Lecture 3: Supersymmetry in Various Dimensions

In the first two lectures we discussed some general features of field theory, but our examples were all from mechanics. The underlying "spacetime" of the theories considered is just time; the "space" in those models is the manifold in which paths take values. In the remaining lectures we turn to field theory proper—spacetimes with positive space dimension. Put differently, the spacetime dimension n is at least 2. Today we present additional general material needed to treat fields in supersymmetric theory; the remaining lectures will consider specific supersymmetric models.

We mathematicians are fond of abstract structures which apply in many different examples. While field theory has such general features—and we try to bring them out as much as possible—the whole subject has many details which resist abstraction. In particular, many properties depend strongly on the spacetime dimension n. Some may find that parts of the remaining lectures resemble lessons in geography or taxonomy, but that reflects a bit the nature of the subject. We encounter this immediately in our discussion of spinors in Minkowski spacetime. It is necessary background to introduce fermion fields in Minkowski spacetime, and the particularities of spinors in various dimensions influence the models which exist. They also enter into the general construction of *superspacetime* and the *superPoincaré group*, which is our second topic. (Of course, 'superspacetime' is usually simply 'superspace'.) The Hilbert space of a supersymmetric *quantum* field theory carries a representation of the superPoincaré group, so we give a general discussion and some examples of such representations. Our discussion of representations of the ordinary Poincaré group in Lecture 1 serves as a model here.

In the second half of the lecture we turn from quantum particles to *classical* fields. We give some general discussion and survey the different supersymmetric field theories. Then we display the classical (nonsupersymmetric) lagrangians for free theories, and briefly discuss the classical field theory (along the lines of Lecture 2). The space of classical solutions is a symplectic vector or affine space. We then see how to get the one particle quantum Hilbert space and the associated Fock space for these free theories. Here the corresponding discussion for free particles in Lecture 1 serves as a model. But in that discussion the symplectic vector spaces we quantized are finite dimensional; here in field theory they are infinite dimensional.

Spinors on Minkowski spacetime

We need to know a bit about the *real* spin representations of the Lorentz group $\mathrm{Spin}(1, n-1)$. This enters both the discussion of spinor fields and also the construction of the superPoincaré group. We summarize some of the basic properties in Table 1. In this table V is the vector representation of $\mathrm{Spin}(1, n-1)$, a real n dimensional vector space with an inner product of signature $(1, n-1)$. We make several comments to help the reader absorb some of the information:

- The first column is the dimension n of spacetime. The reason we stop at $n = 11$ will emerge later. In Lecture 4 we focus completely on theories in dimension 3 or less. In Lecture 5 we

n	Spin $(1, n-1)$	minimal real spin representation S	$\dim S$	V in terms of S^*
1	$\mathbf{Z}/2$	S real	1	$V \cong (S^*)^{\otimes 2}$
2	$\mathbb{R}^{>0}$	S^+, S^- real	1	$V \cong (S^{+*})^{\otimes 2} \oplus (S^{-*})^{\otimes 2}$
3	$SL(2,\mathbb{R})$	S real	2	$V \cong \mathrm{Sym}^2 S^*$
4	$SL(2,\mathbb{C})$	$S_{\mathbb{C}} \cong S' \oplus S''$	4	$V_{\mathbb{C}} \cong S'^* \otimes S''^*$
5	$Sp(1,1)$	$S_{\mathbb{C}} \cong S_0 \otimes_{\mathbb{C}} W$	8	$\bigwedge^2 S_0^* \cong \mathbb{C} \oplus V_{\mathbb{C}}$
6	$SL(2,\mathbb{H})$	$S^{\pm}_{\mathbb{C}} \cong S^{\pm}_0 \otimes_{\mathbb{C}} W$	8	$V_{\mathbb{C}} \cong \bigwedge^2 S_0^{+*} \cong (\bigwedge^2 S_0^{-*})^*$
7		$S_{\mathbb{C}} \cong S_0 \otimes_{\mathbb{C}} W$	16	$\bigwedge^2 S_0^* \cong V_{\mathbb{C}} \oplus \bigwedge^2 V_{\mathbb{C}}$
8		$S_{\mathbb{C}} \cong S' \oplus S''$	16	$S'^* \otimes S''^* \cong V_{\mathbb{C}} \oplus \bigwedge^3 V_{\mathbb{C}}$
9		S real	16	$\mathrm{Sym}^2 S^* \cong \mathbb{R} \oplus V \oplus \bigwedge^4 V$
10		S^+, S^- real	16	$\mathrm{Sym}^2 (S^{\pm})^* \cong V \oplus \bigwedge^5_{\pm} V$
11		S real	32	$\mathrm{Sym}^2 S^* \cong V \oplus \bigwedge^2 V \oplus \bigwedge^5 V$

Table 1: Spinors in Lorentz signature

consider theories in 4 dimensions in detail, and theories in higher dimension in somewhat less detail. Therefore, the reader is well-advised to study $n \leq 4$ carefully.

- Many of us are more familiar with properties of spinors for the compact spin group $\mathrm{Spin}(n)$. There is a mod 8 periodicity—a form of Bott periodicity—and for the Lorentz group here we also find a mod 8 periodicity. For example, the column labeled '$\dim S$' has the property that $\dim S$ for $n+8$ is 16 times $\dim S$ for n. Some properties—for example, the reality conditions—are shifted from the compact case.
- The second column reflects the special isomorphisms among low rank simple Lie groups—they identify the orthogonal groups in dimension at most 6 in terms of other groups. Again, the corresponding list of compact groups is more familiar, as are the exceptional isomorphisms of the complex simple groups. Here we list the exceptional isomorphism for the Lorentz groups, which are noncompact real forms. Notice the neat appearance of $SL(2)$ over the reals, complexes, and quaternions in dimensions $n = 3, 4, 6$. The minimal real spin representations have real dimensions $2, 4, 8$. We will see that dimensions $3, 4, 6$ are the critical dimensions for theories with $2, 4, 8$ supersymmetries. (The maximal dimension in which there is a real spinor representation of dimension 16 is $n = 10$. One might speculate that the octonians could play a role, but to my knowledge nothing concrete along those lines has proved useful. Notice the sequence of dimensions $n = 3, 4, 6, 10$ is closely related to the

dimensions $1, 2, 4, 8$ of the division algebras, which reflects the fact that the Clifford algebra for signature $(1, n-1)$ is closely related to the Clifford algebra for signature $(0, n-2)$.)

