Properties of Force Chain Networks

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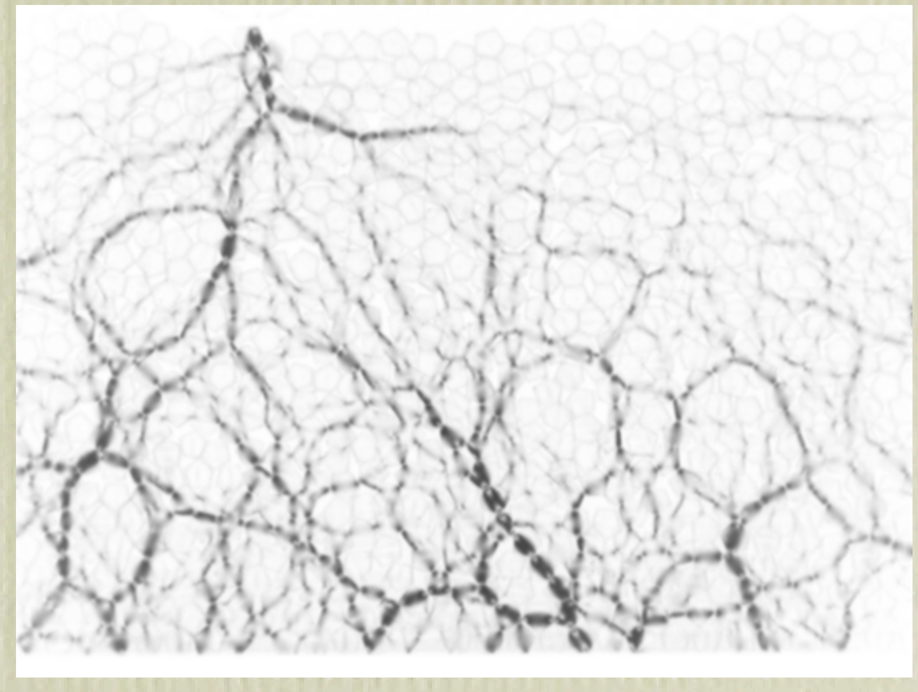

