Kinetic Theory of Granular Gases

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talk, papers available: http://cnls.lanl.gov/~ebn

Los Alamos NATIONAL LABORATORY

Plan

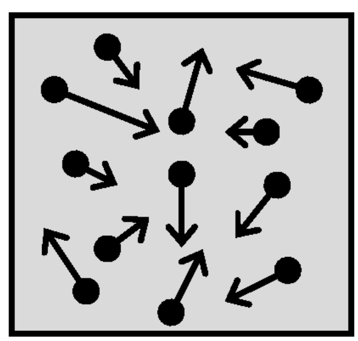
1. Basics: collision rules, collision rates
2. The Boltzmann equation
3. Extreme statistics and the linearized Boltzmann equation
4. Forced steady states
5. Freely cooling states
6. Stationary states and energy cascades
7. Hybrid solutions

Experiments

- **Friction**
  D Blair, A Kudrolli 01
- **Rotation**
  K Feitosa, N Menon 04
- **Driving strength**
  W Losert, J Gollub 98
- **Dimensionality**
  J Urbach & Olafsen 98
- **Boundary**
  J van Zon, H Swinney 04
- **Fluid drag**
  K Kohlstedt, I Aronson, EB 05
- **Long range interactions**
  D Blair, A Kudrolli 01; W Losert 02
  K Kohlstedt, J Olafsen, EB 05
- **Substrate**
  G Baxter, J Olafsen 04

Deviations from equilibrium distribution

Driven Granular gas



- **Vigorous driving**
- **Spatially uniform system**
- **Particles undergo binary collisions**
- **Velocities change due to**
  1. Collisions: lose energy
  2. Forcing: gain energy
- **What is the typical velocity (granular “temperature”)?**

$$T = \langle v^2 \rangle$$

- **What is the velocity distribution?**

$$f(v)$$

Inelastic Collisions (1D)

- **Relative velocity reduced by** $0 < r < 1$

$$v_1 - v_2 = -r(u_1 - u_2)$$

- **Momentum is conserved**

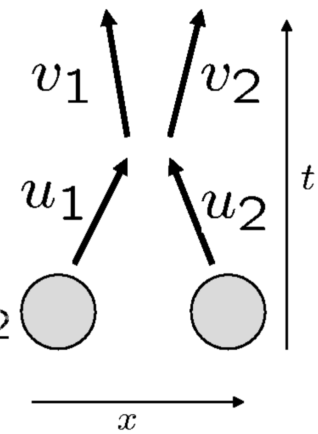
$$v_1 + v_2 = u_1 + u_2$$

- **Energy is dissipated**

$$\Delta E = \frac{1 - r^2}{4}(u_1 - u_2)^2$$

- **Limiting cases**

$$r = \begin{cases} 0 & \text{completely inelastic } (\Delta E = \text{max}) \\ 1 & \text{elastic } (\Delta E = 0) \end{cases}$$



Inelastic Collisions (any D)

- **Normal relative velocity reduced by** $0 < r < 1$

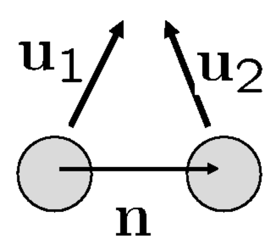
$$(\mathbf{v}_1 - \mathbf{v}_2) \cdot \mathbf{n} = -r(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n}$$

- **Momentum conservation**

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{u}_1 + \mathbf{u}_2$$

- **Energy loss**

$$\Delta E = \frac{1 - r^2}{4}[(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n}]^2$$



- **Limiting cases**

$$r = \begin{cases} 0 & \text{completely inelastic } (\Delta E = \text{max}) \\ 1 & \text{elastic } (\Delta E = 0) \end{cases}$$

Non-Maxwellian velocity distributions

JC Maxwell, Phil Trans Roy. Soc. **157** 49 (1867)