- The third column lists the smallest real spin representation. If for a moment we consider *complex* spin representations, then there is a mod 2 (Bott) periodicity: In odd dimensions there is a unique irreducible complex spin representation and in even dimensions there are two such irreducible half spin representations. In each dimension we ask if the complex spin representation carries a real or quaternionic structure. The answer depends on n (mod 8) and is indicated in the chart. For $n \equiv 1, 2, 3 \pmod 8$ the representation is real, for $n \equiv 4, 8 \pmod 8$ it is complex, and for $n \equiv 5, 6, 7 \pmod 8$ it is quaternionic. (We call the representation complex if it carries no real or quaternionic structure.)
- We write the complexification of a quaternionic spin representation S as $S_{\mathbb{C}} \cong S_0 \otimes_{\mathbb{C}} W$, where W is a fixed two dimensional complex vector space on which the quaternions $\mathbb{H}$ act (on the right). Also, W carries a fixed complex symplectic form. The quaternionic dimension of S is half the complex dimension of S_0. Thus in $n = 6$ dimensions the spin representation is two dimensional quaternionic. Note that we always use 'S' to denote a *real* spin representation, which for $n = 6$ is the underlying 8 dimensional real representation of the 2 dimensional quaternionic representation.
- S is self-dual—$S \cong S^*$—except in dimensions $n \equiv 2, 6 \pmod 8$ where $(S^+)^* \cong S^-$.
- In even dimensions n there are two *complex* half spin representations S^+, S^-. For $n \equiv 2 \pmod 8$ each of them is real and the underlying real representations are inequivalent. For $n \equiv 6 \pmod 8$ each is quaternionic and the underlying real representations are inequivalent. For $n \equiv 4, 8 \pmod 8$ they are complex conjugate (here denoted S' and S'') and the underlying real representations are isomorphic. The half spin representations are often called *chiral*.
- Physicists use the term 'Dirac spinors' for elements in the full complex spin representation, 'Weyl spinors' for elements in a half spin representation for n even, and 'Majorana spinors' for elements in a real spin representation. For example, they say that 'Majorana-Weyl spinors' exist only for $n \equiv 2 \pmod 8$.
- The last column of the table expresses the vector space V in terms of the square of dual spinors. One feature on Minkowski spacetime is that in every dimension there is a *symmetric* equivariant pairing

$$\Gamma\colon S^* \otimes S^* \longrightarrow V. \tag{3.1}$$

This is not true for all real spin representations in other signatures. This pairing enters into our discussion of superspacetime and the superPoincaré group.

- There also exists a symmetric pairing

$$\tilde{\Gamma}\colon S \otimes S \longrightarrow V \tag{3.2}$$

on spinors. We use it to construct the Dirac form which occurs in the kinetic term of the classical lagrangian for spinor fields. The two pairings (3.1) and (3.2) satisfy a Clifford-type relation which is most easily expressed in indices. With respect to bases $\{e_\mu\}$ of V and $\{f^\alpha\}$ of S, we write

$$\Gamma(f_\alpha, f_\beta) = \Gamma^\mu_{\alpha\beta} e_\mu, \qquad \tilde{\Gamma}(f^\alpha f^\beta) = \tilde{\Gamma}^{\mu\alpha\beta} e_\mu.$$

Here $\{f^\alpha\}$ is the dual basis of S^*. Then the Clifford relation is

$$\Gamma^\mu_{\alpha\beta}\tilde{\Gamma}^{\nu\beta\gamma} + \Gamma^\nu_{\alpha\beta}\tilde{\Gamma}^{\mu\beta\gamma} = 2g^{\mu\nu}\delta^\gamma_\alpha, \tag{3.3}$$

where $g^{\mu\nu}$ is the inverse metric. For field theory these symmetric pairings are more fundamental than Clifford multiplication.

- As mentioned earlier, in dimensions $n \not\equiv 2, 6 \pmod 8$ there is a duality pairing

$$\epsilon \colon S \otimes S \longrightarrow \mathbb{R} \tag{3.4}$$

for any minimal real spin representation S. The pairing is skew-symmetric for $n \equiv 3, 4 \pmod 8$ and symmetric for $n \equiv 1, 5, 7, 8 \pmod 8$. In these cases we normalize (3.4) so that the pairings Γ (3.1) and $\tilde{\Gamma}$ (3.2) are related by

$$\tilde{\Gamma}^{\mu\alpha\beta} = \Gamma^\mu_{\alpha'\beta'}\epsilon^{\alpha\alpha'}\epsilon^{\beta\beta'}. \tag{3.5}$$

In dimensions $n \equiv 2, 6 \pmod 8$ there is a nondegenerate pairing of S^+ with S^-; then Γ for $S^\pm$ and $\tilde{\Gamma}$ for $S^\mp$ are related as in (3.5).

- Dimension $n = 2$ is special in that the vector representation V is reducible: $V \cong V^+ \oplus V^-$. This does not happen in any other dimension, and it is the starting point for many special features of 2 dimensional field theory.

Superspacetime and the superPoincaré group

Minkowski spacetime $\check{M}^n$ is the affine space underlying the translation group V. (The "check" notation serves to distinguish ordinary Minkowski spacetime from the super version introduced shortly.) The Poincaré group and the double cover $\check{P}^n$ of its identity component act as a symmetry groups of $\check{M}^n$. In this section we construct super analogues. We introduce a super Lie group of "translations" which is noncommutative. The underlying supermanifold is super Minkowski spacetime, or *superspacetime* for short. Then we define the superPoincaré group as a certain group of symmetries.

To define superspacetime we fix the following data:

$$
\begin{array}{lll}
 & V & \text{vector space with Lorentz metric} \\
 & C \subset V & \text{positive cone of timelike vectors} \\
(3.6) & S & \text{real spin representation of } \mathrm{Spin}(V) \cong \mathrm{Spin}(1, n-1) \\
 & \Gamma\colon S^* \otimes S^* \longrightarrow V & \text{symmetric equivariant pairing}
\end{array}
$$

We require that the pairing satisfy the positivity condition

$$
\begin{aligned}
&\Gamma(s^*, s^*) \in \overline{C}, && s^* \in S^*, \\
&\Gamma(S^*, s^*) = 0 && \text{if and only if } s^* = 0.
\end{aligned} \tag{3.7}
$$

Notice that the choice of C among the two halves of the lightcone is simply a choice of a positive arrow of time.

