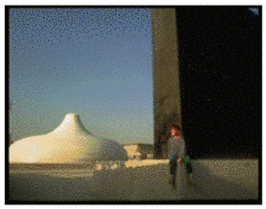
Granular Physics Conference 2005
Kavli Institute for Theoretical Physics

Nonlinear hydrodynamics of a freely cooling granular gas: clustering, singularities and shocks

Baruch Meerson
Hebrew University of Jerusalem



in collaboration with

Efi Efrati - HUJ
Eli Livne - HUJ
Andrea Puglisi - U. Paris-Sud

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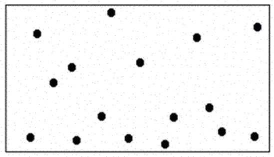
Outline

1. **Clustering instability in a freely evolving granular gas.**
2. **Nonlinear dynamics of clustering in a quasi-1D setting: flow by inertia, attempted singularities, shocks.**
3. **Higher dimensions?**

Gas of inelastic hard spheres: instantaneous binary collisions



σ: particle diameter
m: particle mass

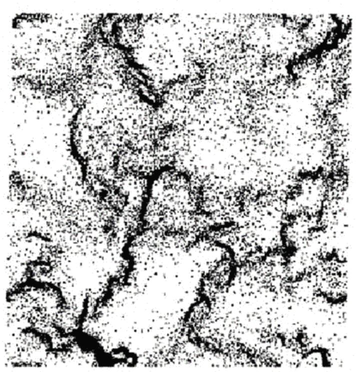
$$\begin{pmatrix} v'_{1\tau} \\ v'_{2\tau} \end{pmatrix} = \begin{pmatrix} v_{1\tau} \\ v_{2\tau} \end{pmatrix}$$

$$\begin{pmatrix} v'_{1n} \\ v'_{2n} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1-r & 1+r \\ 1+r & 1-r \end{pmatrix} \begin{pmatrix} v_{1n} \\ v_{2n} \end{pmatrix}$$

Momentum preserved, part of kinetic energy lost

$0 \le r < 1$ **coefficient of normal restitution**

If left alone, the granular gas "cools", but is that all?



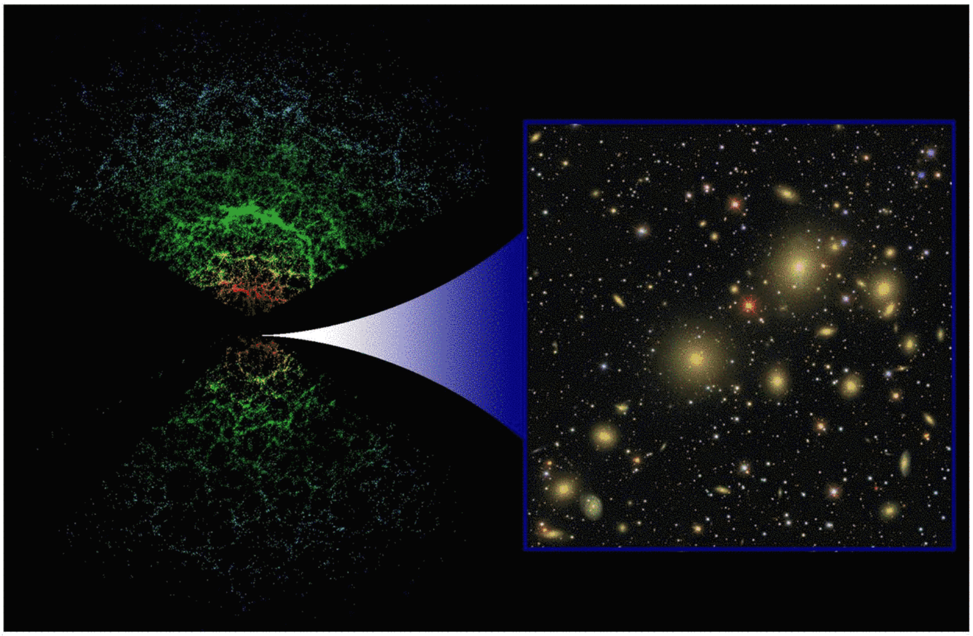
Clustering instability:
formation of clusters of particles, generation of vortices

Goldhirsch and Zanetti (1993), McNamara and Young (1996), …

Homogeneous cooling always unstable in the thermodynamic limit, no matter how small (but non-zero) the inelasticity is.

