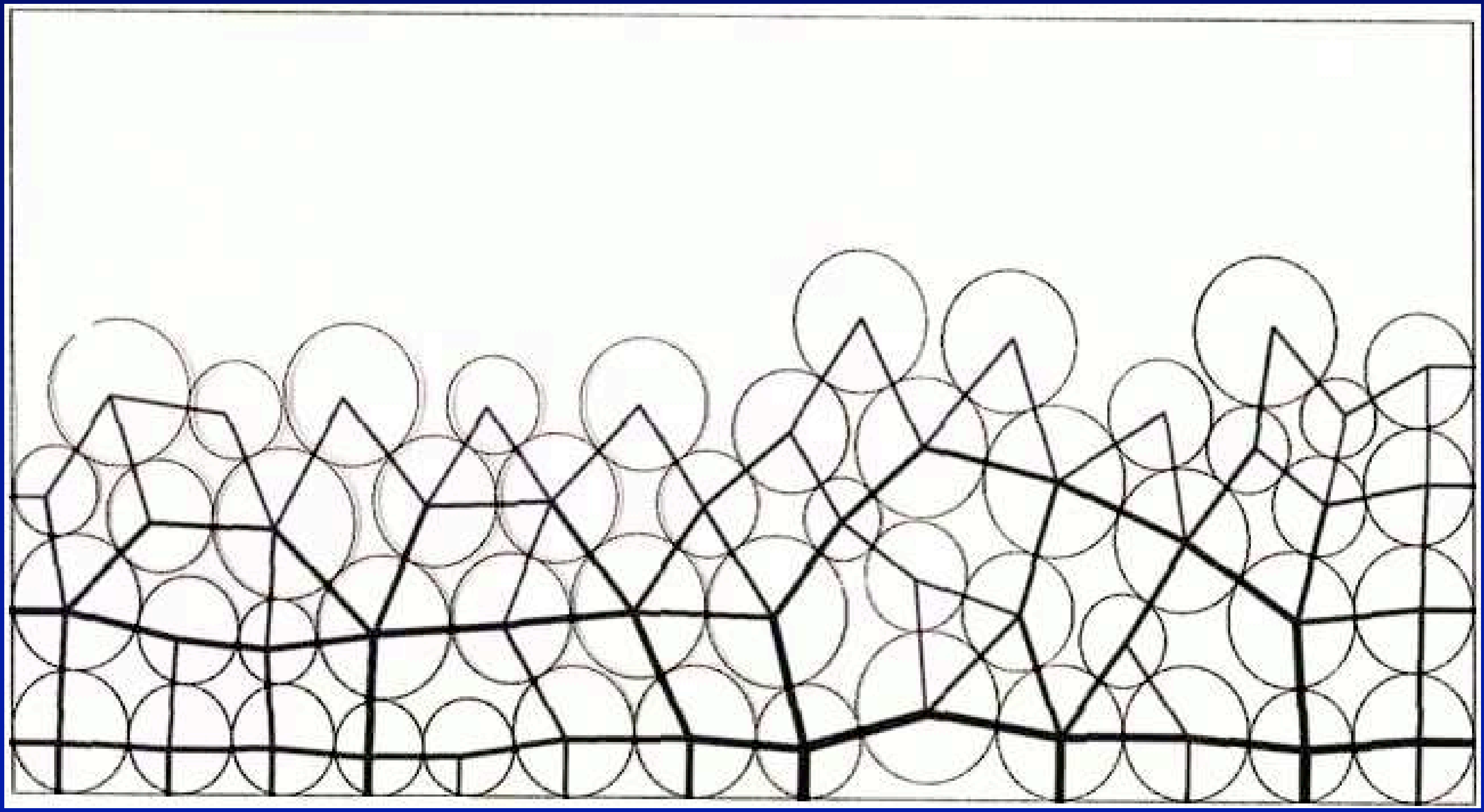
Force Resolution at a Particle



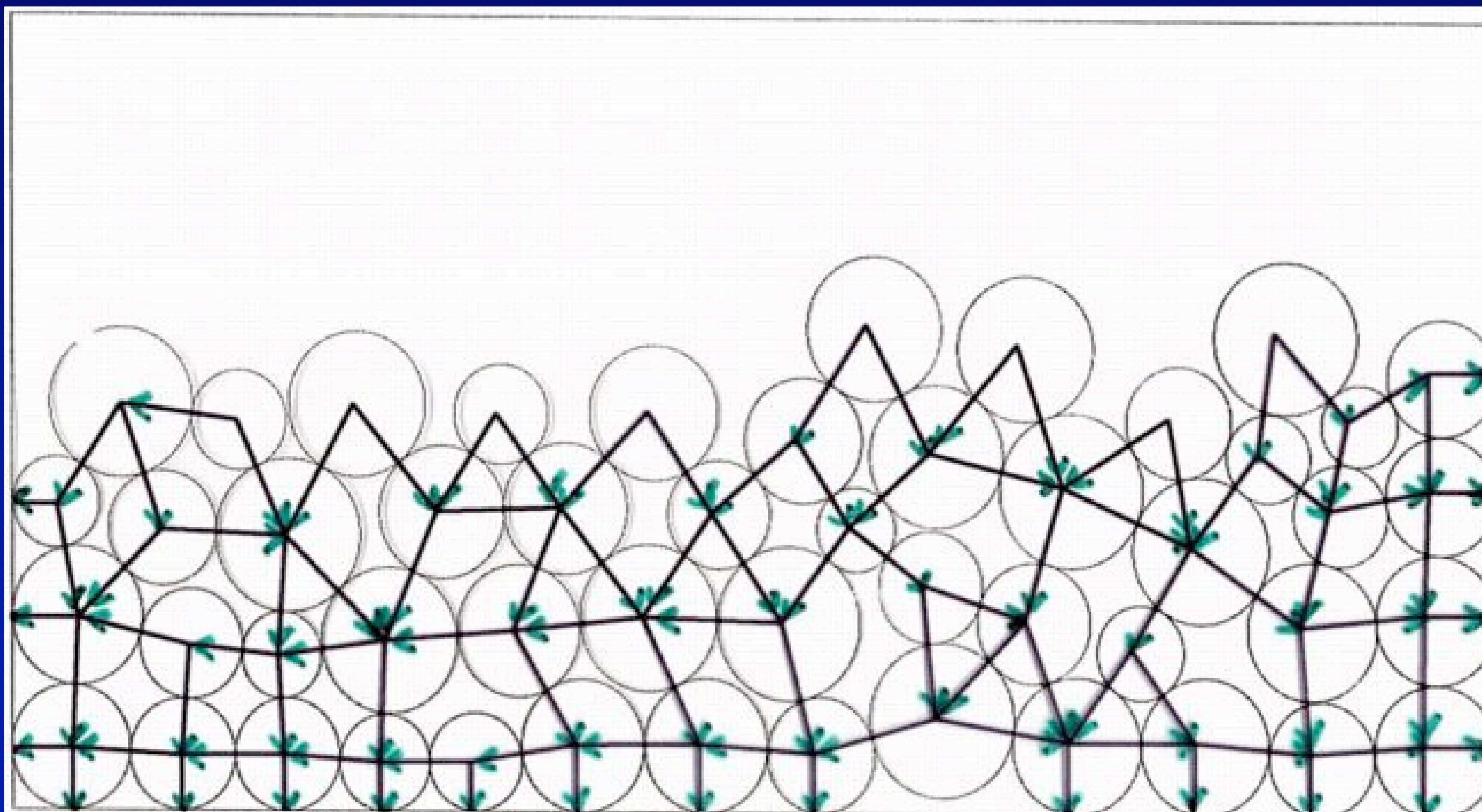
Particle Pile in a Box



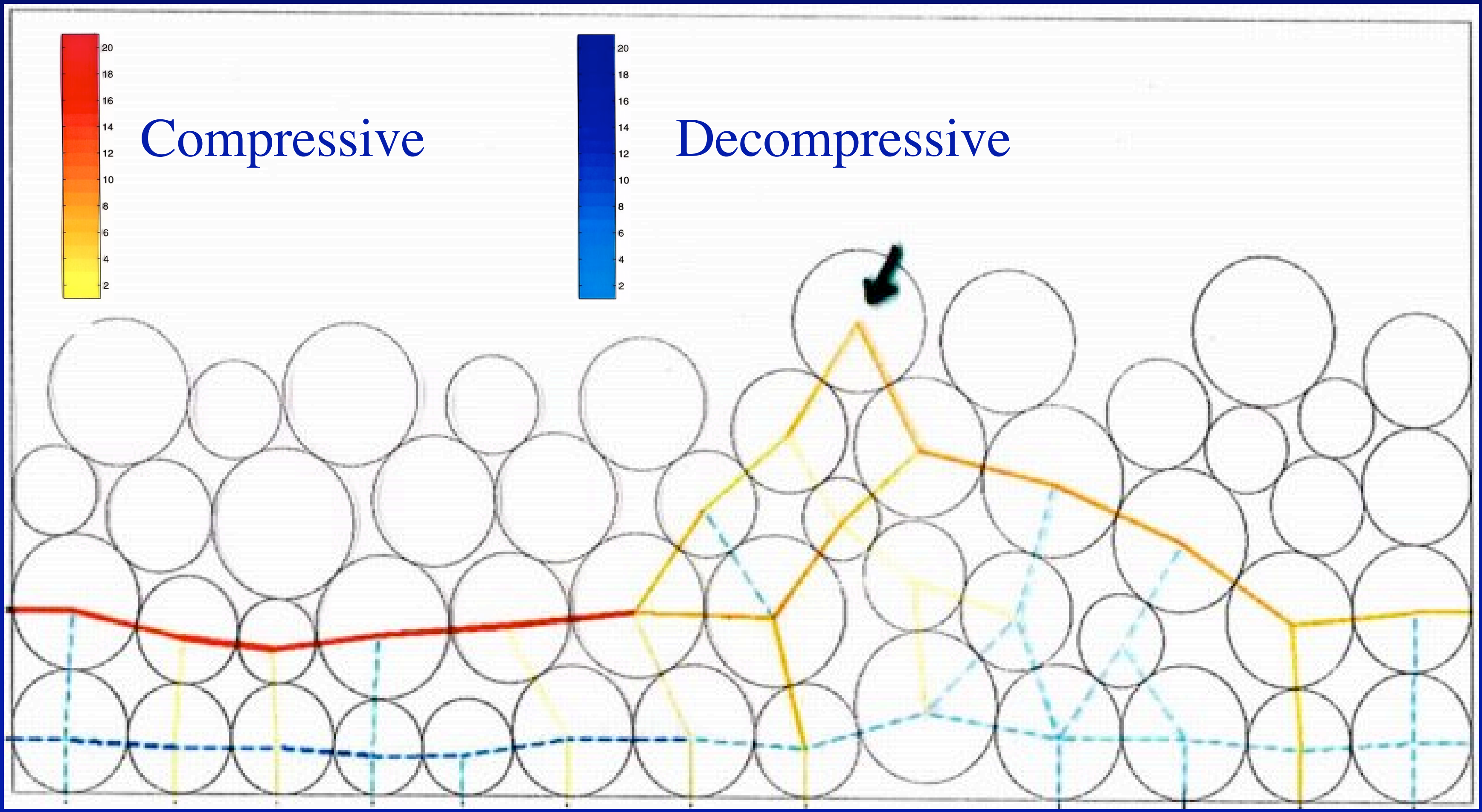
Conventional Force Chain Picture



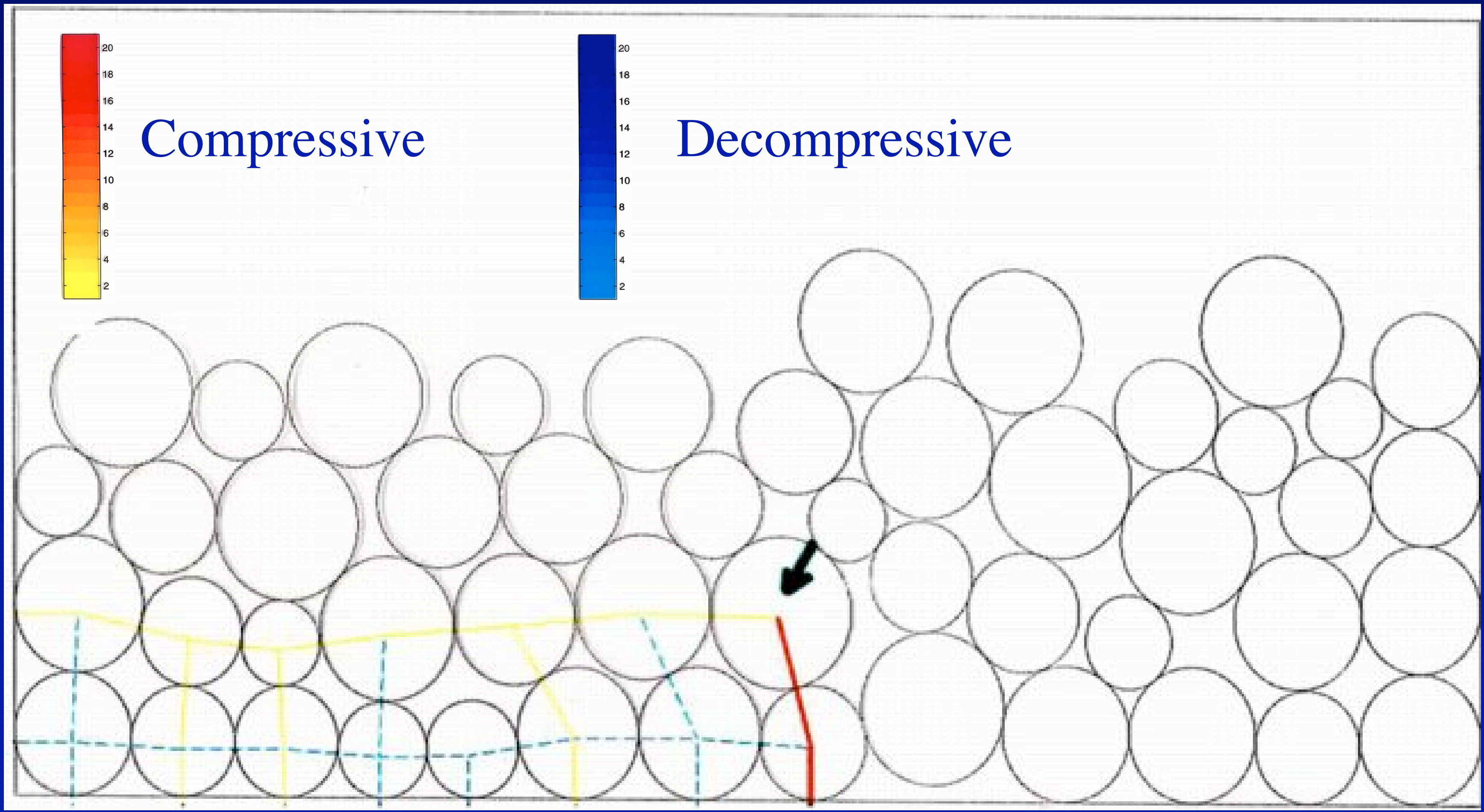
Underlying Directed Topology



