Bounded Collisional Shearing Flows

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Identical, frictionless, spherical particles
Bumpy, frictionless boundaries

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Fluctuations

Particle velocity $\mathbf{c}$

Mean velocity $\mathbf{u} \equiv \langle \mathbf{c} \rangle$

Velocity fluctuation $\mathbf{C} \equiv \mathbf{c} - \mathbf{u}$

Granular temperature $T \equiv \langle C^2 \rangle / 3$

Energy Balance

$$\frac{3}{2}\rho\dot{T} = -\nabla \cdot \mathbf{q} + \mathrm{tr}(\mathbf{t}\nabla\mathbf{u}) - \gamma$$

Rate of change of internal energy

Divergence of the flux of fluctuation energy

Stress power

Rate of collisional dissipation

Constitutive relations

Frictionless, nearly elastic spheres

$$\mathbf{t} = -p\mathbf{1} + 2\mu\hat{\mathbf{D}} \qquad 2\mathbf{D} \equiv \nabla\mathbf{u} + (\nabla\mathbf{u})^T$$

$$p \equiv \rho^s \nu (1 + 4G) T \qquad G \equiv \frac{\nu(2-\nu)}{2(1-\nu)^3}$$

$$\mu \equiv \frac{8J}{5\pi^{1/2}} \rho^s \nu d G T^{1/2} \qquad J \equiv 1 + \frac{\pi}{12}\left(1 + \frac{5}{8G}\right)^2$$

$$\mathbf{q} = -\kappa \nabla T$$

$$\kappa \equiv \frac{4M}{\pi^{1/2}} \rho^s \nu d G T^{1/2} \qquad M \equiv 1 + \frac{9\pi}{32}\left(1 + \frac{5}{12G}\right)^2$$

$$\gamma = \frac{24}{\pi} \frac{\rho^s \nu G}{d} (1 - e) T^{3/2}$$

coefficient of restitution

Constitutive relations
Smooth, nearly elastic spheres

Dense limit, $0.40 \le \nu \le 0.55$

$$\mathbf{t} = -\mathrm{p}\mathbf{1} + 2\mu\hat{\mathbf{D}}$$

$$\mathrm{p} \doteq 4\rho G T \qquad 4\rho G T^{1/2} = \frac{\mathrm{p}}{T^{1/2}}$$

$$\mu \equiv \frac{8J}{5\pi^{1/2}} \rho d G T^{1/2} \qquad J \doteq 1 + \frac{\pi}{12}$$

$$\mathbf{q} \equiv -\kappa \nabla T$$

$$\kappa \equiv \frac{4M}{\pi^{1/2}} \rho d G T^{1/2} \qquad M \doteq 1 + \frac{9\pi}{32}$$

$$\gamma = \frac{24}{\pi^{1/2}} \frac{\rho G}{d} (1 - e) T^{3/2}$$

Simple shear

$$u' = \chi \text{ and } T' = \nu' = 0.$$

$$S = \frac{8J}{5\pi^{1/2}} \rho d G T^{1/2} \chi$$

$$\gamma = \frac{24}{\pi^{1/2}} \frac{\rho G}{d} (1 - e) T^{3/2}$$

$$p = 4\rho G T$$

Energy Balance

$$\cancel{(\kappa T')'} + S\chi - \gamma = 0$$

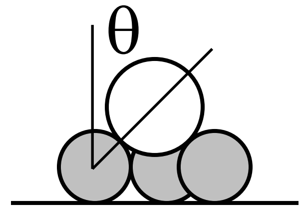
Solve for T: $$T = \frac{J}{15} \frac{d^2 \chi^2}{(1 - e)}$$