Q1: ***Where do the filaments and cellular structure come from?***

The SDSS 3D Universe Map
Credit & Copyright: Sloan Digital Sky Survey Team, NASA, NSF, DOE



Q2: ***Is the resemblance coincidental?***

Hydrodynamic equations for inelastic gases

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0\,,$$

$$n\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = \nabla \cdot \mathbf{P},$$

dilute gas

$$n\left(\frac{\partial T}{\partial t} + (\mathbf{v}\cdot\nabla)T\right) = -\nabla\cdot\mathbf{Q} + \mathbf{P} : \nabla\mathbf{v} - \Gamma\,.$$

P: stress tensor **Q:** heat flux

$\Gamma \sim (1-r^2)\, n^2\, T^{3/2}$: rate of energy loss by inelastic collisions (Haff 1983)

Haff, Jenkins and Richman, Goldhirsch *et al.*, Brey *et al.*, Dufty *et al.*,…

Scale separation: mean free path << hydrodynamic length scale
Often it requires nearly elastic collisions: q=(1-r)/2 << 1

Homogeneous cooling state (HCS)

$$\mathrm{n}(\mathbf{r},t) = \mathrm{n}_0 = \mathrm{const}$$
$$\mathbf{v}(\mathbf{r},t) = 0$$

$$T(\mathbf{r},t) = \frac{T_0}{(1+t/t_0)^2}$$ **Haff (1983)**

$$t_0 = \frac{m^{1/2}}{2\sqrt{\pi} q \sigma n_0 T_0^{1/2}}$$ cooling time

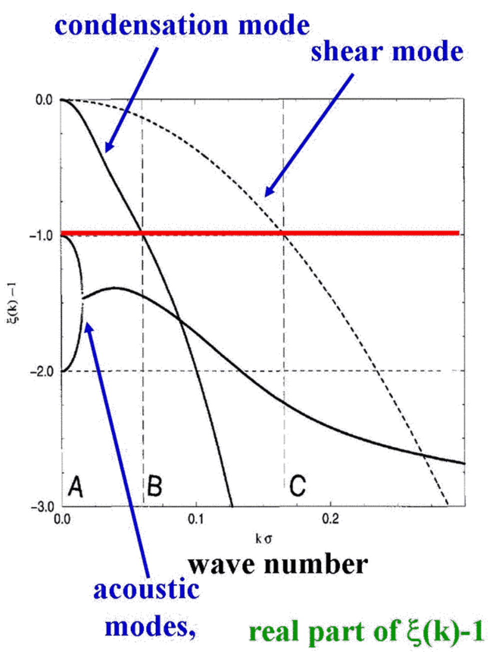
q=(1-r)/2 inelasticity of collisions

Linear theory of instability (McNamara, Goldhirsch *et al.,*)

$$\delta\rho = \delta\rho_k (1+\mathrm{t}/\tau)^{\xi} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}),$$
$$\delta\mathbf{v} = \delta\mathbf{v}_k (1+\mathrm{t}/\tau)^{\xi-1} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}),$$
$$\delta T = \delta T_k (1+\mathrm{t}/\tau)^{\xi-2} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}).$$

Nonlinear stage of instability is hard to follow, even numerically:

1. **Condensation and shear modes strongly coupled**
2. **Features at small scales develop**
3. **Hydrodynamics breaks down at large densities n ~ n_c**



One can avoid two of these three difficulties:

1. Put the particles into a long and narrow 2d box
   $L_x \gg L_y$: shear mode suppressed, coarse-grained quantities depend only on x.

2. Work with a *very* small area fraction:
   density remains, for a long time, much less than n_c *despite clustering*. Large density *contrasts* can be probed.