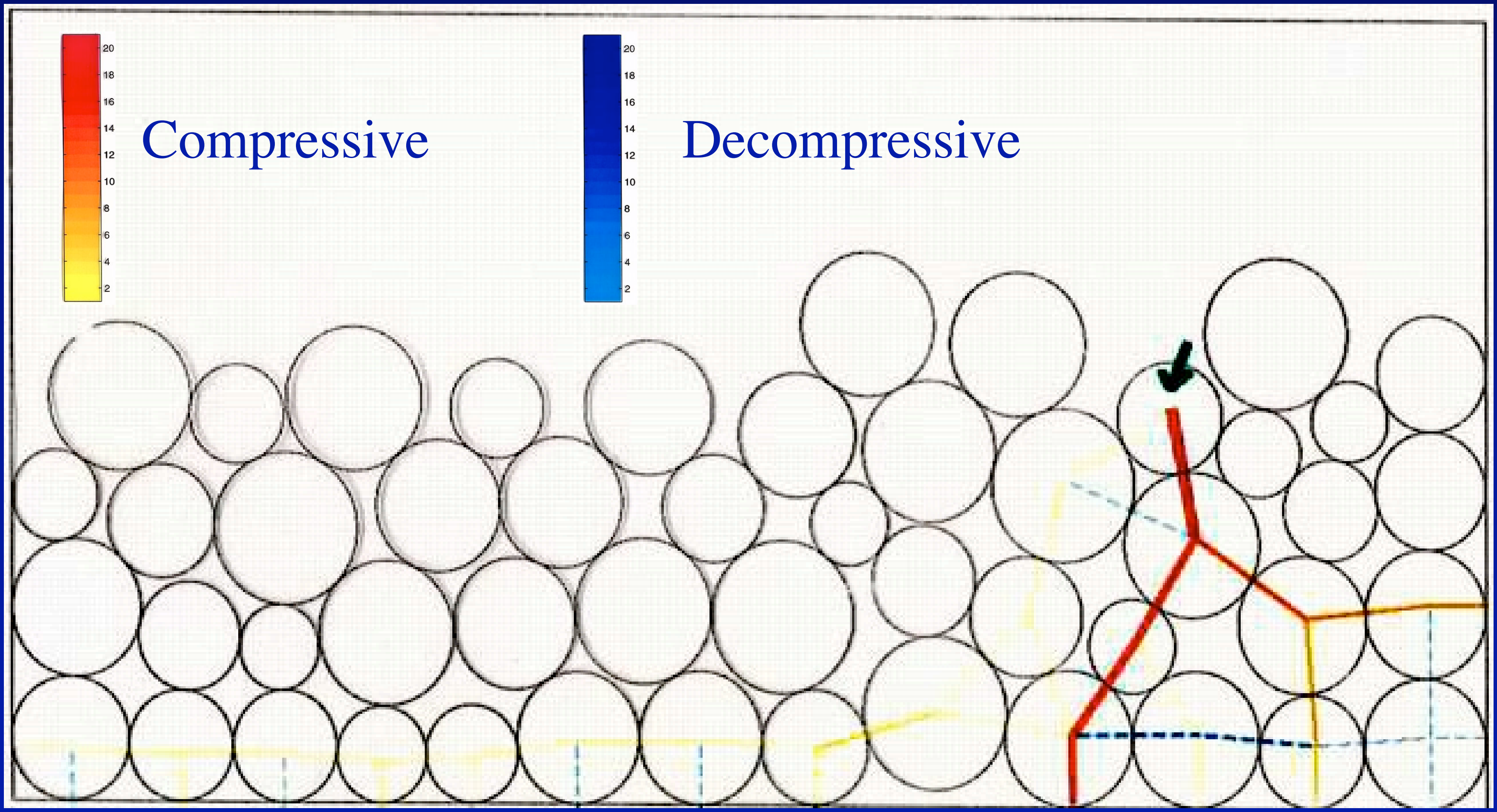
Response to a Force



Response to a Force II



Response to a Force III



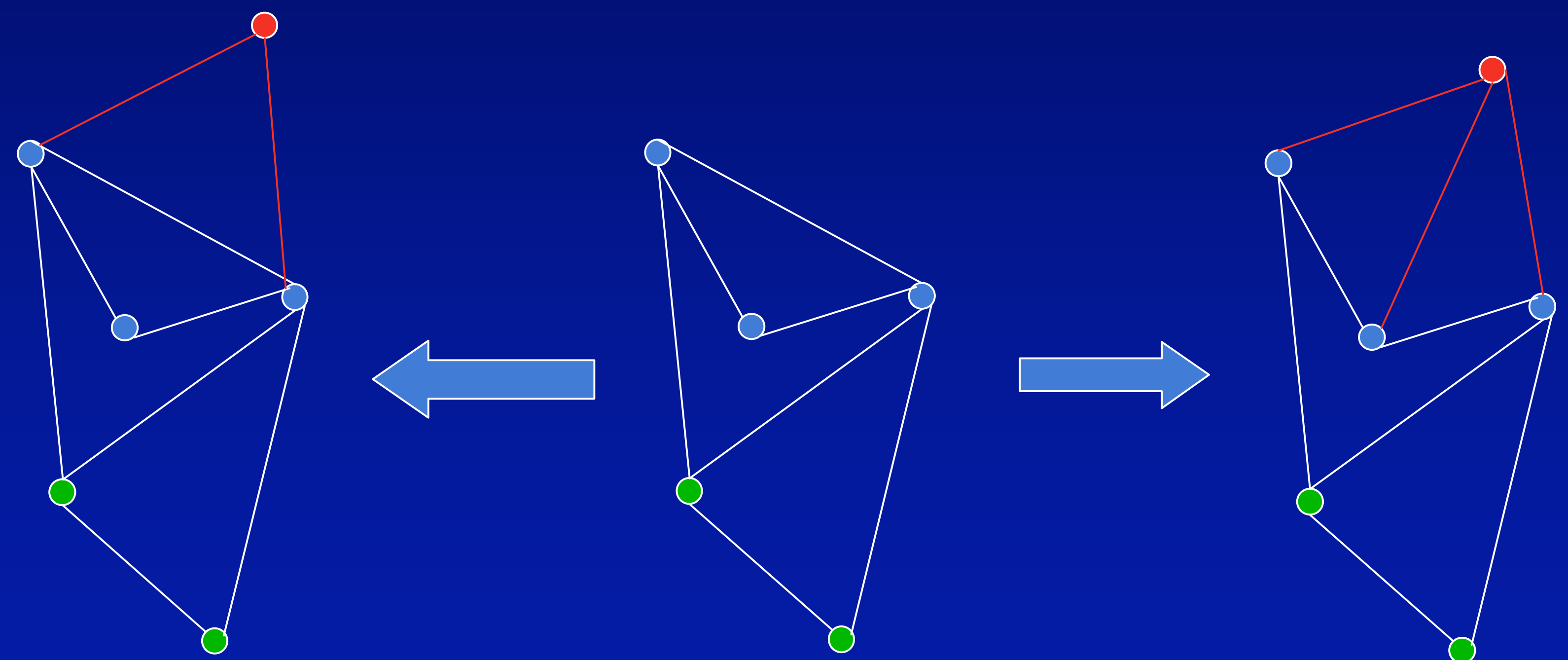
Rigidity Analysis

- Laman's theorem for isostatic graphs: every subgraph has less than or equal to 2 n - 3 edges
- Henneberg construction: every Laman graph can be inductively constructed from Type I and Type II additions of vertices.
  - Type I : add a new node and connect it to 2 existing nodes
  - Type II: subdivide an edge by a new node, and connect it to yet a different node