Experimental Inspiration:
R. Behringer, E. Clement
and their students

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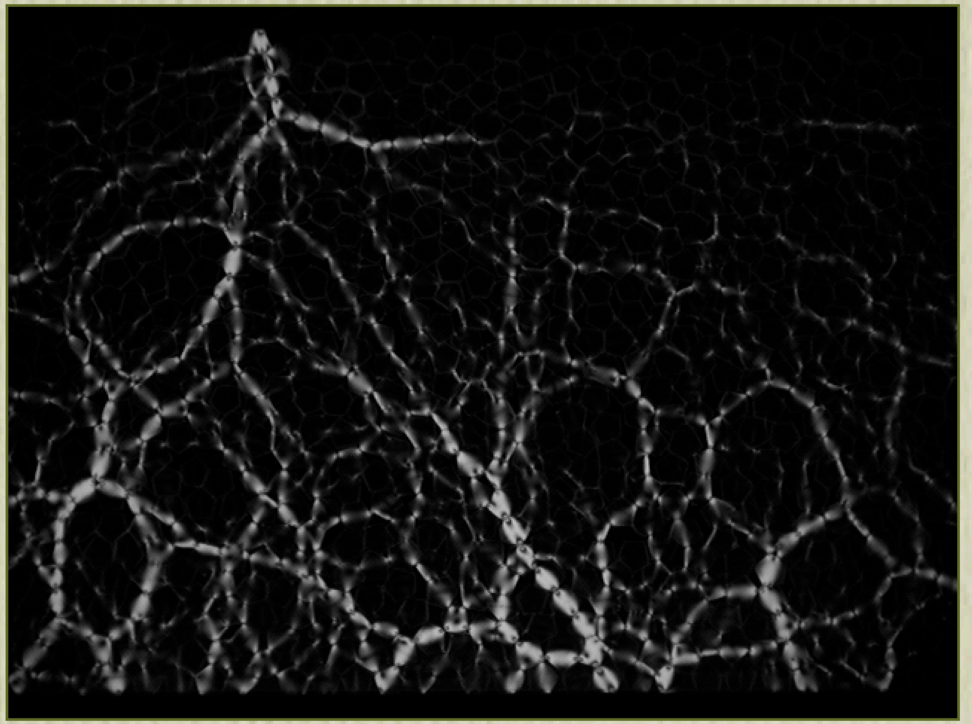
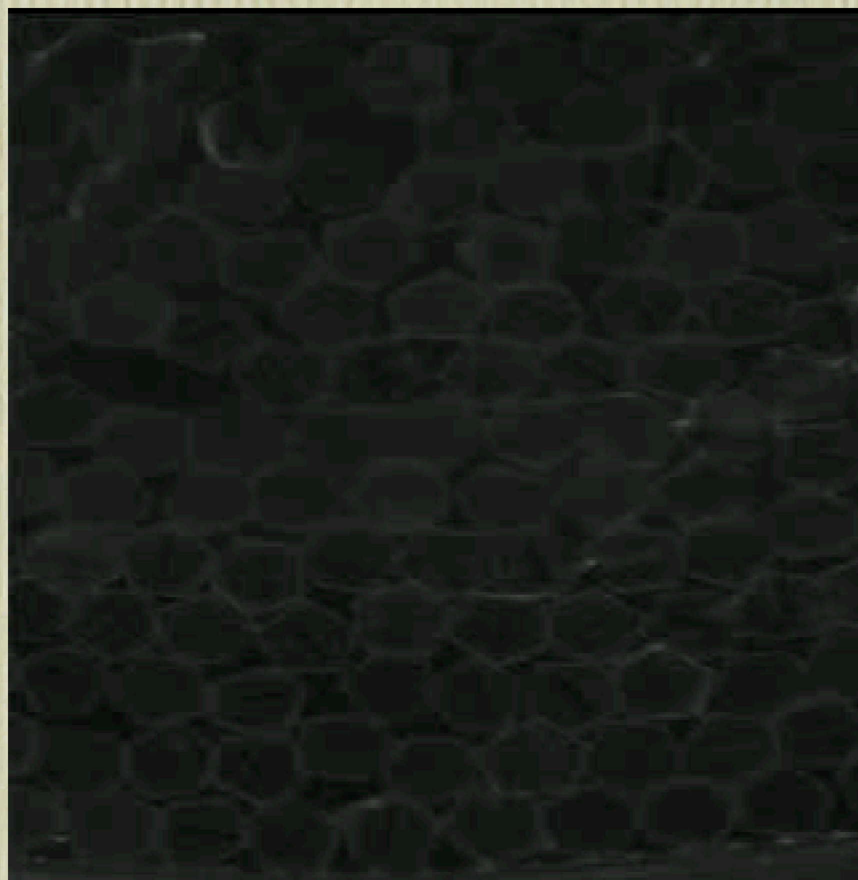
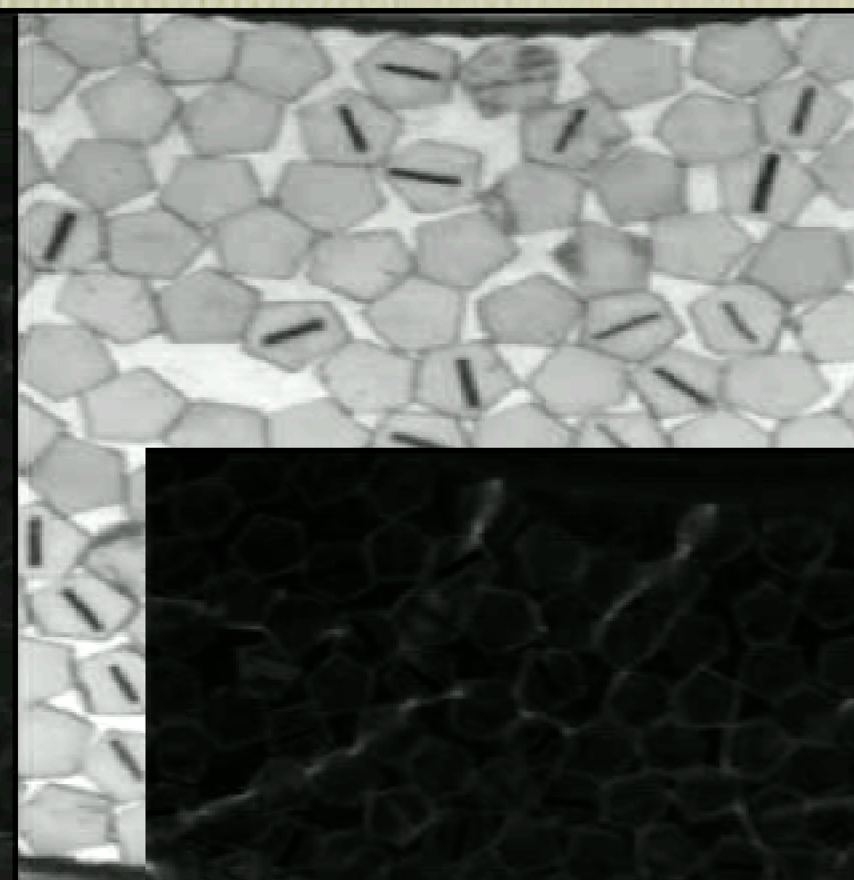
KITP, May 31, 2005