1. **Velocity distribution is isotropic**

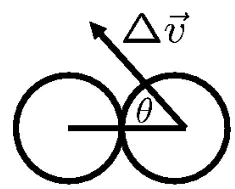
$$f(v_x, v_y, v_z) = f\left(|v|\right)$$

2. **No correlations between velocity components**

$$f(v_x, v_y, v_z) \neq f(v_x)f(v_y)f(v_z)$$

Only possibility is Maxwellian

$$f(v_x, v_y, v_z) \neq C\,\exp\left(-\frac{v_x^2+v_y^2+v_z^2}{2T}\right)$$



Granular gases: collisions create correlations

The Boltzmann equation (1D)

- **Collision rule (linear)** $r = 1-2p, \quad p+q=1$

$$(u_1, u_2) \to (pu_1 + qu_2, pu_2 + qu_1)$$

- **Boltzmann equation (nonlinear and nonlocal)**

$$\frac{\partial f(v)}{\partial t} = \iint du_1 du_2 f(u_1) f(u_2) |u_1 - u_2|^\lambda [\delta(v - pu_1 - qu_2) - \delta(v - u_2)]$$

collision rate gain loss

- **Collision rate related to interaction potential**

$$U(r) \sim r^{-\gamma} \qquad \lambda = 1 - 2\frac{d-1}{\gamma} = \begin{cases} 0 & \text{Maxwell molecules} \\ 1 & \text{Hard spheres} \end{cases}$$

Theory: non-linear, non-local, dissipative

The collision rate

- **Collision rate**

$$K(\mathbf{u}_1,\mathbf{u}_2) = |(\mathbf{u}_1-\mathbf{u}_2)\cdot\mathbf{n}|^\lambda$$

- **Collision rate related to interaction potential**

$$U(r)\sim r^{-\gamma} \qquad \lambda = 1-2\frac{d-1}{\gamma} = \begin{cases} 0 & \text{Maxwell molecules} \\ 1 & \text{Hard spheres} \end{cases}$$

- **Balance kinetic and potential energy**

$$v^2 \sim r^{-\gamma} \quad \Rightarrow \quad r\sim v^{-2/\gamma}$$

- **Collisional cross-section**

$$\sigma \sim v r^{d-1} \quad \Rightarrow \quad \sigma \sim v^{1-\frac{2}{\gamma}(d-1)}$$

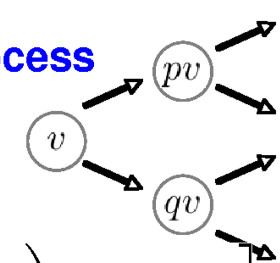
Extreme Statistics (1D)

- **Collision rule: arbitrary velocities**

$$(u_1,u_2)\to(pu_1+qu_2,pu_2+qu_1)$$

- **Large velocities: linear but nonlocal process**

$$v \xrightarrow{v^\lambda} (pv,qv)$$



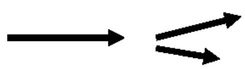
- **High-energies: linear equation**

$$\frac{\partial f(v)}{\partial t} = v^\lambda\left[\underbrace{\frac{1}{p^{1+\lambda}}f\left(\frac{v}{p}\right)}_{\text{gain}} + \underbrace{\frac{1}{q^{1+\lambda}}f\left(\frac{v}{q}\right)}_{\text{gain}} - \underbrace{f(v)}_{\text{loss}}\right]$$

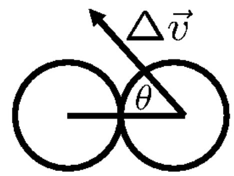
Linear, nonlocal evolution equation

Extreme Statistics (any D)