Henneberg Construction

Type I Step

Type II Step

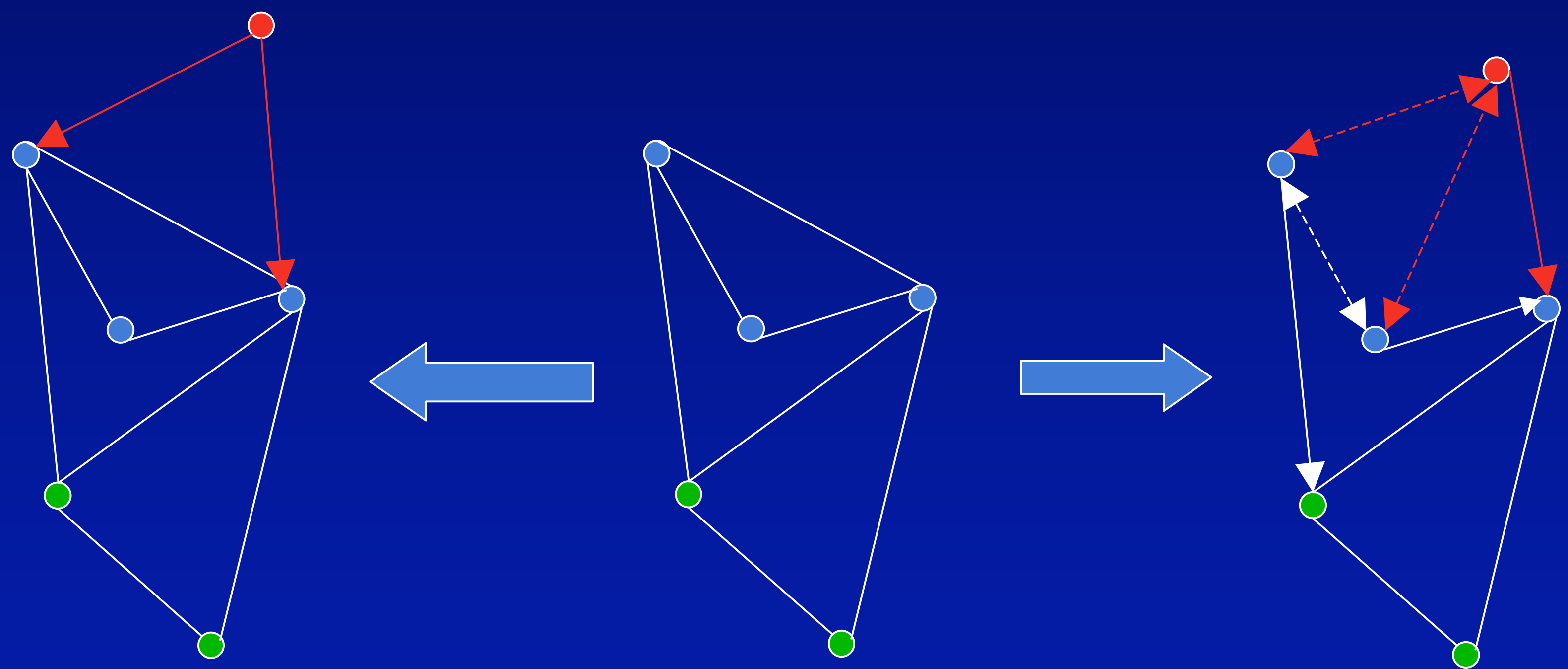


Green circles are pinned
(further towards the boundary in the hierarchy)