Boundary Conditions
bumpy, frictionless, dense flow

Tangential momentum

$$\frac{S}{p} = \left(\frac{\pi}{2}\right)^{1/2} f \frac{v}{T^{1/2}}$$

Slip velocity



Total energy

$$q = Sv - D \quad \text{with} \quad D = ph(1 - e_w)T^{1/2}$$

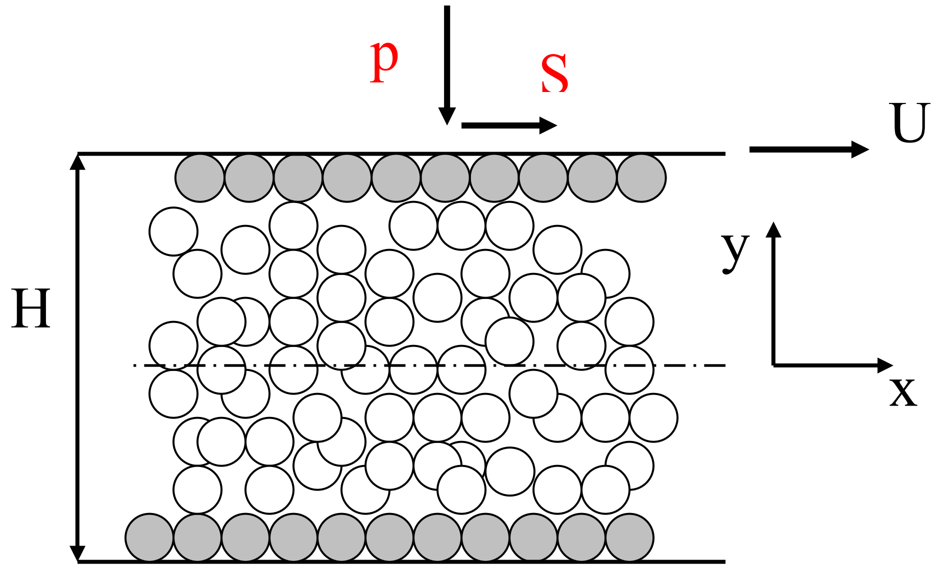
Coefficients

$$f(\theta) \doteq \frac{2}{\theta^2} - \frac{5\pi}{24J} + \frac{25 + 300\sqrt{2} - 7J}{360J}\theta^2$$

$$h(\theta) \doteq 1 + \frac{1}{4}\theta^2$$

e.g. hexagonal close packed: $\theta = \pi / 6$

Steady, Fully-Developed, Dens Flow between Identical, Bumpy Boundaries



$$p = 4\rho G T = \text{ constant, } S = \mu u' = \text{ constant}$$

$$w \equiv T^{1/2} \qquad d^2 w'' - k^2 w = 0$$

$$k^2 \equiv \left[3(1-e) - (5\pi/4J)(S/p)^2\right]/M$$

$$u' = \frac{5\pi^{1/2}}{2J}\frac{w}{d}\frac{S}{p}$$

Solutions

$k^2 > 0$ (internal dissipation)

$$w(y) = w_0 \cosh(ky/d)$$

$$u(y) = w_0 \frac{5\pi^{1/2}}{2J}\frac{S}{p}\frac{1}{k}\sinh\left(ky/d\right)$$

$K^2 \equiv -k^2 > 0$ (internal production)

$$w(y) = w_0 \cos(Ky/d)$$

$$u(y) = w_0 \frac{5\pi^{1/2}}{2J}\frac{S}{p}\frac{1}{K}\sin\left(Ky/d\right)$$

Boundary Conditions at H/2

Fluctuation Energy:

$$w' = (B/d)w$$

$$B \equiv \left[(\pi/2)f(S/p)^2 - h(1-e_w)\right]/\sqrt{2}M$$

Use this with the solution for w:

$$\tanh\left(\frac{kH}{2d}\right) = \frac{B}{k}, \quad k^2 > 0, \ B > 0$$

$$\tan\left(\frac{KH}{2d}\right) = -\frac{B}{K}, \quad K^2 > 0, \ B < 0$$

Boundary Conditions at H/2

Slip:

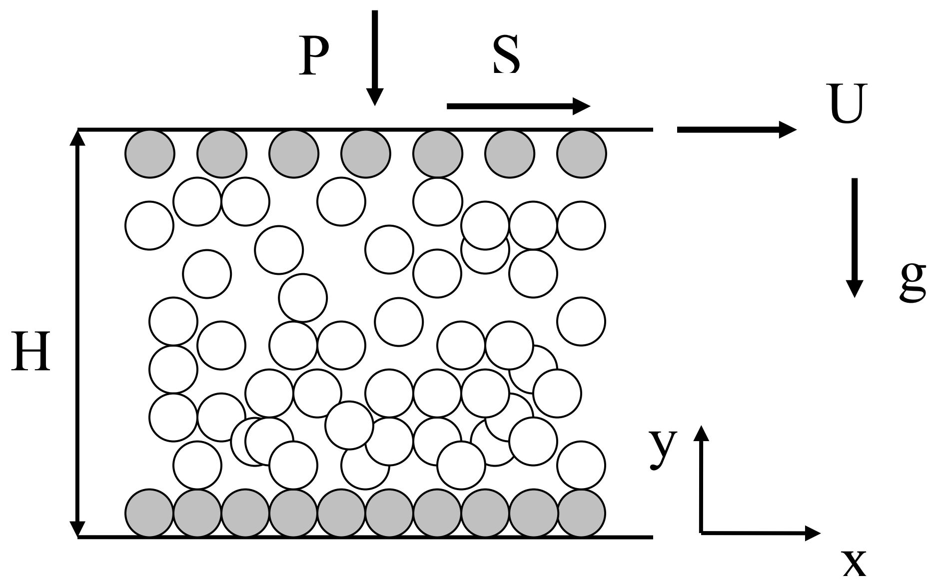
$$U - u_1 = \left(\frac{\pi}{2}\right)^{1/2} f \frac{S}{p} w_1$$

$$u_1 \equiv u(H/2), \; w_1 \equiv w(H/2)$$

$$\frac{w_1}{U} = \frac{1}{\left[\left(\frac{\pi}{2}\right)^{1/2} f + \frac{5\pi^{1/2}}{2J}\frac{B}{k^2}\right]\frac{S}{p}}, \quad k^2 > 0$$

$$\frac{w_1}{U} = \frac{1}{\left[\left(\frac{\pi}{2}\right)^{1/2} f - \frac{5\pi^{1/2}}{2J}\frac{B}{K^2}\right]\frac{S}{p}}, \quad K^2 > 0$$

Steady, Fully-Developed Shearing between Parallel Horizontal Boundaries



Know: V (total volume), P, and U

Predict: S, H, u(y), T(y), ν(y), and p(y)

Boundary conditions at $y = 0$ and $y = H$:

$$p(H) = P$$

$$\frac{v}{T^{1/2}} = \left(\frac{\pi}{2}\right)^{1/2} f \frac{S}{p}$$

$$\mp q = Sv - h(1 - e_w) p T^{1/2}$$

First-Order System

$$S' = 0$$

$$u' = \frac{S}{\mu} \qquad \mu \equiv \frac{8J}{5\pi^{1/2}} \rho^s \nu d G T^{1/2}$$

$$q' = Su' - \gamma \qquad \gamma \equiv \frac{24}{\pi} \frac{\rho^s \nu G}{d} (1-e) T^{3/2}$$

$$T' = -\frac{q}{\kappa} \qquad \kappa \equiv \frac{4M}{\pi^{1/2}} \rho^s \nu d G T^{1/2}$$

$$p' = -\rho^s \nu g \qquad p \equiv \rho^s \nu (1+4G) T$$

$$\left(\frac{d}{d\nu} \left[\rho^s \nu (1+4G) \right] T \nu' = \rho^s \nu (1+4G) \frac{q}{\kappa} - \rho^s \nu g \right)$$

$$I' = \nu \qquad I(y) \equiv \int_0^y \nu(\xi) d\xi \qquad I(0) = 0,\ I(H) = V$$

Conclusion

Boundary value problems can be phrased for steady, fully-developed, shearing flows of frictionless, nearly elastic spheres.

Analytic solutions can be found in the limit of dense flows.

Numerical solutions are easy to obtain for more general flows.