The spin representation S need *not* be the minimal ones listed in Table 1. It may be a sum of several copies of the minimal one(s).

We remark that only the conformal structure underlying the Lorentz metric on Minkowski spacetime enters into this data.

Now define the graded Lie algebra

$$\mathcal{L} = V \oplus S^* \tag{3.8}$$

to be the graded vector space shown with V central and nontrivial bracket

$$[Q_1, Q_2] = -2\Gamma(Q_1, Q_2), \qquad Q_1, Q_2 \in S^*. \tag{3.9}$$

Let $\exp \mathcal{L}$ be the corresponding super Lie group and M the underlying supermanifold. The splitting (3.8) means that as a manifold

$$M \cong V \times \Pi S^*.$$

We make some remarks about this construction of superspacetime M:

- A super Lie group is best understood with auxiliary odd parameters: for each commutative superalgebra R the "R-points"—that is, the group whose Lie algebra is the ordinary Lie algebra $(R \otimes \mathcal{L})^0$—form an ordinary group. It is a good exercise to write the group law in this group using the Hausdorff formula. Since $\mathcal{L}$ is only mildly nonabelian, the formulas are quite simple.
- The positivity condition (3.7) has an important consequence: Minus time translation can be written as a sum of squares of odd elements in the Lie algebra $\mathcal{L}$. (The square of

an odd element Q is $\frac{1}{2}[Q,Q]$.) When we quantize a supersymmetric theory the self-adjoint quantum operator which corresponds to minus time translation is the hamiltonian $\hat{H}$. From this Lie algebra statement we conclude that $-\sqrt{-1}\,\hat{H}$ is the sum of squares of odd skew-adjoint operators, and with our sign conventions this means that $\hat{H}$ is nonnegative. (See the end of Lecture 2 and also the discussion in Lecture 1 surrounding (1.10).) So *the energy (hamiltonian) in a supersymmetric field theory is nonnegative*. This also holds in the classical theory as we will see later, at least in examples.

- The sign conventions in the physics literature vary, certainly, but there is probably unanimity in using a positive sign in (3.9) as opposed to our negative sign! *Caveat emptor!* Beware when comparing our signs to those in any physics reference.
- Consider the left-invariant odd distribution $\mathcal{S}^*$ on M which at the origin is $S^* \subset \mathcal{L}$. We might call it a 'conformal structure' on M. The group of diffeomorphisms of M which preserve $\mathcal{S}^*$ is the *superconformal group*; it contains left translations and extensions to superspacetime of conformal transformations on (open subsets of) Minkowski spacetime.

Introduce bases $\{Q_\alpha\}$ for S^* and $\{e_\mu\}$ for V. We write the symmetric pairing Γ as before:

$$\Gamma(Q_\alpha, Q_\beta) = \Gamma^\mu_{\alpha\beta} e_\mu.$$

Let x^μ be the linear coordinates on V and θ^α the odd linear coordinates on ΠS^* corresponding to the given bases. Then $\{x^\mu, \theta^\alpha\}$ is a global coordinate system on superspacetime M. Let ∂_μ and $\partial/\partial\theta^\alpha$ be the corresponding coordinate vector fields. We will usually express our computations in terms of a global framing of M rather than in terms of these global coordinates. We use the basis $\{\partial_\mu, D_\alpha\}$ of *left* invariant vector fields to write lagrangians, and the commuting basis $\{\partial_\mu, \tau_{Q_\alpha}\}$ of *right* invariant vector fields to write symmetries. Note that ∂_μ is both left and right invariant since V is central. The vector fields D_α and τ_{Q_α} are given by the formulas

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} - \Gamma^\mu_{\alpha\beta}\theta^\beta\partial_\mu \\ \tau_{Q_\alpha} &= \frac{\partial}{\partial\theta^\alpha} + \Gamma^\mu_{\alpha\beta}\theta^\beta\partial_\mu, \end{aligned}$$

as the reader may check. The nontrivial brackets are

$$\begin{aligned} [D_\alpha, D_\beta] &= -2\Gamma^\mu_{\alpha\beta}\partial_\mu \\ [\tau_{Q_\alpha}, \tau_{Q_\beta}] &= +2\Gamma^\mu_{\alpha\beta}\partial_\mu. \end{aligned}$$

The brackets of the left invariant D_α are as in the Lie algebra $\mathcal{L}$; brackets of the right invariant τ_{Q_α} are opposite. Also

$$[D_\alpha, \tau_{Q_\beta}] = 0$$

since left invariant vector fields commute with right invariant vector fields.

The coordinates and vector fields are all *real*. In certain cases, such as the $n = 4$ superspacetime we discuss in Lecture 5, it is convenient to use a global framing by *complex* vector fields.

For $n = 1$ and the one dimensional spin representation S we recover the superspacetime and superalgebra encountered in Lecture 2 when we discussed the superparticle. The even part of the algebra is generated by infinitesimal time translation ∂_t and the odd part by an element Q with

$$[Q, Q] = -2\partial_t.$$

The *superPoincaré algebra* is the graded Lie algebra

$$\mathfrak{p} = \big(V \oplus \mathfrak{so}(V)\big) \;\oplus\; S^*. \tag{3.10}$$

It contains both $\mathcal{L}$ and the ordinary Poincaré algebra as subalgebras. The nontrivial brackets are the representation of $\mathfrak{so}(V)$ on V and on S^* as well as the bracket (3.9) on odd elements. We let P denote the form of the corresponding super Lie group whose group of "real points" is the connected group discussed in Lecture 1 which contains $\mathrm{Spin}(1, n-1)$.

A word about nomenclature: Let $s = \dim S$. Then we usually use '$M^{n|s}$' to denote superspacetime and '$P^{n|s}$' to denote the symmetry group. This fixes V and S up to isomorphism in cases where there is only one minimal real spin representations. But in cases where there are chiral spin representations—for $n \equiv 2, 6 \pmod 8$—we use the notations '$M^{n|(s^+,s^-)}$' and '$P^{n|(s^+,s^-)}$' where the decomposition $s = s^+ + s^-$ indicates the dimension of the spaces of chiral supersymmetries. For example, the supersymmetry group relevant to the superparticle is $P^{1|1}$. As other examples of minimal supersymmetry groups we cite $P^{3|2}$, $P^{4|4}$, and $P^{6|(8,0)}$.