Henneberg Construction

Type I Step

Type II Step



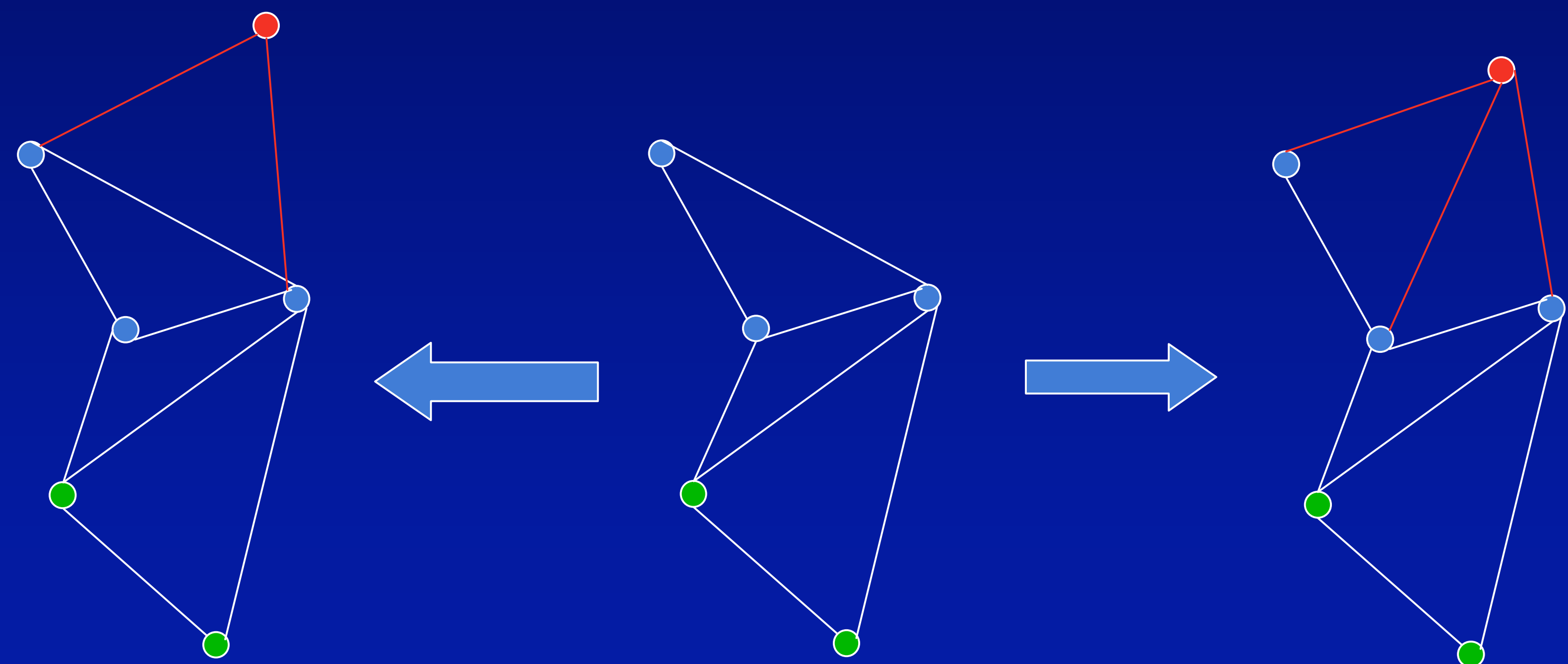
Leads to directed edges

Leads to a rigid group (hypervertex) with directed edges coming from it

Henneberg Construction

Type I Step

Type II Step

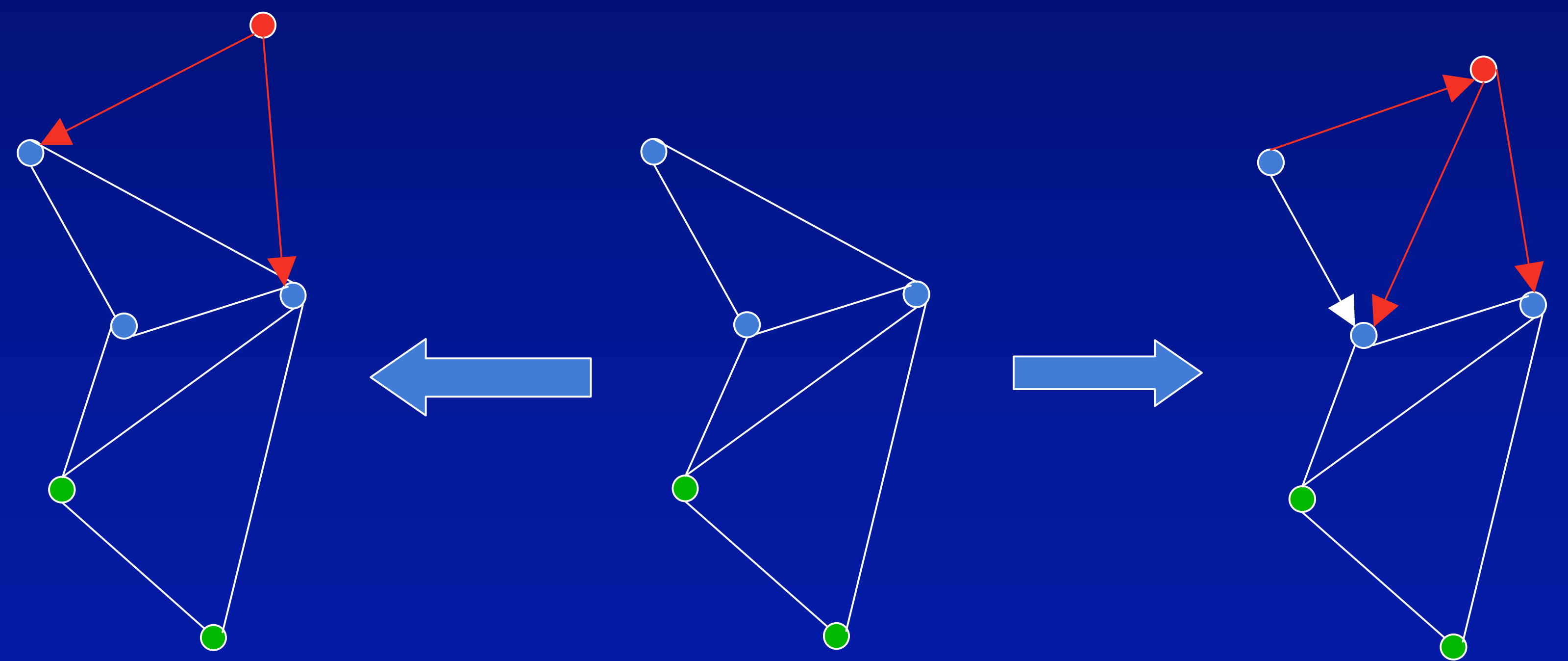


Green circles are pinned
(further towards the boundary in the hierarchy)

Henneberg Construction

Type I Step

Type II Step



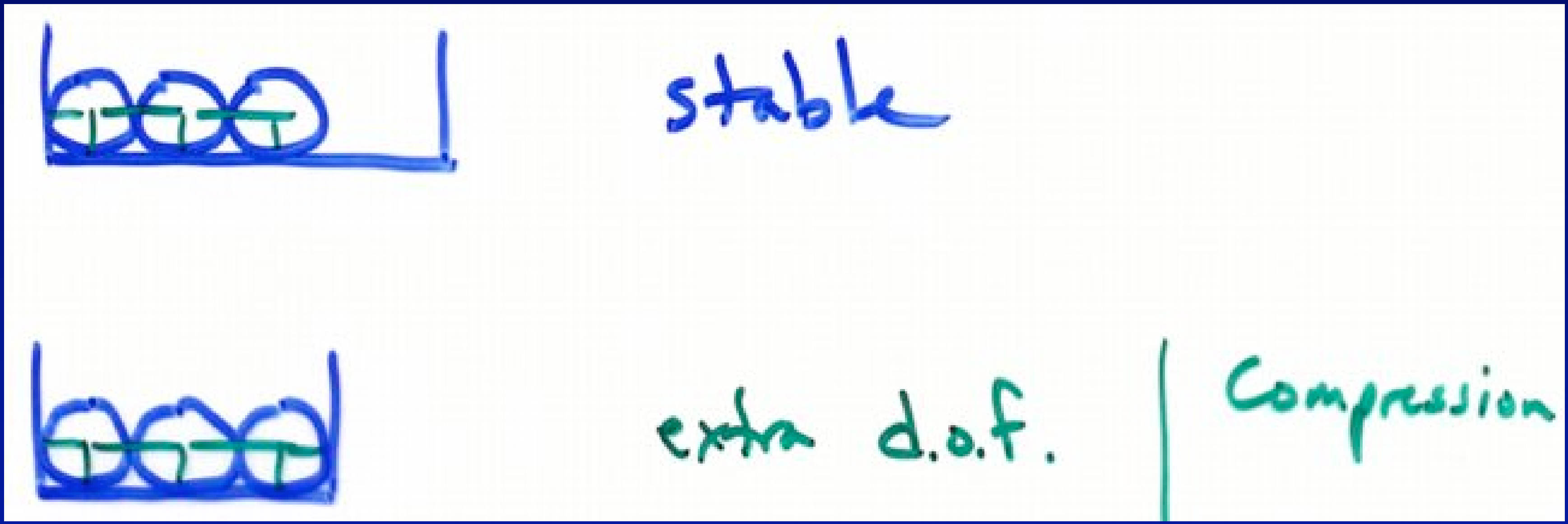
Leads to directed edges

Still leads to directed edges

Rigidity Decomposition

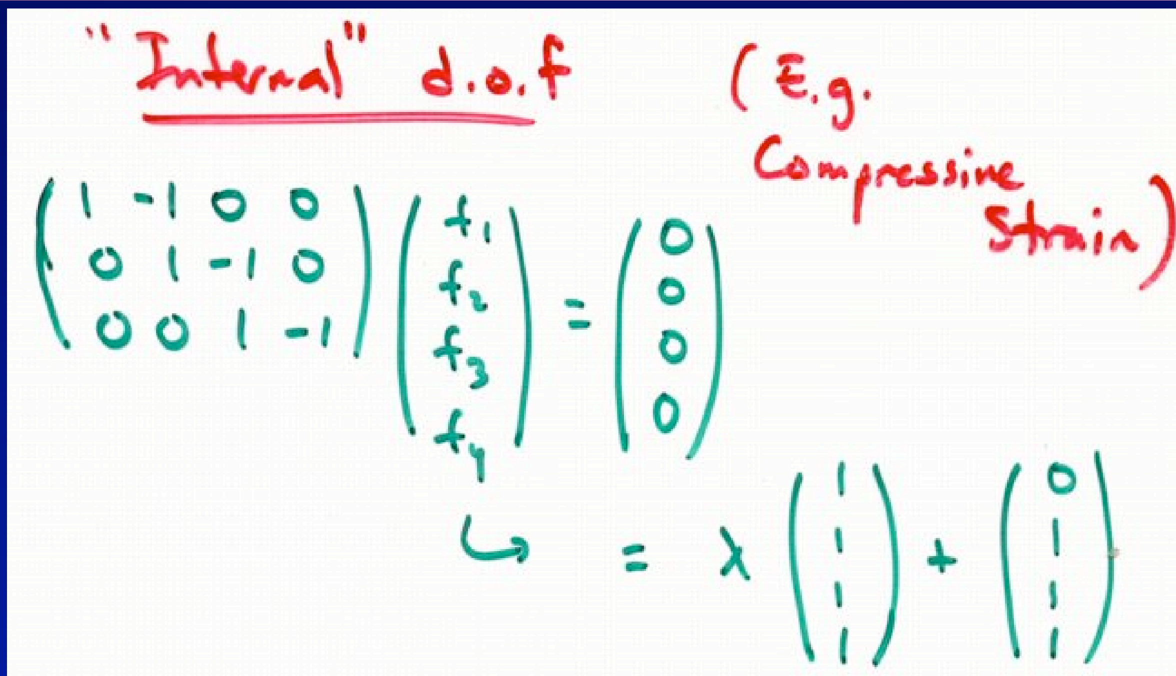
- Rigidity decomposes into particles (vertices) and groups of particles (hypervertices).
- In our gravity settled piles, only Type I steps are needed in the vast majority of Henneberg steps.
- We proceed with the approximation that we only consider graphs that are Type I constructable.

Soft or compressed packings: extra contacts



For this latter rigid configuration,
while the internal stress is controlled by the external wall stress
the force response function is not statically determinate.