- **Collision process: large velocities**



$$v \xrightarrow{(v\cos\theta)^\lambda} (\alpha v, \beta v)$$



- **Stretching parameters related to impact angle**

$$\alpha = (1-p)\cos\theta \qquad \beta = \sqrt{1-(1-p^2)\cos^2\theta}$$

- **Energy decreases, velocity magnitude increases**

$$\alpha^2+\beta^2 \leq 1 \qquad \alpha+\beta \geq 1$$

- **Linear equation**

$$\frac{\partial f(v)}{\partial t} = \left\langle (v\cos\theta)^\lambda \left[\frac{1}{\alpha^{d+\lambda}} f\left(\frac{v}{\alpha}\right) + \frac{1}{\beta^{d+\lambda}} f\left(\frac{v}{\beta}\right) - f(v)\right]\right\rangle$$

Forced steady states: overpopulated tails

T van Noije, M Ernst 97

- **Energy injection: thermal forcing (at all scales)**

$$dv/dt = \eta$$

- **Energy dissipation: inelastic collision**

$$v \longrightarrow (pv, qv)$$

- **Steady state equation**

$$0 = D\frac{d^2 f(v)}{d^2 v} + v^\lambda \left[\frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right) - f(v)\right]$$

- **Stretched exponentials**

$$f(v) \sim \exp\left(-v^{1+\lambda/2}\right)$$

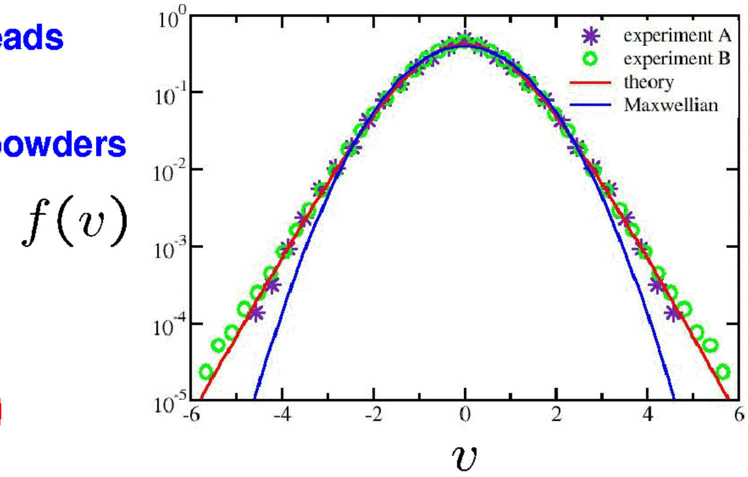
Nonequilibrium velocity distributions

- A **Mechanically vibrated beads**
  F Rouyer & N Menon 00
- B **Electrostatically driven powders**
  I Aronson & J Olafsen 05
- **Gaussian core**
- **Overpopulated tail**

$$f(v) \sim \exp\left(-|v|^{\delta}\right)$$
$$1 \le \delta \le 3/2$$

- **Kurtosis**

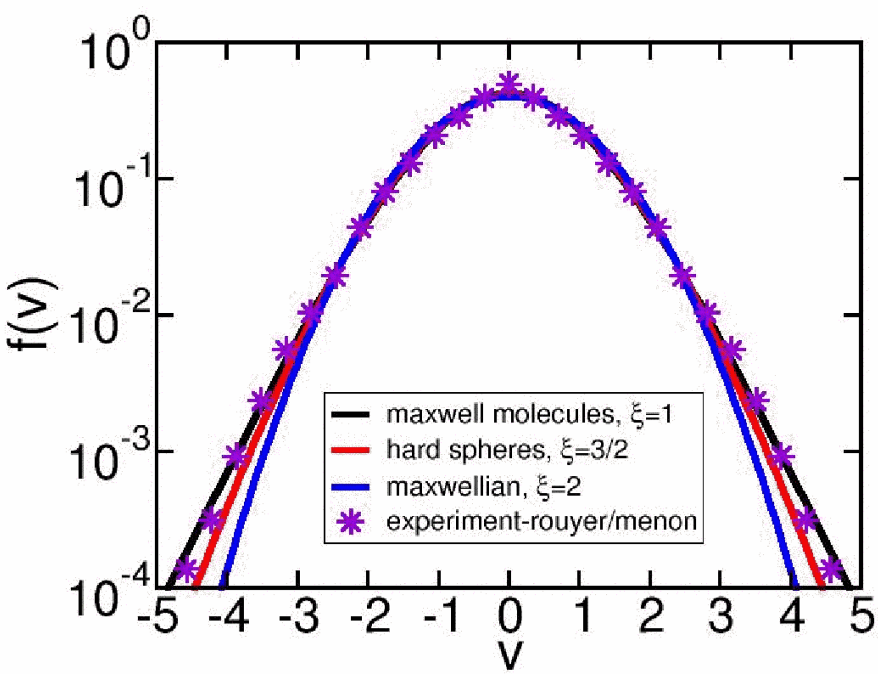
$$\kappa = \begin{cases} 3.55 & \text{theory} \\ 3.6 & \text{experiment} \end{cases}$$