Grains – – Macroscopic Stress

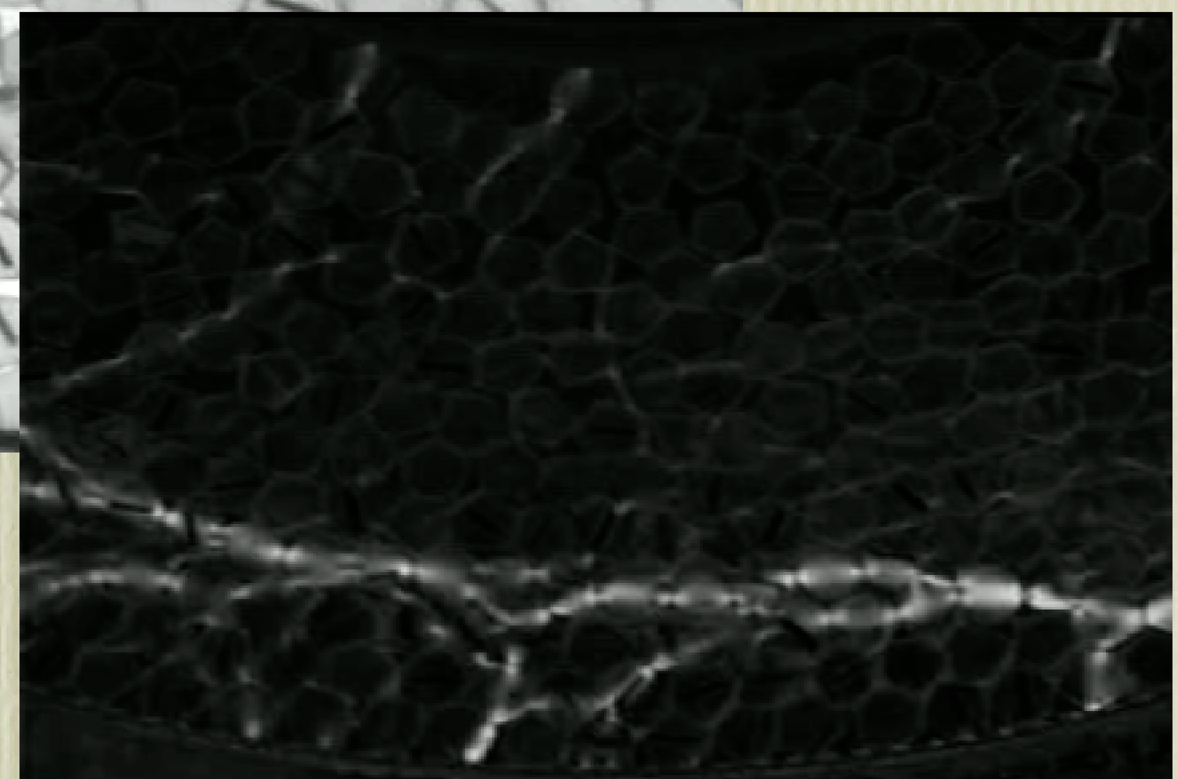


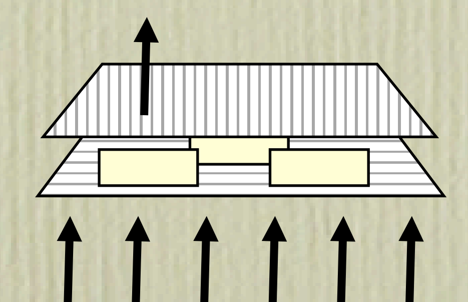
Dense granular material: Couette Shear





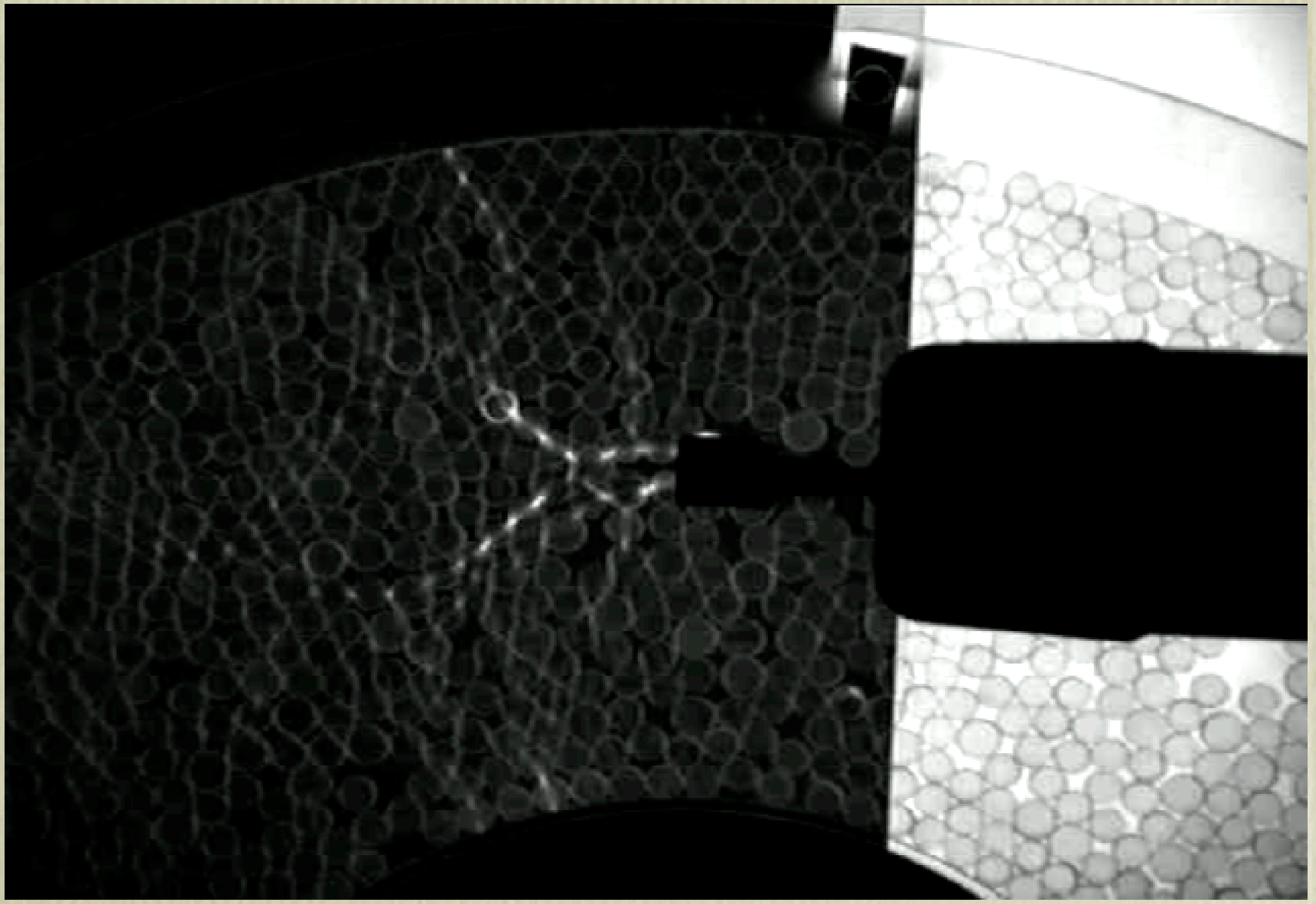
Bob Hartley





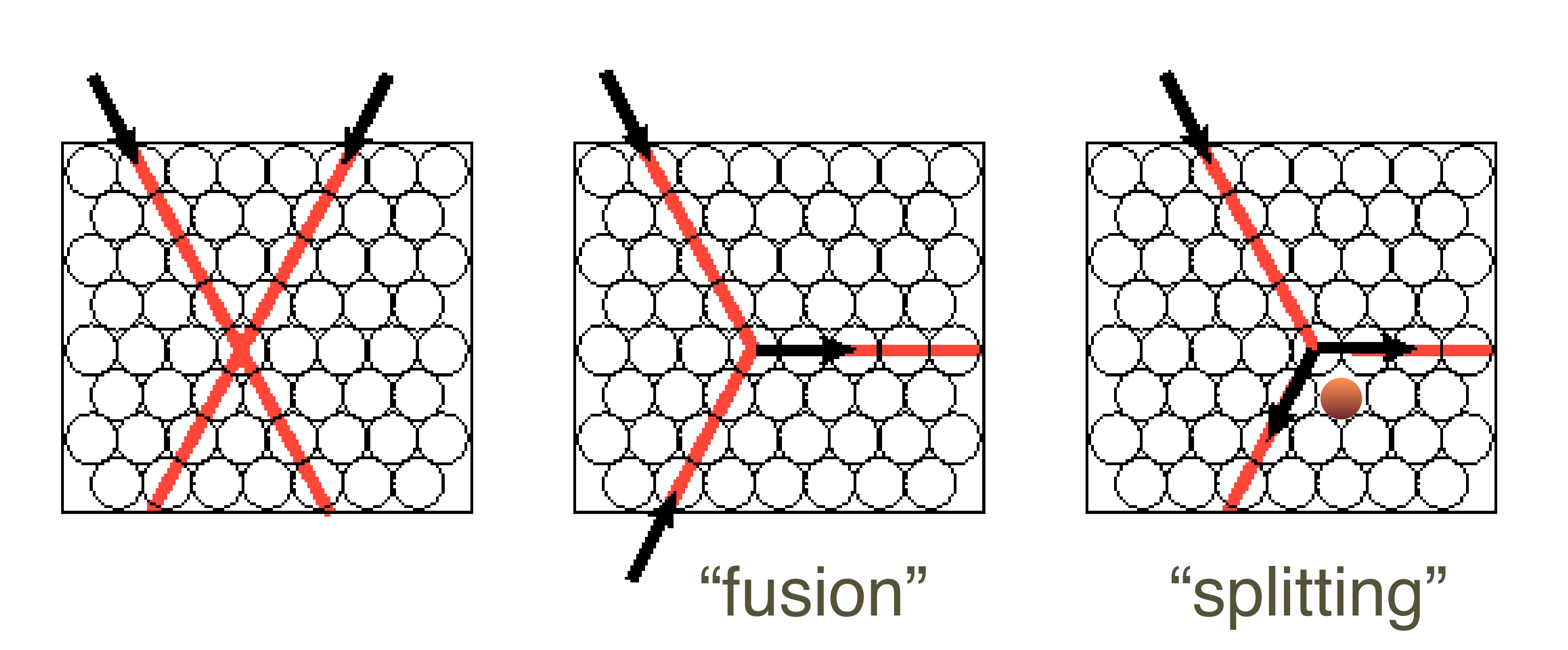
Flow past an obstacle

Junfei Geng



A Theory of Force Chain Networks