Take care of small scales by employng, in numerical solution, Lagrangian coordinates

1d flow of a freely cooling dilute inelastic gas

$$\frac{dn}{dt} + n\frac{\partial \mathrm{v}}{\partial \mathrm{x}} = 0\,, \qquad n\frac{d\,\mathrm{v}}{dt} = \frac{\partial P}{\partial \mathrm{x}},$$

$$n\frac{dT}{dt} = P\frac{\partial \mathrm{v}}{\partial \mathrm{x}} + K\frac{\partial}{\partial \mathrm{x}}\left(T^{1/2}\frac{\partial T}{\partial \mathrm{x}}\right) - \frac{8q}{K}n^2T^{3/2},$$

m=1

$$P = -nT + \frac{K}{4}T^{1/2}\frac{\partial v}{\partial x}$$ the stress field

pressure — shear viscosity

$$K = \frac{2}{\sqrt{\pi}\sigma L_x n_0} \ll 1$$ the Knudsen number, $q \ll 1$

d/dt = ☆/☆t+ v ☆/☆x

Periodic boundary conditions, 0<x<1.

$n \to n/n_0,$
$T \to T/T_0,$
$\mathrm{v} \to \mathrm{v}/T_0^{1/2},$
$x \to x/L_x,$
$t \to tT_0^{1/2}/L_x.$

Efrati, Livne, BM PRL (2005)

Numerical solution of hydrodynamic equations

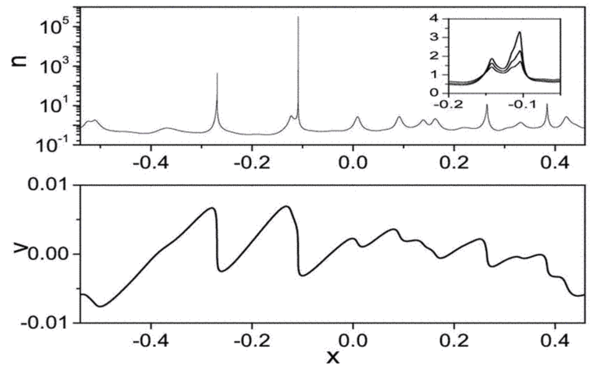


FIG. 1: The density and velocity profiles at scaled time $t = 7.043$, shortly before the major density peak develops singularity. The parameters $K = 4 \cdot 10^{-4}$ and $q = 10^{-2}$ correspond to 79 linearly unstable Fourier modes. 2000 Lagrangian mesh points are used, so the major density peak includes more than 50 mesh points above the density value of $n = 10^2$. The inset shows an earlier density history (at $t = 2$, 3 and 4) of a region around the major density peak.

K=4◈10^{-4}
q=10^{-2}

Multiple narrow density peaks (clusters), steep velocity gradients.

Temperature continues to decay.

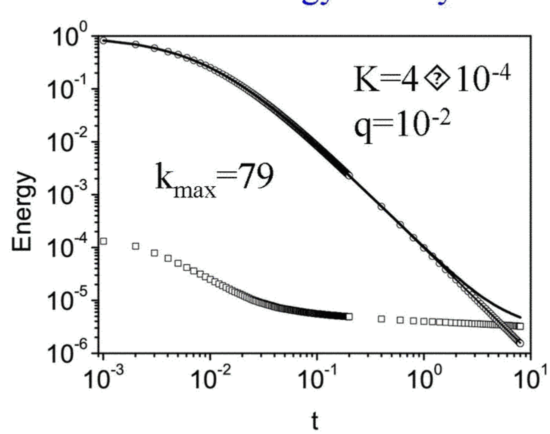
n and ☆v/☆x grow without limit until numerical scheme is unable to accurately follow

Singularity develops in a finite time?

Time history of the total energy of the gas

$$E(t) = \int_{-1/2}^{1/2} \left(nT + \frac{1}{2} n v^2 \right) dx \ .$$

thermal energy density   macroscopic kinetic energy



Thick line: E(t)

Circles: thermal energy

Squares: macroscopic kinetic energy

Thin line: Haff's law

Mach number >>1

Macroscopic kinetic energy approaches a constant

Thermal energy continues to follow Haff's law

The blow-up caused by "freezing up" of the stress field

$$P = -nT + \frac{K}{4} T^{1/2} \frac{\partial \mathrm{v}}{\partial x} \to 0 \quad \text{as} \quad T \to 0.$$

As a result, a 1D flow by inertia sets in:

$$\frac{\partial \mathrm{v}}{\partial t} + \mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{x}} = 0, \qquad \frac{\partial n}{\partial t} + \frac{\partial}{\partial \mathrm{x}}(n\mathrm{v}) = 0,$$

while the temperature becomes irrelevant.
This *flow of non-interacting particles* develops a finite-time singularity.