Our notation follows the modern convention of labeling supersymmetric theories by the dimension of the space of supersymmetries—the number of Q_α. There is another system which refers to "$N = 1$" supersymmetry, "$N = 2$" supersymmetry, etc. Here the nomenclature is a bit dimension dependent. '$N = 1$' usually refers to the minimal amount of supersymmetry, '$N = 2$' to twice as much, and so on. Thus $N = 1$ supersymmetry in $n = 4$ dimensions refers to the supersymmetry group $P^{4|4}$, $N = 2$ supersymmetry to $P^{4|8}$, and $N = 4$ supersymmetry to $P^{4|16}$. This nomenclature is uneven in dimensions $n \equiv 2, 6 \pmod 8$ with chiral supersymmetry and is also not completely standard for $n = 1$. For example, $N = 1$ supersymmetry in $n = 2$ dimensions usually refers to $P^{2|(1,1)}$ whereas $N = 1$ supersymmetry in $n = 6$ dimensions usually refers to $P^{6|(8,0)}$.

There are inclusions of the supersymmetry groups obtained by setting some spatial translations to be trivial. Table 2 indicates sequences of algebras with a given number of supersymmetries (i.e., a given number of Q_α). These are the symmetry groups of supersymmetric field theories. The *classical* theories are often related by *dimensional reduction*, which is the process of reducing the dimension of a classical field theory by restricting to fields which are constant in a subspace of spatial directions. Notice that in dimensions $n \equiv 2, 6 \pmod 8$ we only obtain *nonchiral* theories by dimensional reduction, that is, theories with equal numbers of chiral supersymmetries. This is a well-known property of spin representations: No multiple of the half spin representation in dimension n extends to a representation of the spin group in higher dimensions.

# SUSYs	SUSY groups
1	$P^{1\|1} \hookrightarrow P^{2\|(1,0)}$
2	$P^{1\|2} \hookrightarrow P^{2\|(1,1)} \hookrightarrow P^{3\|2}$
4	$P^{1\|4} \hookrightarrow P^{2\|(2,2)} \hookrightarrow P^{3\|4} \hookrightarrow P^{4\|4}$
8	$P^{1\|8} \hookrightarrow P^{2\|(4,4)} \hookrightarrow P^{3\|8} \hookrightarrow P^{4\|8} \hookrightarrow P^{5\|8} \hookrightarrow P^{6\|(8,0)}$
16	$P^{1\|16} \hookrightarrow P^{2\|(8,8)} \hookrightarrow \dots \hookrightarrow P^{6\|(8,8)} \hookrightarrow \dots \hookrightarrow P^{10\|(16,0)}$
32	$P^{1\|32} \hookrightarrow P^{2\|(16,16)} \hookrightarrow \dots \hookrightarrow P^{6\|(16,16)} \hookrightarrow \dots \hookrightarrow P^{10\|(16,16)} \hookrightarrow P^{11\|32}$

Table 2: The supersymmetry groups

We remark that in the context of dimensional reductions of classical field theories it is more proper to view Table 2 in terms of *subquotients* rather than inclusions. If we dimensionally reduce a theory on superspacetime M by considering fields constant along a spatial translation τ, then we may as well view fields as functions on $M/\mathbb{R}\tau$. If P is the superPoincaré group of M, the superPoincaré group of $M/\mathbb{R}\tau$ is the subquotient

$$(\text{normalizer of } \mathbb{R}\tau \text{ in } P) \;/\; \mathbb{R}\tau.$$

Central extensions and R-symmetry

There are two algebraic features of the supersymmetry groups which play an important role in some supersymmetric field theories: central extensions and outer automorphisms.

Recall that one important fact about the Lorentz signature is that in the symmetric square of a real spin representation we always find a copy of the vector representation. This is used to construct superspacetime (see (3.6)). For any real spin representation we decompose the symmetric square of the dual as

$$\operatorname{Sym}^2 S^* \cong V \;\oplus\; \mathbb{R}^c \;\oplus\; \bigoplus_i \bigwedge\nolimits^{p_i} V, \tag{3.11}$$

where $p_i \geq 1$. We can add the second term to the supersymmetry algebra (3.10):

$$\tilde{\mathfrak{p}} = \big(V \oplus \mathbb{R}^{c'} \oplus \mathfrak{so}(V)\big) \;\oplus\; S^*, \tag{3.12}$$

for some $c' \le c$. We use the decomposition (3.11) to define the $\mathbb{R}^{c'}$ component in the bracket of odd elements and postulate that $\mathbb{R}^{c'}$ be central. It is easy to check that the Jacobi identity is satisfied. Thus $\tilde{\mathfrak{p}}$ is a central extension of $\mathfrak{p}$ and we have a corresponding central extension at the level of groups.

Elements of the central $\mathbb{R}^{c'}$ are called *central charges*. These central extensions arise in supersymmetric classical field theories. Recall that in any field theory (on Minkowski spacetime) the space $\mathcal{M}$ of classical solutions is a symplectic manifold. In a supersymmetric field theory there is an action of $\mathfrak{p}$ on $\mathcal{M}$ by infinitesimal symplectic transformations. When we lift these infinitesimal symmetries to functions, then we typically get a central extension by locally constant functions. These locally constant functions are the classical central charges. The Poisson brackets of the Noether charges and central charges obey an algebra opposite to that of (3.12).

We can also include any of the other terms in (3.11) in the even part of the superalgebra, but they are not central; they have nontrivial brackets with the Lorentz algebra $\mathfrak{so}(V)$. They play the role of central charges for *extended objects* in field theories, such as the "D-branes" which currently play a prominent role in string theory. (These extended supersymmetry algebras appeared earlier in the supergravity literature.) Although we will not meet them in these lectures, for reference we give the complete decomposition of $\operatorname{Sym}^2 S^*$ for the minimal S in Table 3. We also include two nonminimal examples in dimension $n = 10$; they correspond to two of the superstring theories (and also supergravity theories).