- Chains are injected at boundaries.
- Chains are straight and carry only compressive stress
- Chains are characterized by intensity f and direction θ .
- Direction of arrow is determined by boundary conditions.

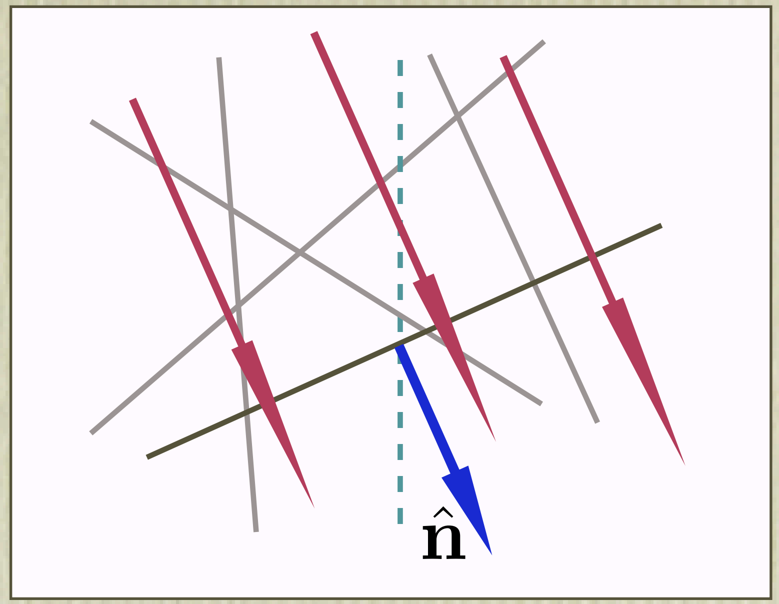


Schematic -- not to be taken literally.