The flow-by-inertia problem is soluble:

$$\mathrm{v}(\mathrm{x},\mathrm{t}) = \mathrm{v}_0(\xi), \qquad n(\mathrm{x},\mathrm{t}) = \frac{\mathrm{n}_0(\xi)}{1 + \mathrm{t}\,\mathrm{dv}_0(\xi)/\mathrm{d}\xi},$$

$\mathrm{v}_0(\xi)$ and $\mathrm{n}_0(\xi)$: "initial" profiles, $\mathrm{x} = \xi + \mathrm{v}_0(\xi)\,\mathrm{t}$

The finite-time singularity of both n(x,t), and

$$\frac{\partial \mathrm{v}(x,t)}{\partial x} = \frac{\mathrm{dv}_0(\xi)/\mathrm{d}\xi}{1 + \mathrm{t}\,\mathrm{dv}_0(\xi)/\mathrm{d}\xi},$$

occurs when the denominator becomes zero for the first time.
Universal scaling behavior close to singularity

Special hydro simulations with a *single* singularity:

$$n(x,t=0)=1\,,\quad T(x,t=0)=1,$$
$$v(x,t=0)=a\sin(2\pi x),\quad a=-0.05,$$

$K=4\times10^{-4}$
$q=10^{-2}$

a single (longest) Fourier mode for the velocity

Overall flow:

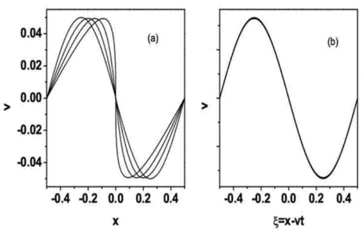


FIG. 3: The numerically computed velocity is shown versus x (a) and versus $\xi = x - vt$ (b) at times 1, 2 and 3.225 (the profiles in figure a steepen as the time progresses). Also shown in figure b is the initial profile (7). All the curves in figure b coincide within 1.5%. The simulation parameters are $K = 4\cdot10^{-4}$ and $q = 10^{-2}$.

Close to singularity:

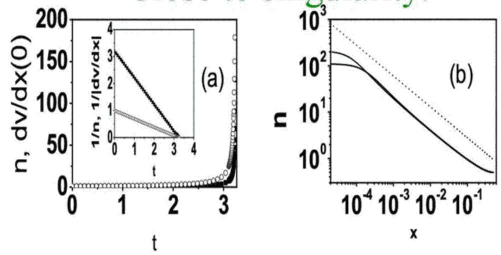
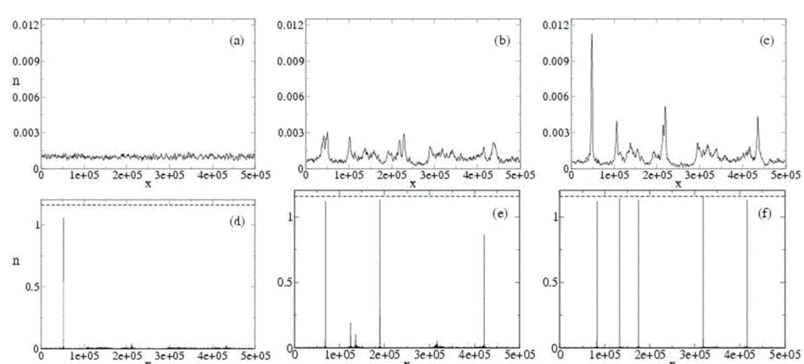


FIG. 4: Numerically computed values of $|\partial v/\partial x|$ (filled squares) and n (empty circles) at x=0 versus time (a). The inset shows the respective inverse values. Figure b depicts the spatial profiles of n at time moments 3.209 and 3.225. The straight line is a $x^{-2/3}$ dependence; it is given for reference. The parameters are the same as in Fig. 3.

How are the singularities smoothed?
What is the late-time relaxation dynamics?