Example. We will encounter the classical version of this example in Lecture 4. We consider $P^{2|(1,1)}$, so the (self-dual) real spin representation $S^+ \oplus S^-$. We have

$$\operatorname{Sym}^2(S^+ \oplus S^-)^* \cong V^+ \ \oplus\ V^- \ \oplus\ \mathbb{R}.$$

(Recall that the vector representation $V \cong V^+ \oplus V^-$ is reducible in $n = 2$ dimensions.) Let $\partial_\pm$ be basis elements of $V^\pm$ and $Q_\pm$ basis elements of $(S^\pm)^*$. Then the nontrivial brackets in $\tilde{\mathfrak{p}}^{2|(1,1)}$ are

$$\begin{aligned} [Q_+, Q_+] &= -2\partial_+ \\ [Q_-, Q_-] &= -2\partial_- \\ [Q_+, Q_-] &= Z, \end{aligned} \tag{3.13}$$

where Z is the central charge. Note $\mathfrak{p}^{2|(1,1)}$ is obtained by setting $Z = 0$.

The group $P^{2|(1,1)}$ sits as a subgroup in $P^{3|2}$, as indicated in Table 2. In the larger group the bracket $[Q_+, Q_-]$ is -2 times infinitesimal translation in the spatial direction which has been set to zero in passing from $P^{3|2}$ to $P^{2|(1,1)}$. In general, for any $P^{n|s}$ which is obtained from a larger $P^{n'|s}$ by dimensional reduction, there is a central extension with $n' - n$ central charges. All of the central

n	SUSY group based on S^*	$\mathrm{Sym}^2 S^*$
1	$P^{1\|1}$	V
2	$P^{2\|(1,0)}$	V^+
2	$P^{2\|(0,1)}$	V^-
3	$P^{3\|2}$	V
4	$P^{4\|4}$	$V \oplus \bigwedge^2 V$
5	$P^{5\|8}$	$V \oplus \mathbb{R} \oplus 3 \bigwedge^2 V$
6	$P^{6\|(1,0)}$	$V \oplus 3 \bigwedge^3_+ V$
6	$P^{6\|(0,1)}$	$V \oplus 3 \bigwedge^3_- V$
7	$P^{7\|16}$	$V \oplus 3\,\mathbb{R} \oplus \bigwedge^2 V \oplus 3 \bigwedge^3 V$
8	$P^{8\|16}$	$V \oplus 2\,\mathbb{R} \oplus \bigwedge^3 V \oplus \bigwedge^4 V$
9	$P^{9\|16}$	$V \oplus \mathbb{R} \oplus \bigwedge^4 V$
10	$P^{10\|(16,0)}$	$V \oplus \bigwedge^5_+ V$
10	$P^{10\|(0,16)}$	$V \oplus \bigwedge^5_- V$
10	$P^{10\|(32,0)}$	$3\,V \oplus \bigwedge^3 V \oplus 3 \bigwedge^5_+ V$
10	$P^{10\|(0,32)}$	$3\,V \oplus \bigwedge^3 V \oplus 3 \bigwedge^5_- V$
10	$P^{10\|(16,16)}$	$2\,V \oplus \mathbb{R} \oplus \bigwedge^2 V \oplus 3 \bigwedge^4 V \oplus \bigwedge^5 V$
11	$P^{11\|32}$	$V \oplus \bigwedge^2 V \oplus \bigwedge^5 V$

TABLE 3: DECOMPOSITION OF $\mathrm{Sym}^2 S^*$

charges in Table 3 are accounted for by dimensional reduction, and it is possible that central charges always arise this way.

The connected group of outer automorphisms of the superPoincaré group is called the R-symmetry group. Since we will not use it in these lectures, we only comment that the action is trivial on the Poincaré subgroup. In other words, R-symmetries only act nontrivially on the Q_α. We will see an example when we study $P^{4|4}$ in Lecture 5.

Representations of the superPoincaré group

Recall from Lecture 1 that the Hilbert space of a quantum field theory in n dimensions is a nonnegative energy unitary representation of the group $\check{P}^n$. Quantum particles are irreducible representations. In a supersymmetric theory this representation extends to a unitary representation of $P^{n|s}$ for some s depending on the amount of supersymmetry. The irreducible representations are not irreducible when restricted to the subgroup $\check{P}^n$, but rather break up as a finite sum. Such collections of particles are called supersymmetric *multiplets*. The particle content of a supersymmetric theory is organized into such multiplets. There is a general story, but to understand supersymmetric quantum field theories one needs to learn the taxonomy of multiplets in various dimensions with various amounts of supersymmetry. We begin with the general theory and work through the examples relevant to Lecture 4. In Lecture 5 we will return to compute more examples which are relevant there.

Recall that in interacting theories the spins of particles are constrained to be at most 2, and if there is no gravity then at most 1. Werner Nahm observed that this severely constrains the possible supersymmetric theories, and this condition is the reason why Table 1 stops at $n = 11$. We will see later in this lecture how the quantization of classical fields leads to irreducible representations of $\check{P}^n$. Thus it is natural to use the analysis of supersymmetry particle multiplets to constrain and predict (interacting) supersymmetric classical field theories. In this way Nahm predicted some supergravity theories which were constructed later.

Consider the supersymmetry group $P^{n|s}$. (We understand $s = (s^+, s^-)$ if $n \equiv 2, 6 \pmod 8$.) As in (1.???) we consider the subgroup V of translations:

$$1 \to V \to P^{n|s} \to \mathrm{Spin}(1, n-1) \ltimes \Pi S^* \to 1$$

Suppose we are given an irreducible positive energy representation of $P^{n|s}$. Upon restriction to V we obtain an irreducible orbit of infinitesimal characters of some mass $m \geq 0$ in V^*, as in Figure 2. The little groups

$$\begin{aligned} m > 0: &\qquad \mathrm{Spin}(n-1) \ltimes \Pi S^* \\ m = 0: &\qquad \mathrm{Spin}(n-2) \ltimes \Pi S^* \end{aligned} \tag{3.14}$$