Excellent agreement between theory and experiment

balance between
collisional dissipation,
energy injection from walls

Comparing kinetic theories



Freely cooling states: temperature decay

Haff, JFM 1982

- **Energy loss** $\Delta T \sim (\Delta v)^2$
- **Collision rate** $\Delta t \sim 1/(\Delta v)^\lambda$
- **Energy balance equation**

$$\frac{\Delta T}{\Delta t} \sim -(\Delta v)^{2+\lambda} \quad \Rightarrow \quad \frac{dT}{dt} \sim -T^{1+\lambda/2}$$

- **Temperature decays, system comes to rest**

$$T \sim t^{-2/\lambda} \quad \Rightarrow \quad f(v) \to \delta(v)$$

System comes to rest

Temperature decay: dimensional analysis

- **Collision rate**

$$K \sim (\Delta v)^\lambda \sim v^\lambda$$

- **Collision rate inversely proportional to time**

$$K \sim t^{-1} \quad \Rightarrow \quad v \sim t^{-1/\lambda}$$

Freely cooling states: similarity solutions

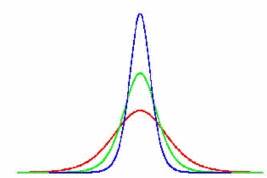
Esipov, Poeschel 97

- **Linearized equation**

$$\frac{\partial f(v)}{\partial t} = v^\lambda \left[\frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right) - f(v)\right]$$

- **Similarity solution**

$$f(v) \to t^{1/\lambda}\Phi(vt^{1/\lambda})$$



- **Steady state equation**

$$\frac{1}{\lambda}[\Phi(z)+z\frac{d}{dz}\Phi(z)] = z^\lambda \left[\frac{1}{p^{1+\lambda}}\Phi\left(\frac{z}{p}\right) + \frac{1}{q^{1+\lambda}}\Phi\left(\frac{z}{q}\right) - \Phi(z)\right]$$

- **Stretched exponentials (overpopulation)**

$$\Phi(z) \sim \exp\left(-z^\lambda\right)$$

An exact solution

- **One-dimensional Maxwell molecules**
- **Fourier transform obeys a closed equation** $F(k) = \int dv\, e^{ikv} f(v)$

$$F(k) = F(pk)F(qk)$$

- **Exponential solution**

$$F(k) = \exp(-v_0|k|)$$

- **Lorentzian velocity distribution**

$$f(v) = \frac{1}{\pi v_0}\frac{1}{1+(v/v_0)^2}$$

Nontrivial stationary states do exist!

Are there nontrivial steady states?

- **Stationary Boltzmann equation**

$$0 = \iint du_1 du_2 f(u_1) f(u_2) |u_1 - u_2|^\lambda [\delta(v - pu_1 - qu_2) - \delta(v - u_2)]$$

collision rate gain loss

Naive answer: NO!