MD simulations, measuring hydro fields

Typical simulation: $L_x=5\times10^5 >> L_y=25$, $n_0=10^{-3}$, q=0.04, periodic BCs in x-direction, reflecting BCs in y-direction

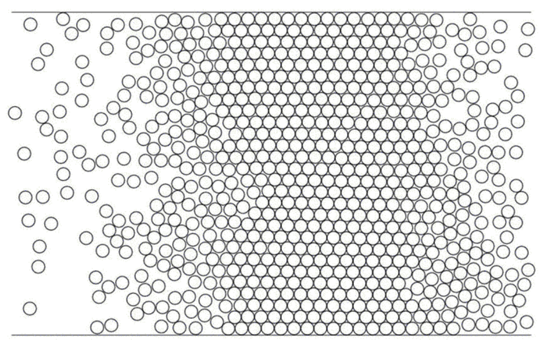


Density history: density growth stops when close packing is reached.

BM and A. Puglisi, EPL (2005)

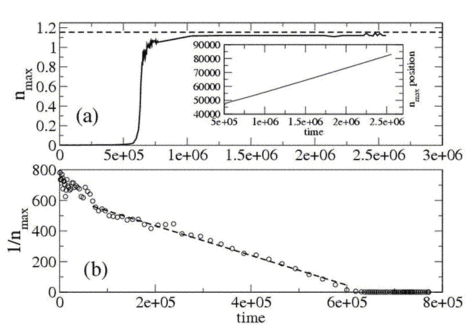
Fig. 1 – The density profiles at scaled times 0 (a), 274 622 (b), 549 554 (c), 771 055 (d), 1.74153×10^6 (e), and 2.53252×10^6 (f). The dashed line in Figs. d-f marks the hexagonal close packing density.

A snapshot of the system
in the region of a density peak

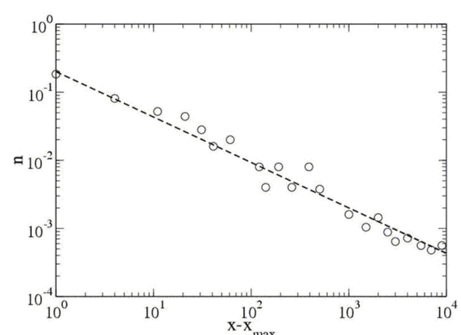


MD simulations are in good agreement with
hydrodynamic predictions before singularity

$n_{max} \sim (t_0 - t)^{-1}$



$n \sim |x - x_{max}|^{-2/3}$



The mean velocity history

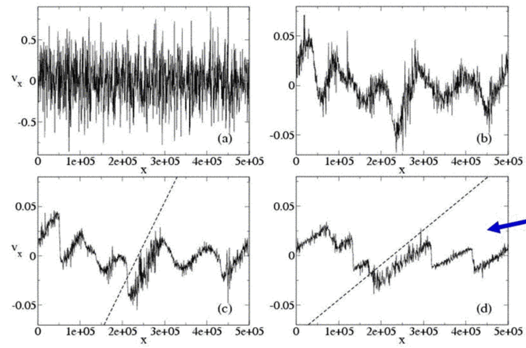
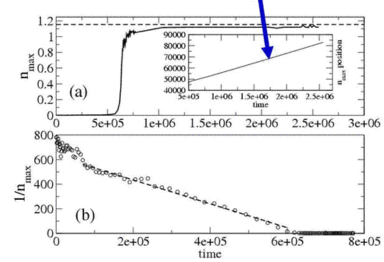


Fig. 2 – The horizontal velocity profiles at times 0 (a), 274622 (b), 771055 (c), and 2.53252×10^6 (d). The straight lines in Figs. c and d have a slope $1/(t+C)$, where $C = 3.8 \times 10^5$.

Like "mature" shocks in the Burgers equation!



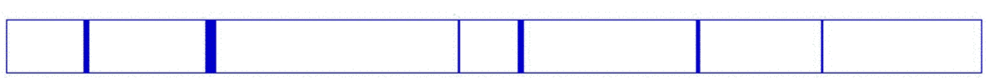
$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \nu \frac{\partial^2 v}{\partial x^2}, \quad \nu \to 0$$

Late-time dynamics describable by the inviscid limit of the Burgers equation:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \nu \frac{\partial^2 v}{\partial x^2}, \quad \nu \to 0, \qquad \frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n v) = 0, \quad (1)$$

Long-time relaxation (coarsening) is identical to the *decaying Burgers turbulence*. Related to the ballistic agglomeration model **(Carnevale, Pomeau and Young 1990).** Scaling laws:

$$M \sim t^{2/3},\ V \sim t^{-1/3},\ N \sim t^{-2/3} \quad \Rightarrow \quad E \sim t^{-2/3}$$



Ben-Naim, Chen, Doolen and Redner (1999) observed that model (1) is valid in a strictly 1D setting: point-like inelastic particles moving on a line.