now include odd parts as well. (Recall that the little group is the reductive part of the stabilizer.) The representation of $P^{n|s}$ is induced from a projective (graded) finite dimensional unitary representation of the little group—as before, we form a homogeneous (graded) hermitian vector bundle over the orbit and take the Hilbert space of sections. The fact that the representation is projective is due to the nontrivial bracket on S^*. More precisely, at $\lambda \in V^*$ we seek a representation of the central extension constructed using the quadratic form q_λ on S^* given by

$$q_\lambda(Q_1, Q_2) = \lambda\big([Q_1, Q_2]\big).$$

Recall the positivity condition (3.7). Since λ is in the closure of the positive dual lightcone, then in view of (3.9) we conclude that q_λ is negative semidefinite. In the massive case the form is negative definite, whereas in the massless case it has a kernel. Except for the exceptional case $s = 1$ the kernel has dimension $s/2$. (This follows by splitting V into the sum of a two dimensional Lorentz space and a negative definite complement, thus reducing to the $n = 2$ dimensional case and an explicit computation.) In the massless case we work with the quotient of S^* by the kernel, and then in both cases we obtain a negative definite space. A representation of (3.14) is then a graded Clifford module

$$W = W^0 \oplus W^1$$

for S^* (modulo the kernel in the massless case) together with an intertwining action of the appropriate Spin group. Our sign conventions for graded Hilbert spaces and self-adjoint operators lead to a slightly different construction than usual.

A few remarks before the examples:

- In this construction we end up considering spinors of spinors—a Clifford module for the vector space of spinors. Also, keep in mind that S is a *real* space, whereas the representation of the little group we construct—the spinors of spinors—is *complex*. The space S is a representation of $\mathrm{Spin}(n-1)$ or $\mathrm{Spin}(n-2)$ by restriction of the $\mathrm{Spin}(1, n-1)$ action.
- Usually $\dim S$ is even and we construct the unique irreducible graded Clifford module by complexifying S and choosing a lagrangian splitting, as described in (1.???).
- The irreducible massless representations tend to be smaller than the irreducible massive representations because of the drop in dimension. We will not see that in our example below since s is very small.
- The representation theory of the supersymmetry groups with central charges is similar. We present it through a simple example below. The important concept of a *BPS particle* is introduced there.
- Allowable representations in a local quantum field theory are constrained to be CPT invariant, as described in Lecture 1.
- We choose the grading of W—which piece is even and which piece odd—according to the spin-statistics principle.
- The even and odd pieces of a Clifford module have equal dimensions. This means that *a supersymmetry multiplet has an equal number of bosonic degrees of freedom and fermionic degrees of freedom*. (The number of degrees of freedom is the dimension of the representation of the little group.) There is an exception for chiral theories in $n = 2$, however. For example, there exists a theory with symmetry group $P^{2|(1,0)}$ with one (chiral) fermion and no bosonic degrees of freedom.

Example 1 **($P^{3|2}$).** We first review the representations of the group $\check{P}^3$. For massive representations the little group is $\mathrm{Spin}(2)$, and its irreducible complex representations are one dimensional and labeled by the spin j. All of the representations are CPT invariant. The spin $j = 0$ representation is called a *massive scalar particle*, the spin $j = 1/2$, $j = -1/2$ representations *massive spinor* *particles*, and the spin $j = 1$, $j = -1$ representations *massive half-vector particles*. (The *massive vector particle* is the sum of the $j = 1$ and $j = -1$ representations.) In the massless case the little group is $\mathrm{Spin}(1) \cong \mathbb{Z}/2$. There are two irreducible complex representations, each of dimension one. The trivial one is the *massless scalar particle* and the nontrivial one is the *massless spinor particle*.

Why isn't there a massless vector particle? We'll answer that question at the end of the lecture.

Now we turn to the supersymmetry group $P^{3|2}$. In the massive case we have in addition to the little group $Spin(2)$ a negative definite Clifford algebra generated by S^*, which here has dimension 2. After complexification we choose a basis $\{Q_+, Q_-\}$ where $\mathrm{Spin}(2)$ acts on $Q_\pm$ by $j = \pm 1/2$. A physicist describes the Clifford module as follows. Fix a *vacuum* vector $|0\rangle$ and postulate

$$Q_-|0\rangle = 0.$$

Then the Clifford module is

$$W = \mathbb{C} \cdot |0\rangle \;\oplus\; \mathbb{C} \cdot Q_+|0\rangle.$$

To define the action of $\mathrm{Spin}(2)$ we need to specify a spin j for $|0\rangle$; then the spin of $Q_+|0\rangle$ is $j + 1/2$.

To summarize: A massive multiplet for $P^{3|2}$ is a pair of particles of adjacent spins j and $j + 1/2$.

	-1	$-1/2$	0	$1/2$	1
massive scalar		1	1		
massive scalar			1	1	
massive vector	1	1		1	1

Table 4: Massive multiplets for $P^{3|2}$

We show some possibilities in Table 4, which indicates the number of representations with a given spin j which occur in each multiplet. We only list multiplets with spin at most 1; they are the relevant multiplets for supersymmetric theories without gravity. The multiplets are usually named after the highest spin boson. There are two *massive scalar multiplets* because of the two different possibilities for a massive spinor. In addition to the *massive vector multiplet* shown, there are also *massive half-vector multiplets*.

At the end of this lecture we will learn how to construct representations from fields. It is instructive to see how the two different spinor particles arise—the difference comes from the sign in the mass term in the lagrangian. We will not consider massive vectors in these lectures, but it is amusing to see how the massive half-vectors can be written in terms of fields—the 3 dimensional Chern-Simons term enters.

Now we consider the massless multiplets. In this case the quadratic form on S^* is degenerate, and the quotient by the kernel is one dimensional. The irreducible complex Clifford module $W = W^0 \oplus$

W^1 is still two dimensional. This is atypical; with more supersymmetry massless representations have fewer degrees of freedom than massive ones. The little group $\mathrm{Spin}(1) \cong \mathbb{Z}/2$ acts trivially on W^0 and nontrivially on W^1. So there is a unique massless multiplet, the *massless scalar multiplet,* which contains one scalar particle and one spinor particle.