Appearance of Elastic Modes



- The elastic mode (1,1,1,1) appears in an amount λ
  - λ depends on particle rigidities, etc.,
- The response is a linear superposition of (left & right) directed response functions

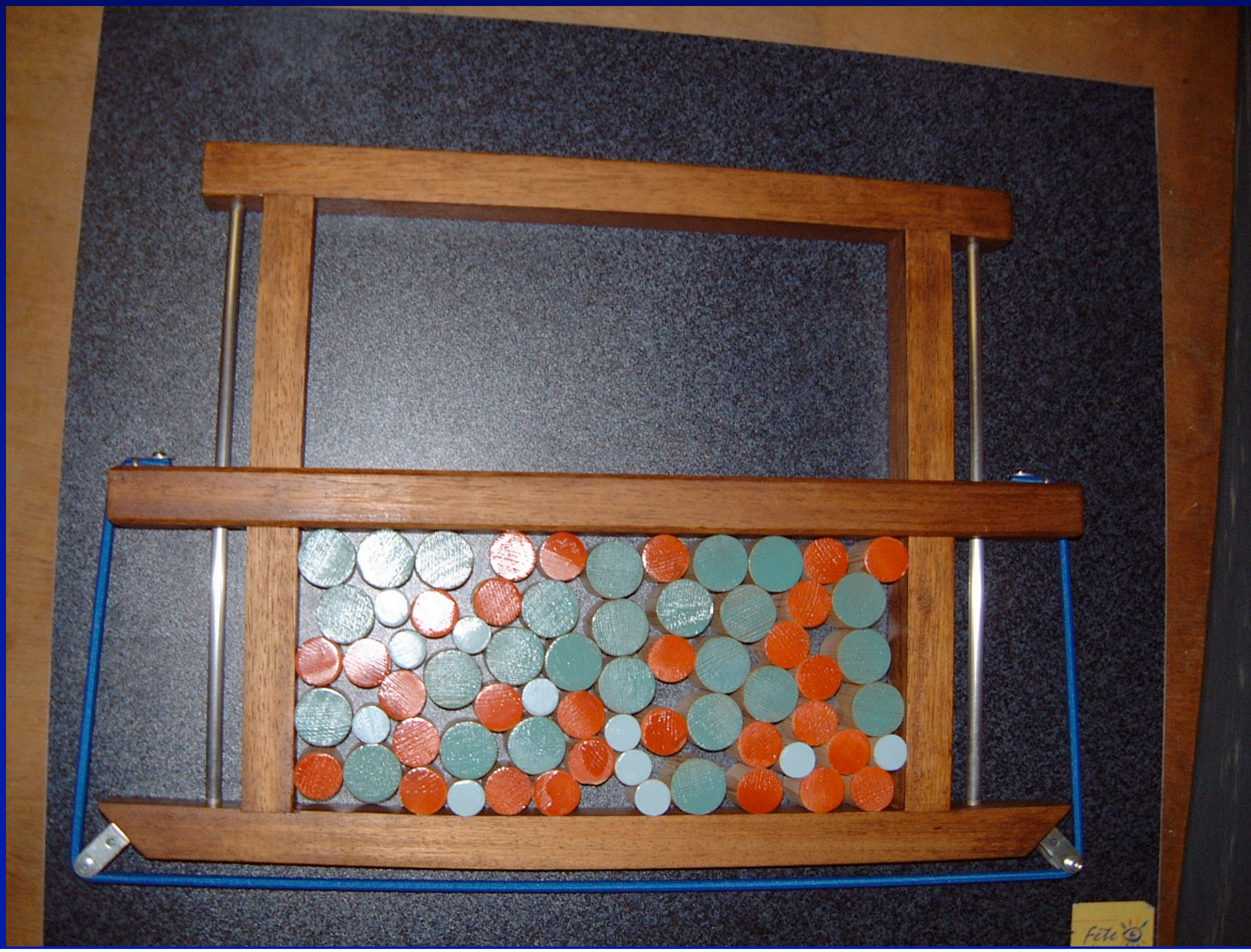
Conclusions for Force Chain Hierarchy

- Isostatic response function have a hierarchical topology that depends only on packing network
- This hierarchy can be built recursively from the boundary, with irreducible rigidity groups usually consisting of a single particle
- In this case, force chain links are directional
- Boundaries play a role--compare with continuum advanced & retarded Green's functions
- Each particle will have exactly 2 outgoing bonds & not just in sequential deposition.

Consequences of Force Chain Hierarchy

- Stress or force propagation occurs as a series of "scattering" events
  - Results in a non-phenomenological "Q-model" or Boltzmann eq^n that depends on outgoing pair geometries
  - Response functions, not force chains (their sum), are fundamental quantity in scattering theory
- Stress-field is localized and directed
  - signature of hyperbolicity
- Stress-indeterminacy leads to elastic modes that appear gradually with increasing contacts
  - sum of random amplitudes explains transition from exponential to Gaussian distribution of forces

Experiment or Toy?



Suggested Experiments

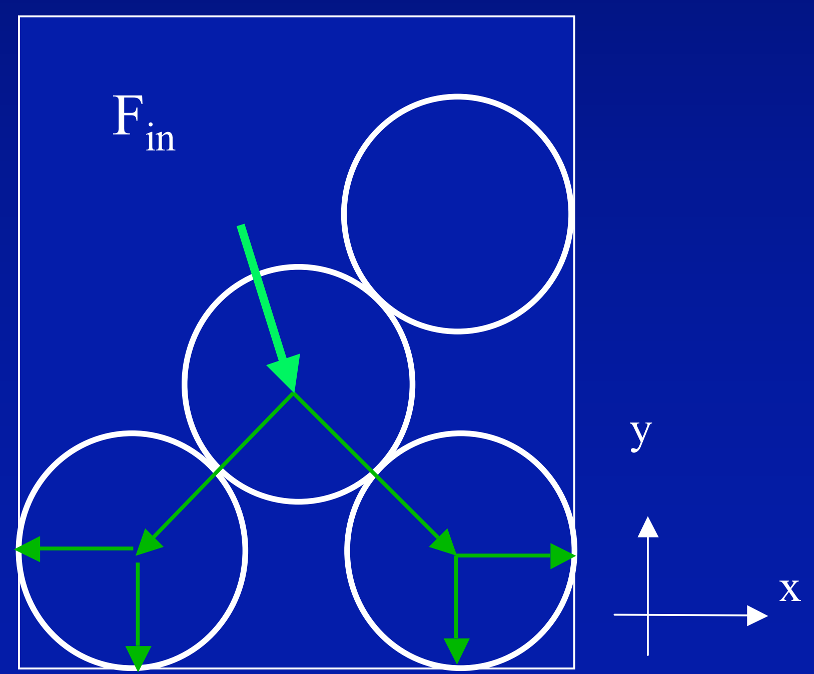
- Topological analysis of contact networks in experimental packs
  - gravity stabilized
  - compression stabilized
- Examine transition from hyperbolicity to elasticity with extra contacts by compression or softness
  - force distribution changes
- Measure force scattering kernel
  - relate correlation lengths to topological features