What about a fully multi-dimensional instability?

We argue that the stress tensor freezes up, flow by inertia sets in:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{0}\,, \qquad \frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{v}) = 0\,.$$

The energy equation, at late times, decouples.

Why?

1. The *linear* theory predicts that the Mach number should grow with time
2. MD simulations in 2D (**Nie, Ben Naim and Chen 2002**): at late times ***"typical local velocity fluctuations are small compared with the typical velocity"***

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{0}\,, \qquad \frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{v}) = 0\,. \qquad (1)$$

Exact solution:

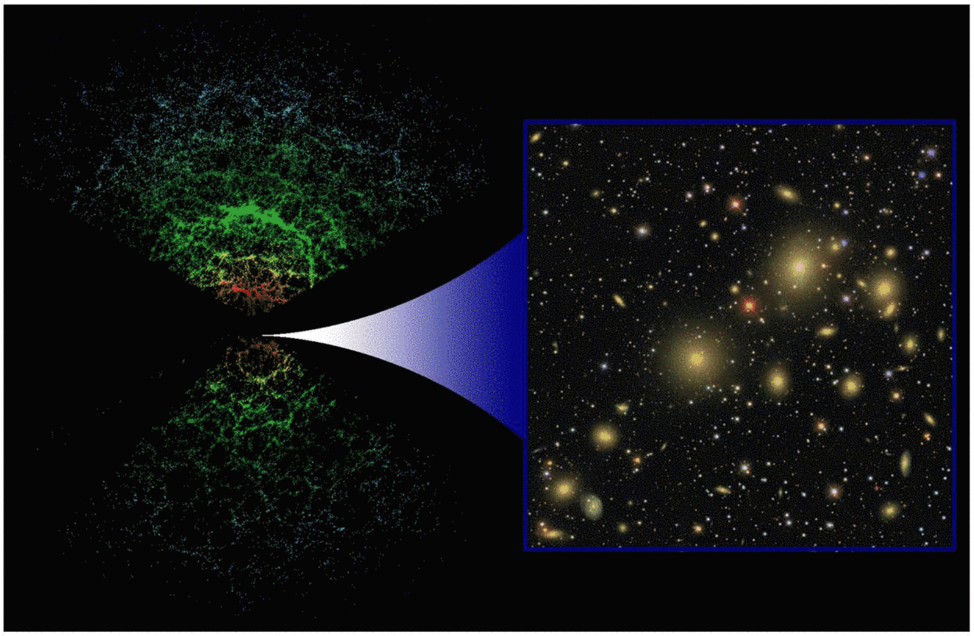
$$\mathbf{v}(\mathbf{x},t) = \mathbf{v_0}(\xi), \qquad n(\mathbf{x},t) = \frac{n_0(\xi)}{\left|\delta_{ij} + t\,\partial v_{0_i}(\xi)/\partial \xi_j\right|}\,,$$

$$\mathbf{x} = \xi + t\,\mathbf{v_0}(\xi) \qquad (2)$$

$\star v_i/\star x_j$ and n(**r**,t) blow up in a finite time. A full classification of singularities: **Arnold, Bruce.** Similar equations appear in cosmology (the Zeldovich approximation), in the context of large-scale matter distribution in the universe.

"Zeldovich pancakes" in 3D, "Zeldovich filaments" in 2D, cellular topology

The SDSS 3D Universe Map

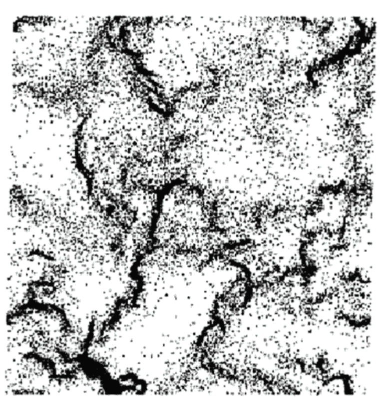


The resemblance is *not* coincidental

What happens beyond the singularity in 2D?