Example 2 ($\tilde{P}^{2|(1,1)}$). This example illustrates the representation theory for a centrally extended supersymmetry group. We refer to (3.13) for the bracketing relations in the central extension $\tilde{\mathfrak{p}}^{2|(1,1)}$ of the symmetry algebra $\mathfrak{p}^{2|(1,1)}$. From these brackets we easily compute

$$\frac{1}{2}[Q_+ \pm Q_- \, , \, Q_+ \pm Q_-] = -\partial_t \pm Z, \tag{3.15}$$

where

$$\partial_t = \partial_+ + \partial_-$$

is infinitesimal time translation. Note that the left hand side of (3.15) is the square of an odd element. Thus the quantum operators $\hat{H} \pm \hat{Z}$ corresponding to the right hand side of (3.15) are nonnegative; in other words,

$$\hat{H} \geq |\hat{Z}|. \tag{3.16}$$

Here $\hat{H}$ is the quantum hamiltonian. In an irreducible supersymmetry representation the operator $\hat{Z}$ is a constant since Z is central. For $Z = 0$ the inequality (3.16) is a special case of the general argument that the quantum hamiltonian in a supersymmetric theory is nonnegative. But with the central charge we have a stronger bound. The Poincaré invariant statement is the *BPS bound*: the *mass* is bounded below by the absolute value of the central charge. Furthermore, if we have equality—the mass equal to the absolute value of the central charge—then the quadratic form on the odd part of the little group has a half dimensional kernel, just as for massless representations. In this case that is evident in (3.15). So we obtain special irreducible massive multiplets—*BPS multiplets*—which typically have fewer degrees of freedom than the usual massive multiplets. If this is the case, then they are stable under perturbations. Hence these BPS representations are an important source of stable particles in supersymmetric theories.

The states in a BPS representation are annihilated by 1/2 of the supersymmetry, in this example by $Q_+ + Q_-$ or $Q_+ - Q_-$. There are more complicated situations where a different fraction of the supersymmetry annihilates a representation, which is also termed BPS.

In Lecture 4 we will see a classical field configuration which satisfies a classical version of the BPS condition for the supersymmetry group $\tilde{P}^{2|(1,1)}$: It solves the equations of motion and is annihilated by 1/2 of the classical supersymmetry.

A quick tour of fields and classical field theories

We restrict the discussion to field theories formulated on Minkowski spacetime $\check{M} = \check{M}^n$. Then a field is a function (or tensor field or spinor field—in general, a section of a fiber bundle) on $\check{M}$. In our discussion and in all examples of field theories we consider we assume the spin-statistics connection. For a classical field the *spin* is determined by the representation of the Lorentz group under which it transforms. Thus differential forms have integer spin whereas spinor fields have half-integer spin. Spin-statistics asserts that fields of integer spin are classical bosons, whereas fields of half-integer spin are classical fermions. The space of fields is a supermanifold whose underlying manifold of real points consists of classical bosons.

In Minkowski spacetime the space of fields is *real* and the lagrangian of a unitary field theory is also *real*. The analytic continuation to Euclidean space—which we do not consider in these lectures—may not be real.

The catalog of fields which arise is brief, and it is helpful to survey it briefly if for no other reason than to establish vocabulary. We proceed according to spin, beginning with fields of integer spin—bosons. Here we discuss both linear and nonlinear fields. For linear fields we have the possibility of writing free (noninteracting) lagrangians, and we will do that in the next section. Nonlinear fields are necessarily interacting.

The simplest *spin 0* field is a *real scalar field* $\phi\colon \check{M} \to \mathbb{R}$. The space of all real scalar fields is a real vector space. Quite generally, a *scalar field* is a map $\phi\colon M \to X$ for some manifold X. To write down a kinetic term we need a Riemannian metric on X.

One example of a linear *spin 1* field is a 1-form $\alpha \in \Omega^1(\check{M})$. The space of these fields is again a real vector space. We can take instead the space of connections with abelian structure group $\mathbb{R}^{>0}$ or $\mathbb{T}$, the circle group. If we fix a principal bundle P with one of these structure groups, then the space of connections on P is a real affine space whose associated vector space of translations is isomorphic to the space of 1-forms. A connection is also called a *gauge field*. These theories have *gauge equivalence* or *local symmetry*: we work on the quotient of the space of fields by gauge transformations. For 1-forms this means that the space of fields is the real vector space $\Omega^1(\check{M})/d\Omega^0(\check{M})$. The "space" of abelian connections is really a category—already the bundles on which the connections live form a category—and the space of connections modulo gauge equivalence is the affine space of equivalence classes in the category. There is a nonlinear version of these spin 1 fields: the space of gauge equivalence classes of connections with some nonabelian structure group G. There are other *linear* spin 1 fields as well, namely p-form fields with $1 < p < n$. Again there is a gauge equivalence and the true space of fields is the quotient $\Omega^p(\check{M})/d\Omega^{p-1}(\check{M})$. There is also an affine version analogous to abelian connections, but it is more difficult to describe. However, there is not a known nonlinear or nonabelian analog of p-form fields. The p-form fields with $p \geq 2$ often arise in supergravity theories in dimensions $n \geq 6$. We will not encounter them in these lectures.

Finally, a *spin 2* field is a traceless symmetric tensor field or a metric. It is the basic field in theories of gravity. Here too there is local symmetry: the group of local symmetries is the diffeomorphism group.

Turning now to fermions, or fields of half-integral spin, a *spin 1/2* field is a *spinor field* $\psi\colon \check{M} \to S$ for some real spin representation S of $\mathrm{Spin}(1, n-1)$. This is the second use for the information in Table 1; the first was in the construction of superspacetime. Here too it is important that we use *real* spin representations, since fields in Minkowski spacetime are real. The representation S need not be minimal. There is no nonlinear version of fermion fields.

A *spin 3/2* field is called a *Rarita-Schwinger field*. It is a function $\chi\colon \check{M} \to (S \otimes V^*)_0$, where S is a minimal real spin representation and the subscript means that we take the highest weight irreducible piece of the tensor product. These fields only arise in supergravity theories. There is a local symmetry as well: we work on the quotient by the image of the covariant derivative of spinor fields $\check{M} \to S$.