II. Statistics

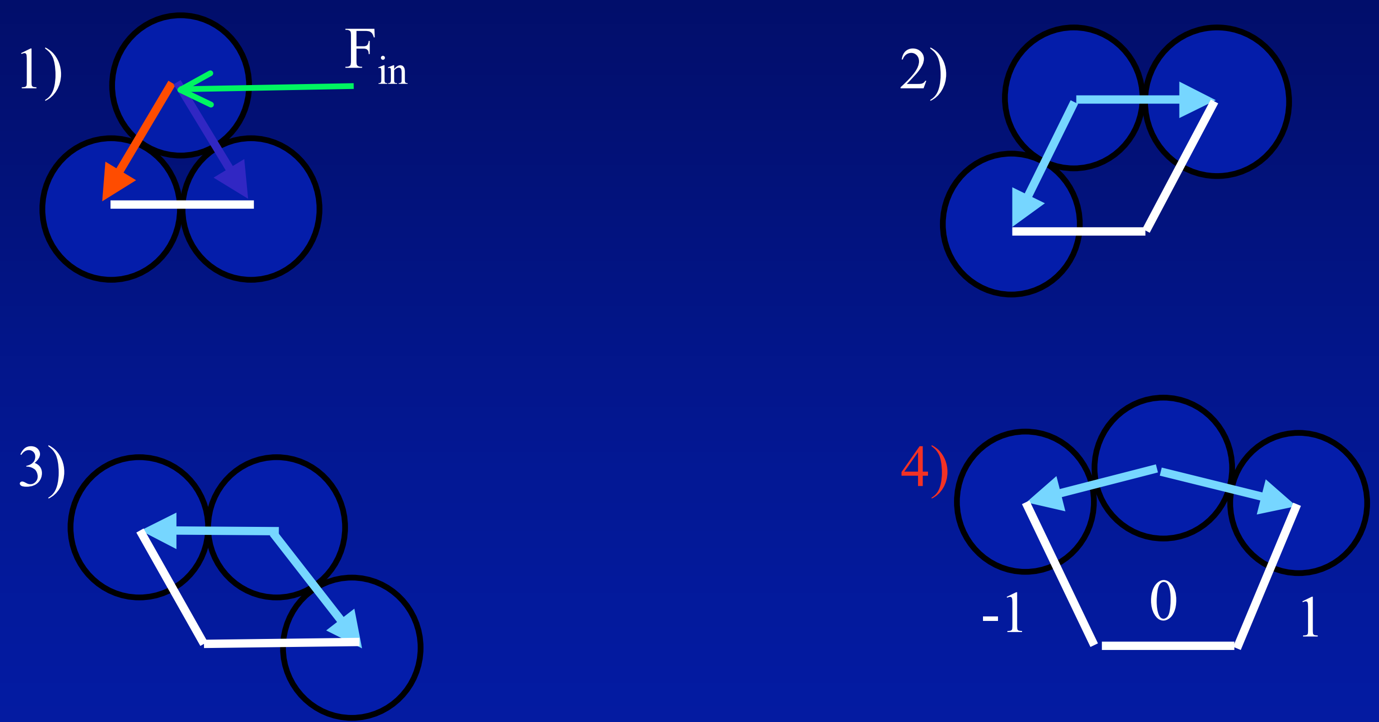
Let me count the ways….

Simple Model: Random Sequential Deposition of Smooth Hard Disks

- Each particle deposited is a little larger than the previous one-> well defined contact geometry
- Approximately a lattice



Analytic Theory: Response Function

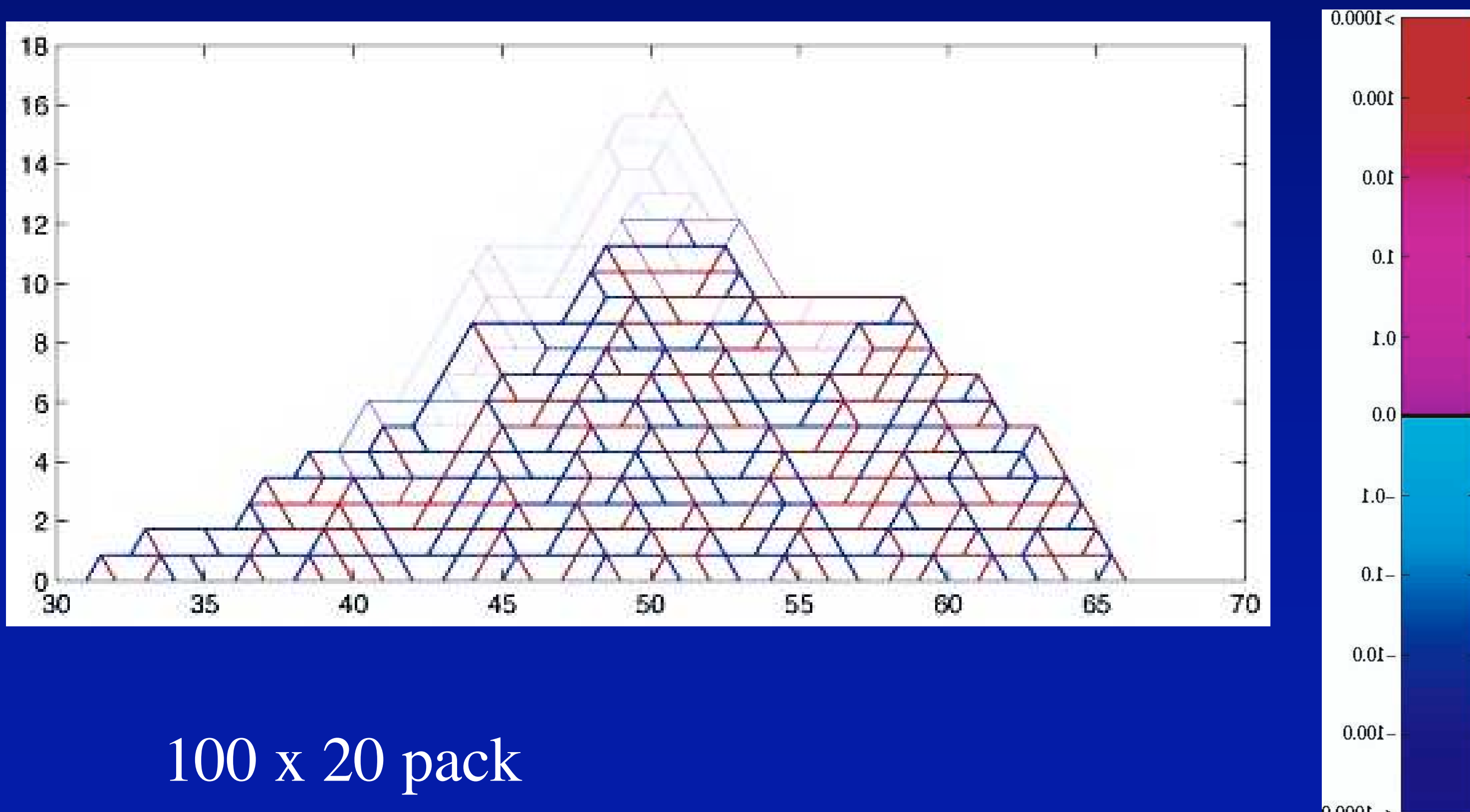


The four support geometries (types of scattering center)

The steady-state pack surface growth statistics can also be calculated.