- **According to the energy balance equation**

$$\frac{dT}{dt} = -\Gamma$$

- **Dissipation rate is positive**

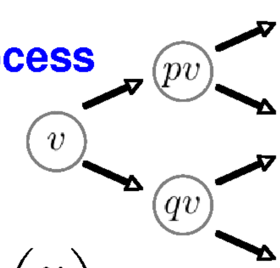
$$\Gamma > 0$$

Cascade Dynamics (1D)

- **Collision rule: arbitrary velocities**

$$(u_1, u_2) \to (pu_1 + qu_2, pu_2 + qu_1)$$

- **Large velocities: linear but nonlocal process**



$$v \to (pv, qv)$$

- **High-energies: linear equation**

$$f(v) = \frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right)$$

loss gain gain

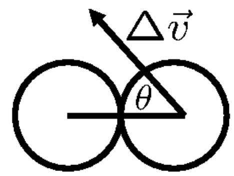
- **Power-law tail**

$$f(v) \sim v^{-2-\lambda}$$

Cascade Dynamics (any D)

- Collision process: large velocities

$$v \to (\alpha v, \beta v)$$



- Stretching parameters related to impact angle

$$\alpha = (1-p)\cos\theta \qquad \beta = \sqrt{1-(1-p^2)\cos^2\theta}$$

- Energy decreases, velocity magnitude increases

$$\alpha^2+\beta^2 \le 1 \qquad \alpha+\beta \ge 1$$

- Steady state equation

$$f(v) = \left\langle \frac{1}{\alpha^{d+\lambda}} f\left(\frac{v}{\alpha}\right) + \frac{1}{\beta^{d+\lambda}} f\left(\frac{v}{\beta}\right) \right\rangle$$

Power-laws are generic

- Velocity distributions always has power-law tail

$$f(v) \sim v^{-\sigma}$$

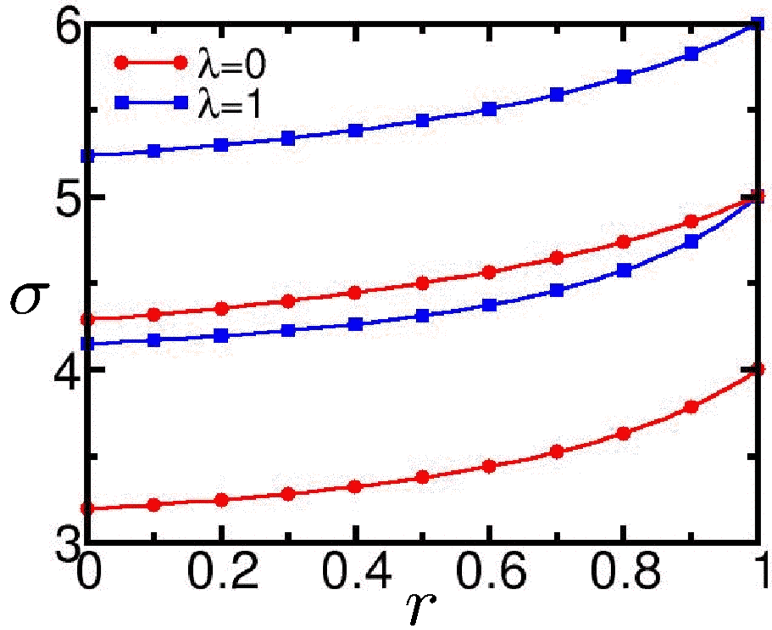
- Exponent varies with parameters

$$\frac{1-{}_2F_1\left(\frac{d+\lambda-\sigma}{2},\frac{\lambda+1}{2},\frac{d+\lambda}{2},1-p^2\right)}{(1-p)^{\sigma-d-\lambda}} = \frac{\Gamma(\frac{\sigma-d+1}{2})\Gamma(\frac{d+\lambda}{2})}{\Gamma(\frac{\sigma}{2})\Gamma(\frac{\lambda+1}{2})}$$

- Tight bounds $1 \le \sigma-d-\lambda \le 2$
- Elastic limit is singular $\sigma \to d+2+\lambda$