Variables and Physical Quantities

- **Chain density variables**

$P(f, \hat{\mathbf{n}}) \equiv$ Density of chains with strength f



- **Material Properties:**

The probability per unit length that a chain will split:

$$\lambda^{-1} \phi_s(\mathbf{f}|\mathbf{f}_1, \mathbf{f}_2)$$

in: $\mathbf{f}$; out: $\mathbf{f}_1, \mathbf{f}_2$

The probability that two intersecting chains will fuse:

$$Y \phi_f(\mathbf{f}|\mathbf{f}_1, \mathbf{f}_2)$$

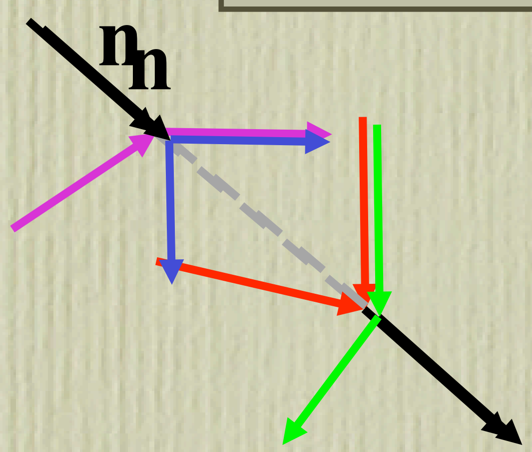
in: $\mathbf{f}_1, \mathbf{f}_2$; out: $\mathbf{f}$

“Boltzmann” Equation for Force Chains

$$(\hat{\mathbf{n}} \cdot \nabla) P(\mathbf{f}) = -\frac{1}{\lambda} P(\mathbf{f}) + \frac{2}{\lambda} \int d^2\mathbf{f}_1 d^2\mathbf{f}_2 \, \phi_s(\mathbf{f}|\mathbf{f}_1, \mathbf{f}_2) P(\mathbf{f}_1)$$

$$-Y P(\mathbf{f}) \int d^2\mathbf{f}_1 d^2\mathbf{f}_2 \, \phi_f(\mathbf{f}_1|\mathbf{f}, \mathbf{f}_2) P(\mathbf{f}_2)$$

$$+\frac{Y}{2} \int d^2\mathbf{f}_1 d^2\mathbf{f}_2 \, \phi_f(\mathbf{f}|\mathbf{f}_1, \mathbf{f}_2) P(\mathbf{f}_1) P(\mathbf{f}_2)$$



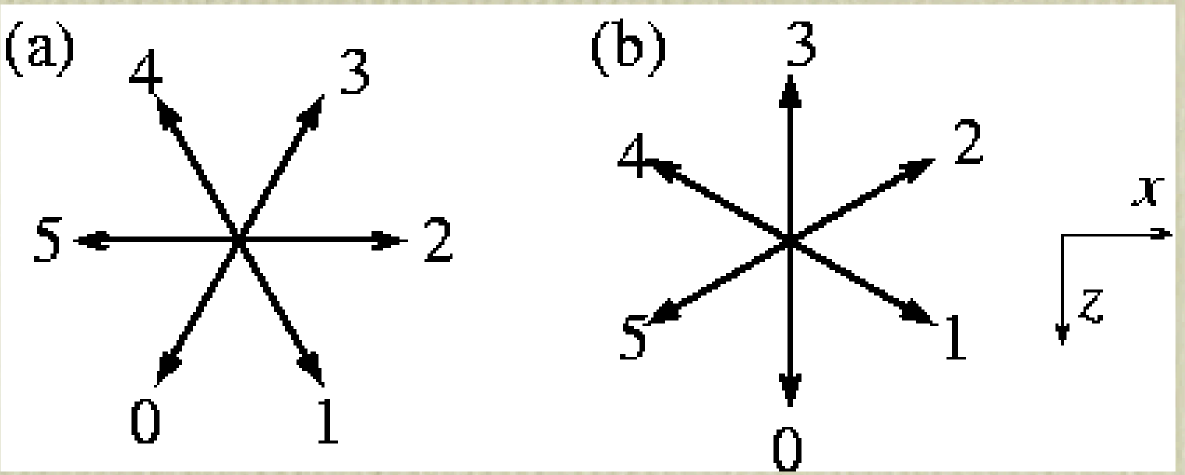
For an isotropic material:

$$\phi(\mathbf{f}_a|\mathbf{f}_b, \mathbf{f}_c) = \delta(\mathbf{f}_a - \mathbf{f}_b - \mathbf{f}_c)\Theta(f_a, f_b, f_c)\psi(\theta_b - \theta_a, \theta_c - \theta_a)|\sin(\theta_c - \theta_b)|$$

Linear theory is divergent! ($P(\mathbf{f}) = 0$ is unstable.)