Direct Simulation of Pack

- Single Instance Response Function

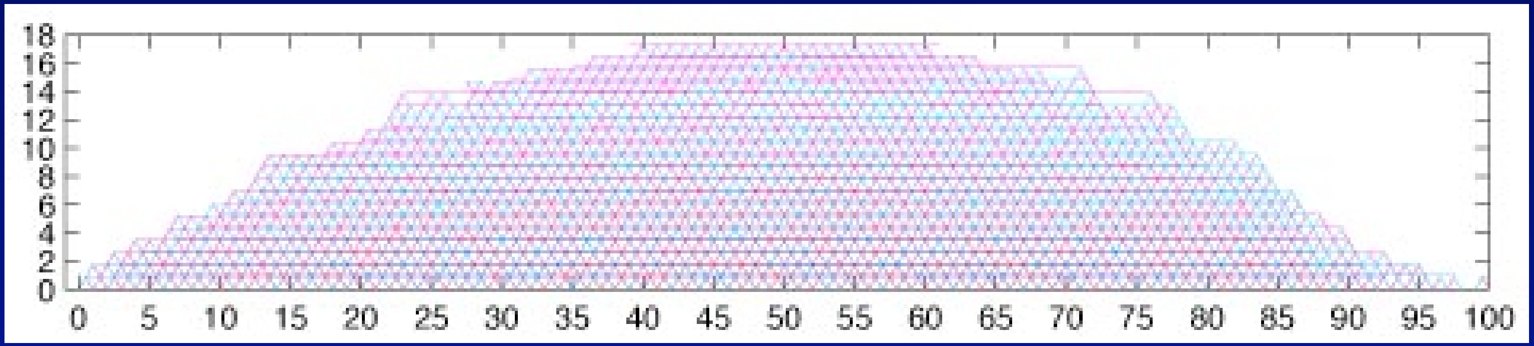


Compression

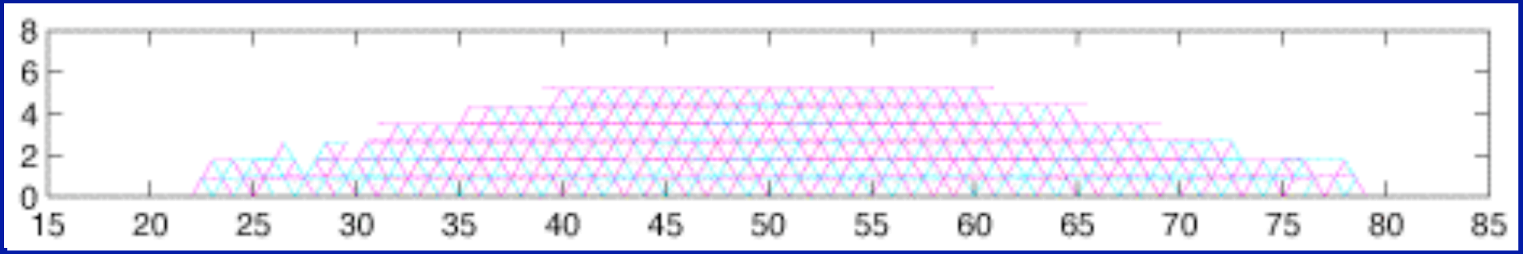
Decompression

100 x 20 pack

Ensemble Average: Large Fluctuations



100,000 instances

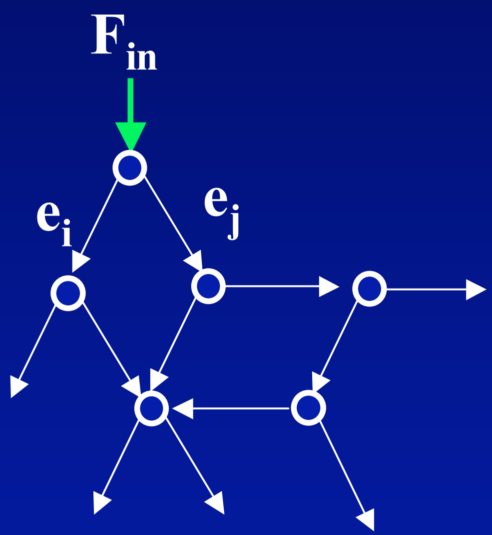


1,000,000 instances

Force Scattering: Boltzmann Equation on Quenched Fields

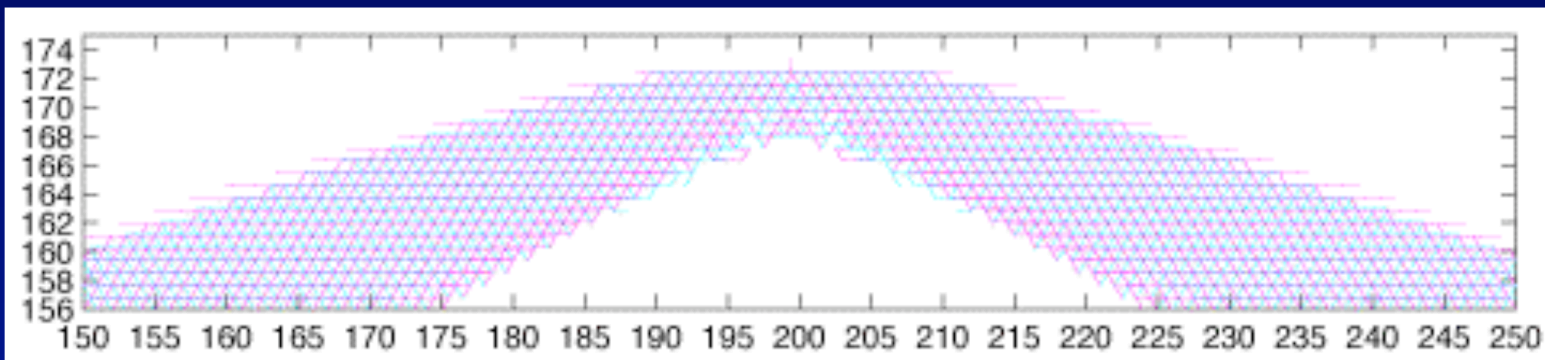
A force applied in the system is thus resolved sequentially by "scattering centers" fixed in space:

Correlated Boltzmann Equation on Quenched Fields

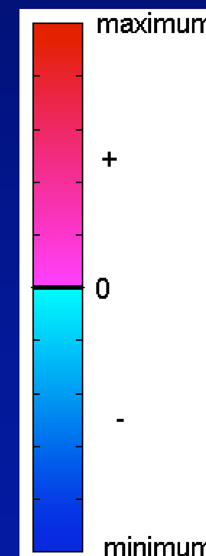


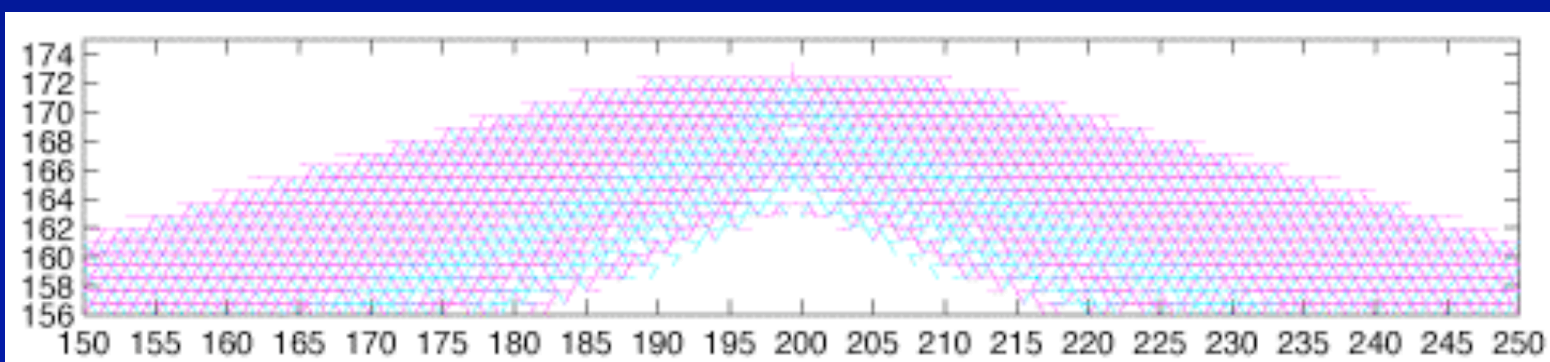
Analogous to wave scattering in inhomogeneous media, except "sequentially" doesn't correspond to any time:
Summing over sequential events gives force response function

Average Response Function (Analytic Approximations)



Uncorrelated Scattering





Correlated Scattering

Conclusions for Lattice Model

- A simple exactly enumerable case was studied
- Fluctuations are exponentially large
  - consistent with random multiplicative process Moukarzel J. Phys '02
  - but stably formed packs likely reorganize to avoid large stresses
- Off-axis bimodal response function was found
- Spatial correlations can significantly alter the response function
- Nearest neighbor correlation was inadequate for matching experiment beyond a couple of layers

Acknowledgements

- Tawiwan Kangsadan &
  Jeremy Lechman
- Particle Science & Technology Group, CSM
- Research Corporation
  - Research Innovation Award RI0161, 1997