Dissipation rate always divergent
Energy finite or infinite

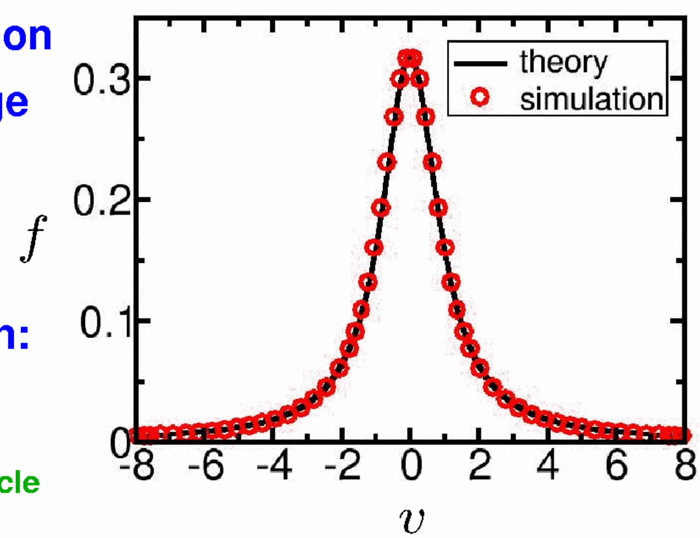
The characteristic exponent σ



σ varies with spatial dimension, collision rules

Monte Carlo Simulations

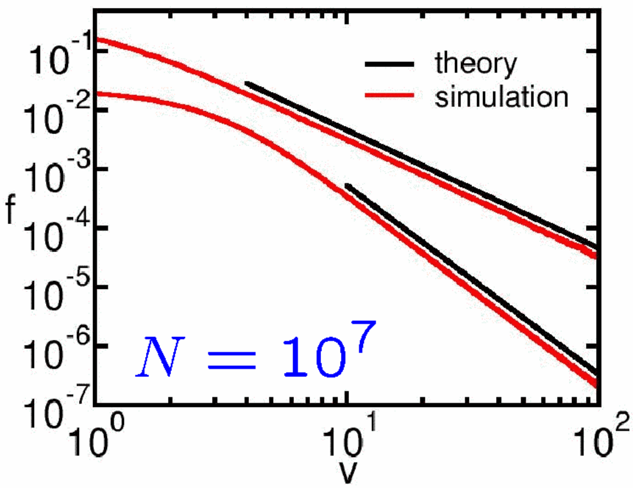
- **Compact** initial distribution
- Inject energy at very large velocity scales only
- Maintain constant total energy
- **"Lottery"** implementation:
  - Keep track of total energy dissipated, E_T
  - With small rate, boost a particle by E_T



Excellent agreement between theory and simulation

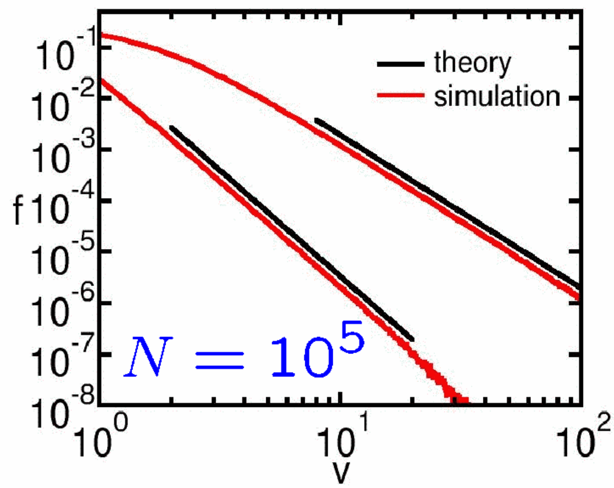
Further confirmation

Maxwell molecules (1D, 2D)