Without loss of generality: $Y = \lambda = 1$

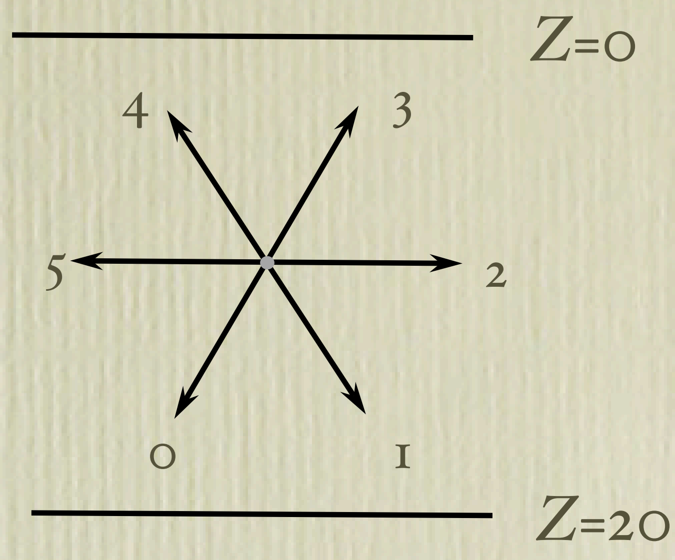
A Solvable Special Case

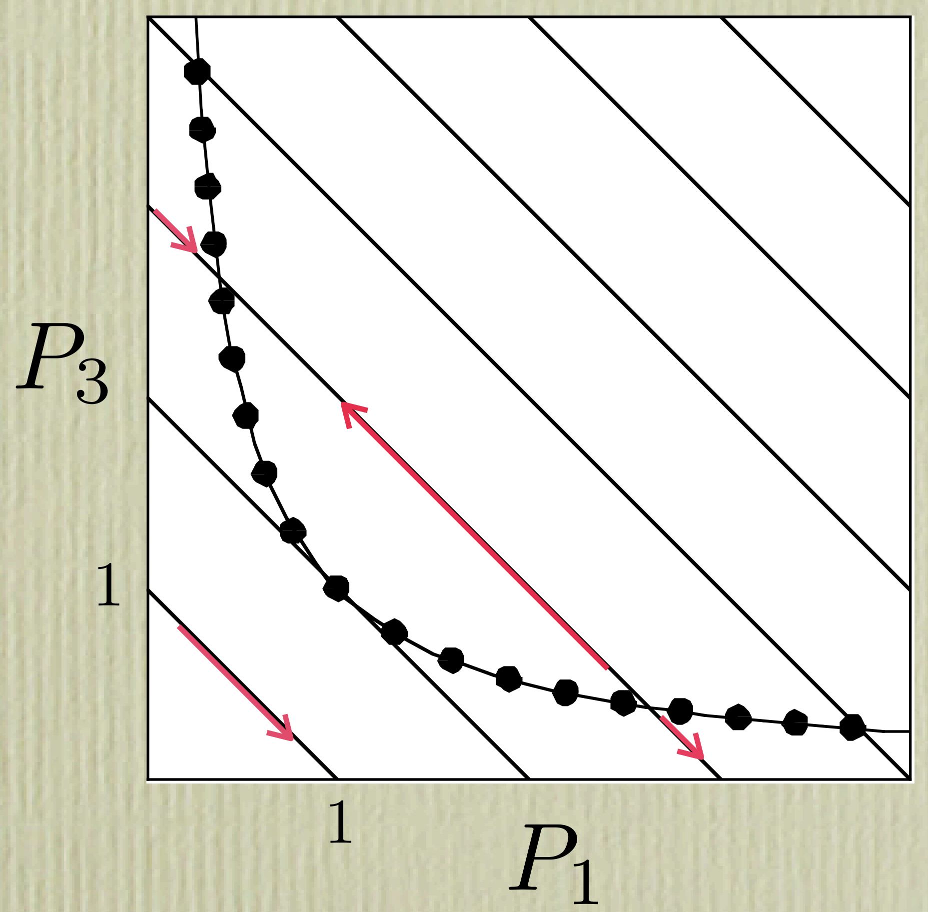


- Assume all chains are on one of the hexagonal directions and all injected chains have same f.
  - Splittings and fusions always form symmetric, 120^{o} vertices.
  - $P(f,\theta)$ becomes a discrete set P_n
- **For uniform horizontal loading:**
  - "Boltzmann" equation → set of coupled, nonlinear ODE's.