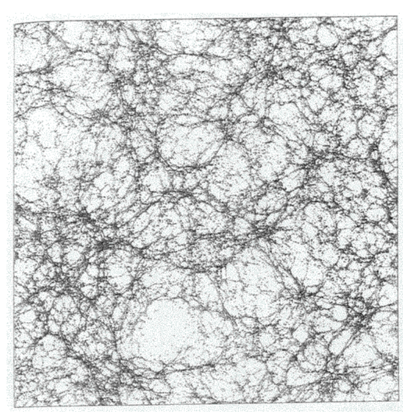
A (potential) 2D Burgers equation?

MD simulations



Goldhirsch and Zanetti (1993)

Burgers model

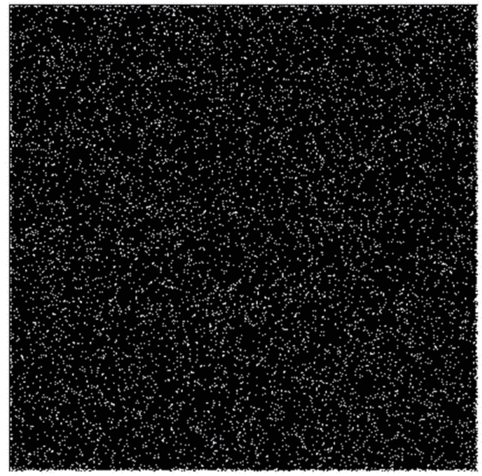


Vergassola (1995)

The v-field is *not* potential, so *Burgers eqn. inapplicable*

Barrow and Saich (1993):
"High-density regions should be high vorticity regions"

Seems to agree with MD simulations of a freely cooling inelastic gas



Movie courtesy of T. Pöschel

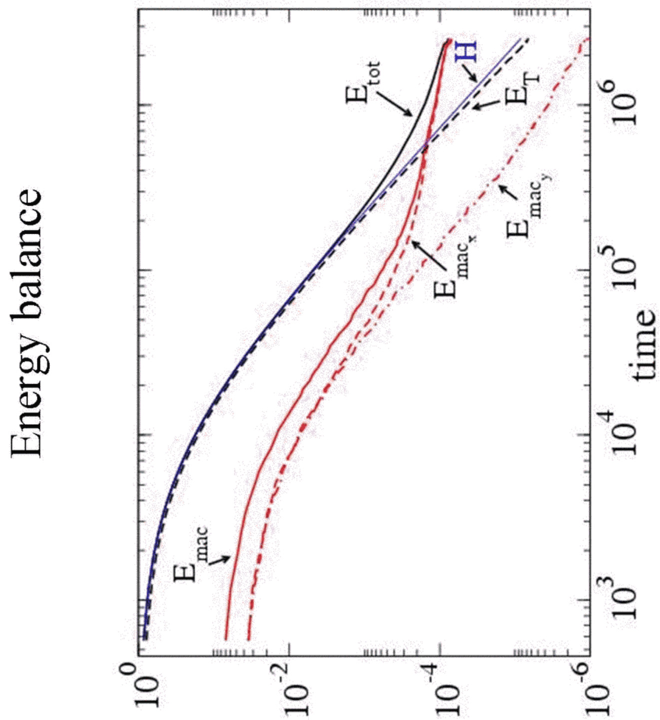
Q3: ***What is the role of vorticity?***

Summary and open questions

1. ***Quasi-1D clustering instability of inelastic gases:***

   Late times: flow by inertia
   arrest of singularities at close packing
   inviscid Burgers model: a proper late-time continuum description.

2. ***Does flow by inertia set in in higher dimensions?*** **(yes, work in progress)**
   Arrest of singularities at close packing? **(yes, work in progress)**
   Coarse-grained model for late times?
   Role of vorticity? (Potential-flow) Burgers model unsatisfactory.


Energy balance
time
E_tot
H
E_T
E_mac
E_mac_x
E_mac_y
10^0
10^-2
10^-4
10^-6
10^3
10^4
10^5
10^6