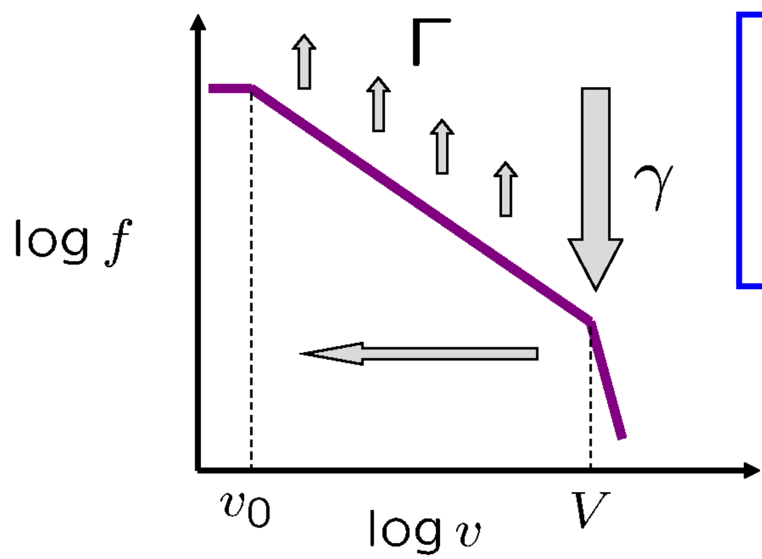
d	theory	simulation
1	2	1.995
2	3.19520	3.19

Hard spheres (1D, 2D)



d	theory	simulation
1	3	2.994
2	4.14922	4.15

Injection, cascade, dissipation



Experimental realization?
Energetic particle "shot" into static medium

Energy balance

$$\Gamma \sim \gamma V^2$$

- ❖**Energy is injected at large velocity scales**
- ❖**Energy cascades from large velocities to small velocities**
- ❖**Energy dissipated at small velocity scales**

Energy balance

- Energy injection rate γ
- Energy injection scale V
- Typical velocity scale v_0
- Balance between energy injection and dissipation

$$\gamma \sim V^\lambda (V/v_0)^{d-\sigma}$$

- For "lottery" injection: injection scale diverges with injection rate

$$V \sim \begin{cases} \gamma^{-1/(2-\lambda)} & \sigma < d+2 \\ \gamma^{-1/(\sigma-d-\lambda)} & \sigma > d+2 \end{cases}$$

with Ben Machta (Brown)