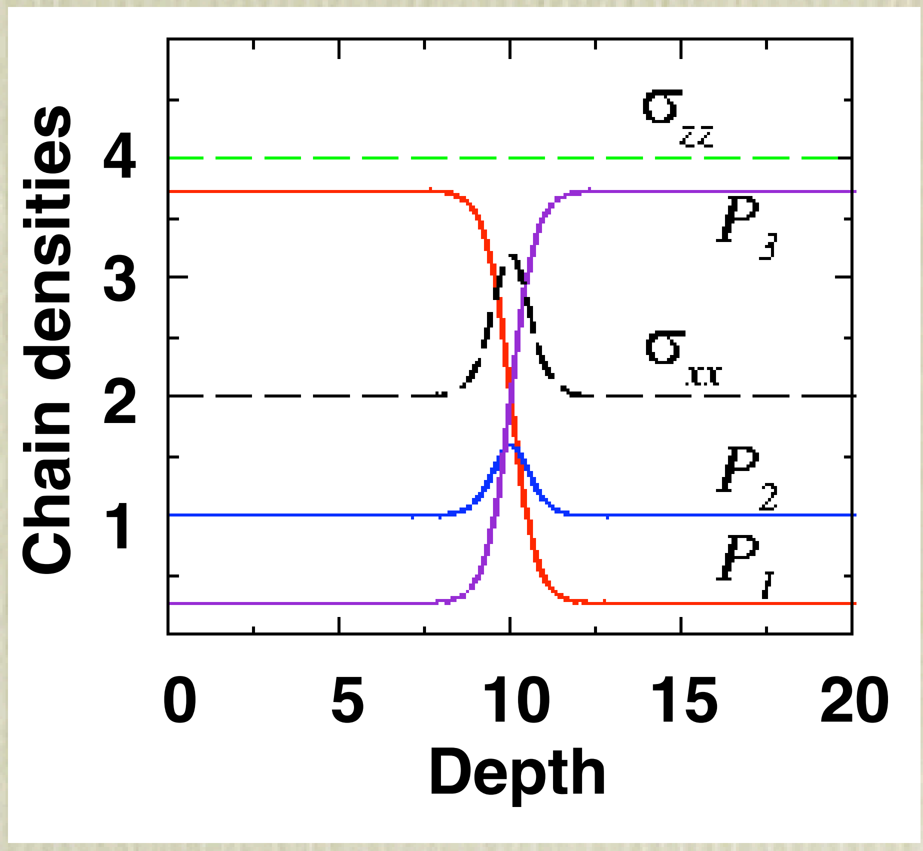
No x-dependence ; Reflection symmetry

$$\cos(\theta_n)\partial_z P_n = -P_n + P_{n-1} + P_{n+1} + (P_{n-1}P_{n+1} - P_n P_{n+2} - P_n P_{n-2})$$

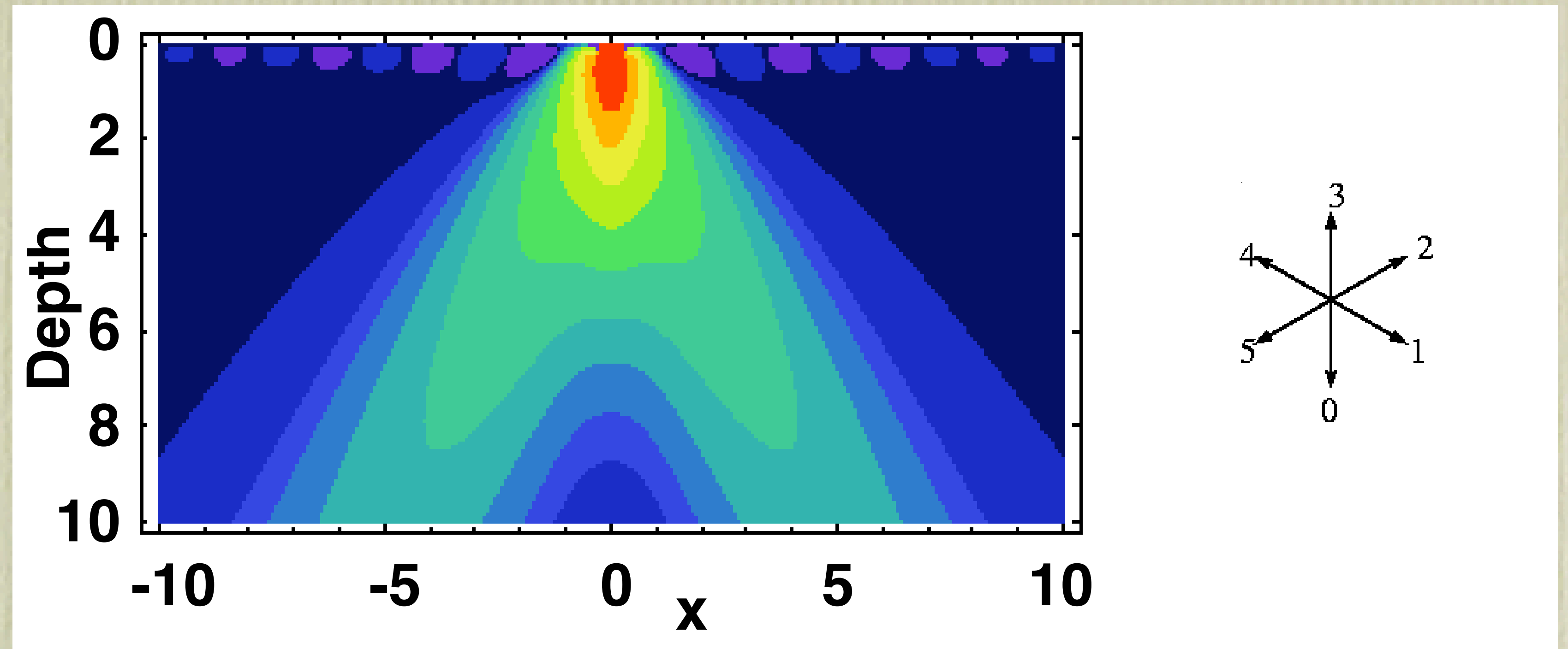




Boundary conditions:
Specify P1 at top and P3 at bottom.