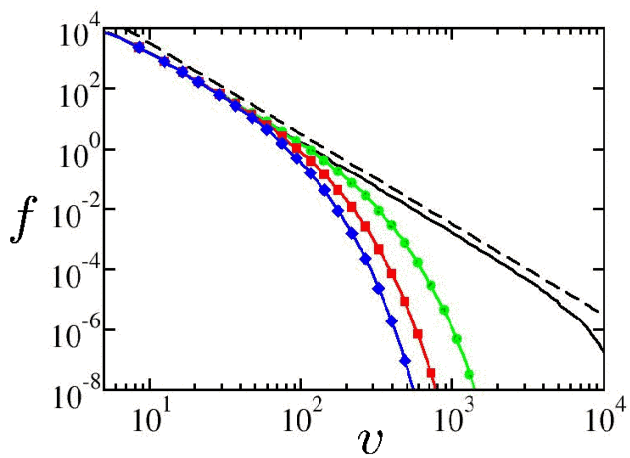
Self-similar collapse

- Self-similar distribution

$$f(v,t) \simeq v^{-\sigma} \Phi\left(\frac{v}{V(t)}\right)$$

- Cutoff velocity decays

$$V(t) \sim t^{-1/\lambda}$$



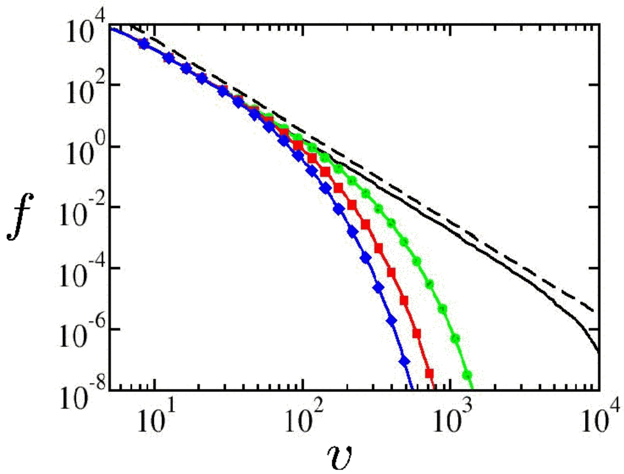
- Scaling function

$$\Phi(x) = \sum_{n=1}^{\infty} A_n \exp\left[-(2^n x)^\lambda\right] \qquad A_n = \prod_{\substack{k=1 \\ k\neq n}}^{\infty} \frac{1}{1-2^{\lambda(n-k)}}$$

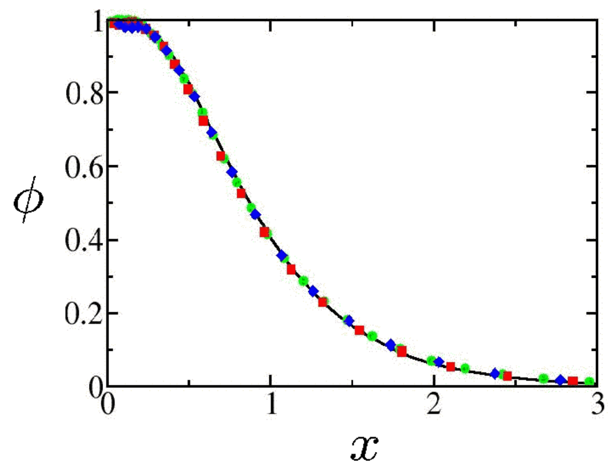
Hybrid between steady-state and time dependent state

Numerical confirmation

Velocity distribution



Scaling function



A third family of solutions exists

Conclusions

- New class of nonequilibrium stationary states
- Energy cascades from large to small velocities
- Power-law high-energy tail
- Energy input at large scales balances dissipation
- Associated similarity solutions exist as well
- Temperature insufficient to characterize velocities
- Experimental realization: requires a different driving mechanism

Outlook

- Spatially extended systems
- Spatial structures
- Polydisperses granular media
- Experimental realization

E. Ben-Naim and J. Machta, Phys. Rev. Lett. **94**, 138001 (2005)

E. Ben-Naim, B. Machta, and J. Machta, cond-mat/0504187

Deviation from Maxwell-Boltzmann

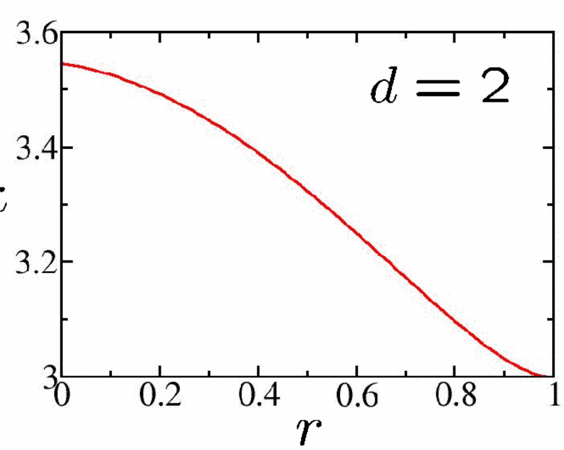
- Kurtosis κ

$$\kappa = \langle v^4 \rangle / \langle v^2 \rangle^2$$

$$\kappa = 3 + \frac{18(1-r)^2(1+r)}{33 - 25r + 3r^2 - 3r^3}$$

- Restitution coefficient r

$$\Delta E \propto \left(1 - r^2\right) (\Delta v)^2$$



1. Velocity distribution independent of driving strength
2. Stronger dissipation yields stronger deviation

Exact solution of Maxwell's kinetic theory: thermal forcing balances dissipation

EB, Krapivsky 02