Response Function (for vertical star)



- Ordinary elastic response for shallow depths.
- Two diffusively broadening peaks at large *z*. (Hyperbolic!)
  - Propagation direction distinct from star vector directions.

Continuum of Directions

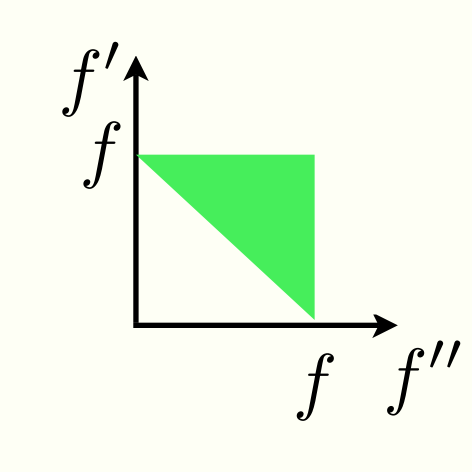
Goal: Find at least one pair of splitting and fusion kernels for which at least one solution of the Boltzmann equation can be found.

- Normalizations of the kernels:

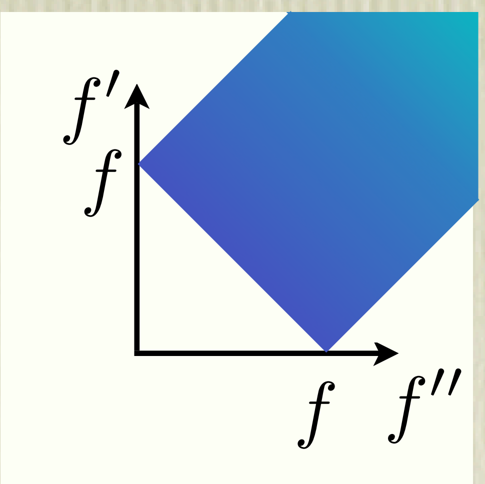
$$\max \left[\int d^2 f' \phi_f(\mathbf{f}'|\mathbf{f}'', \mathbf{f}) \right] = 1$$

$$\int d^2 f' d^2 f'' |\sin(\theta' - \theta'')| \phi_s(\mathbf{f}|\mathbf{f}', \mathbf{f}'') = 2$$

- Assume kernels **and solution** are isotropic:
  - Splitting:



Fusion:



- Sum rule required to avoid divergence in chain density:

$$2\,\lambda Y \int_0^\infty df\, f\, P(f) = 1$$

Continuum of Directions

Goal: Find at least one pair of splitting and fusion kernels for which at least one solution of the Boltzmann equation can be found.

- Simplified equation for a homogeneous and isotropic network:

$$\frac{3}{\lambda} P(f) = \frac{4}{\lambda} \int_f^\infty df' \, \frac{P(f')}{f'} + Y \int_0^\infty df' \, P(f') \int_{|f'-f|}^{f'+f} df'' \, P(f'')$$

- Asymptotics:
  - Small forces: $P(f) \sim f^{-4/3}$
  - Large forces: $P(f) \sim f^{-3/2} \, e^{-f}$
- Full solution: **#*?$%*€?!**

Some Open Questions

- Are discrete DFCN’s generic?
  - Does orientational order induce propagation of peaks?
  - Does the discreteness of chains intensities cause artifacts?
- Can we do numerical simulations?
  - Is there a straightforward algorithm that generates DFCN’s with the statistics described in the master equation?
- How are boundary conditions determined in real systems?
  - In a 2D experiment, which fixed point is picked out?
  - Can we measure $P(\mathbf{f},\mathbf{r})$ in experiments?
- How can we include gravity?
- 3D models: Icosahedral with tetrahedral vertices

References

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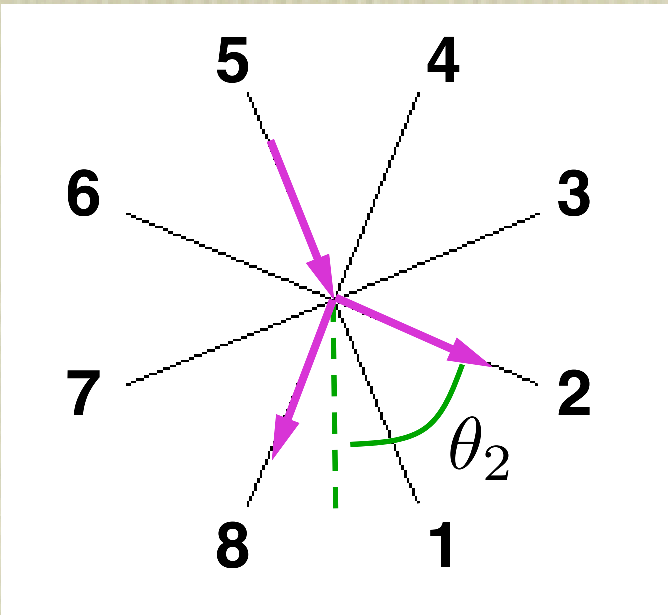
The 8-fold Way

Splittings and fusions always at 90^{o}



- Force intensities: $f_m = \sqrt{2^m}$
- Fixed point (homogeneous) solutions:

$$P_n(f_m) = p^{-f_m^2 + f_m \cos\theta_n}$$

Response function: **?!*&#$!**

Lots of weird stuff to worry about here …