To describe a field theory in Minkowski spacetime we might first list the field content: scalar fields with values in such-and-such a manifold, a gauge field with gauge group so-and-so, spinor fields of this-or-that particular type, and so on. A *bosonic theory* is one whose fields are all bosonic. There is a standard classification of theories according to the field content. A bosonic theory with only scalar fields is called a *σ-model*. If the scalar fields take values in a linear space it is called a *linear σ-model*, otherwise a *nonlinear σ-model*. A *gauge theory* includes a gauge field (connection) and possibly also scalar fields. A *pure* gauge theory is a theory with only a gauge field; theories which also include scalar fields can be described as the *coupling* of a gauge theory with a σ-model. One form of coupling is that the scalar field might be a section of a fiber bundle associated to the connection field. For example, we might have a theory with an abelian connection, say with structure group $\mathbb{T}$, and a complex scalar field which is a section of an associated complex line bundle. A theory of *gravity* has the spin 2 metric and then possibly gauge fields and scalar fields as well.

We can also construct theories with fermions only or with both bosons and fermions. The theories we consider in these lectures are *supersymmetric*; a supersymmetry group acts as a group of symmetries of the classical theory. Since supersymmetries exchange even and odd fields, such theories necessarily have both bosons and fermions. The theories are named after the highest spin boson field. For example, a *supersymmetric σ-model* contains only scalar fields and spinor fields. A *supersymmetric gauge theory* contains gauge fields and spinor fields; it may contain scalar fields as well. Finally, a *supergravity* theory contains a metric and Rarita-Schwinger field, and it may contain fields of lower spin as well. For example, the two supergravity theories in $n = 10$ dimensions with $s = 32$ supersymmetries contain fields of all the different possible spins.

A supersymmetric theory with more than the minimal number of supersymmetries is often said to have *extended supersymmetry*, and the corresponding supersymmetry groups are referred to as *extended supersymmetry groups*. These theories often, but not always, come by *dimensional re-*

duction from a theory in a higher spacetime dimension with the same number of supersymmetries. (We dimensionally reduce a classical theory by restricting to fields which are constant along some spatial directions. We give some examples in the Lecture 4.) Theories with minimal supersymmetry are simpler, and if a theory with extended supersymmetry comes by dimensional reduction from one with minimal supersymmetry, then it is easier to understand that way. There are theories with extended supersymmetry which are *not* dimensional reductions, most notably theories with extended *chiral* supersymmetry in dimensions $n \equiv 2, 6 \pmod 8$. (These theories have supersymmetry group $P^{n|(s^+,s^-)}$ and more than the minimal amount of supersymmetry.)

# SUSY	maximal dimension	σ-model	gauge theory	supergravity
1	2	S	S	S
2	3	S	S	S
4	4	S (Kähler)	S	S
8	6	✓(hyperkähler)	S	✓
16	10		✓	✓
32	11			✓

Table 5: Supersymmetric theories

In Table 5 we catalog supersymmetric theories according to the number of supersymmetries. In the second column we indicate the maximal spacetime dimension n in which the theory may be formulated; theories in lower dimension are obtained by dimensional reduction. Keep in mind that there are exceptional theories with extended supersymmetry—like the chiral theories just mentioned—which do not come by dimensional reduction. The chiral theories in dimensions 2,6,10 listed in Table 5 have the minimal amount of supersymmetry. In the maximal dimension each theory listed in the table has minimal supersymmetry. An 'S' indicates that there is a superspacetime formulation; a '✓' indicates that the theory exists, but there is no adequate superspacetime formulation (off-shell); and blank spot in the table means that there is no theory. The blank spots can be predicted from the representation theory of the supersymmetry group and the constraint on the spin of interacting quantum particles, as we indicated in the last section. A few remarks about the table:

- In Lecture 2 we discussed two formulations of the superparticle. First, we described the theory in *components*, that is, in terms of ordinary fields. Then we gave a super(space)time formulation in terms of *superfields*. All theories with a superspacetime formulation (marked 'S' in Table 5) may be described in terms of either superfields or in terms of component fields. Theories with no superspacetime formulation only have a description in terms of "component fields", though these fields are not the components of anything—they are just fields on ordinary Minkowski spacetime. We will see that the component formulation of theories which have a superspacetime formulation often includes *auxiliary fields* which enter the lagrangian algebraically. Theories without a superspacetime formulation have no auxiliary fields.

- Theories with a superspacetime formulation have manifest supersymmetry in the superspacetime formulation. The supersymmetry algebra closes off-shell. In the component formulation the supersymmetry is not manifest, but the algebra still closes off-shell. (For that we need to include the auxiliary fields if there are any.) If there is no superspacetime formulation, then the supersymmetry algebra does not close off-shell. (Recall the example of the coupled harmonic oscillators in Lecture 2 for an analog of these phenomena in ordinary classical mechanics.)
- The superspacetime formulations are most useful for understanding the supersymmetry, since in this way it is manifest. However, to see the physics of the theory it is best to work in components. Some computations are easier in superspacetime, some easier in components. If there is a superspacetime formulation, then it is useful in the quantum theory to give *a priori* constraints on the possible quantum corrections. Such corrections must respect the supersymmetry, and the constraints imposed are more easily seen in superspacetime.
- The most commonly used superspacetimes are those in dimensions $n \leq 4$ with $s \leq 4$ supersymmetries. For theories with more supersymmetry, we can still use the $s = 4$ superspacetime to keep part of the supersymmetry manifest. Nevertheless, superspacetimes with $s > 4$ have striking applications.
- We indicated that there is a superspacetime formulation of supersymmetric gauge theories with 8 supersymmetries, but this only applies to *pure* gauge theories. The superspacetime formulation in 6 dimensions is a bit deficient in some ways; the reduction to 4 dimensions is better
- There are superspacetime formulations of σ-models with 8 supersymmetries for special kinds of hyperkähler manifolds, but none which works in general.

From classical fields to quantum particles

Finally, we indicate how to pass from classical fields on Minkowski spacetime $\check{M}^n$ to positive energy unitary representations of the symmetry group $\check{P}^n$. We use the *free* lagrangian. A lagrangian is *free* if it leads to a *linear* classical equation of motion. For a free supersymmetric theory the construction we describe gives a representation of the appropriate supersymmetry group. We restrict our discussion to real scalar fields, spinor fields, and abelian connections. The story for other free fields (p-form fields, Rarita-Schwinger fields, and symmetric tensor fields) is similar. We leave the easy computations to the reader as exercises in the classical field theory yoga described in the previous lecture.

Let $\check{M} = \check{M}^n$ denote n dimensional Minkowski spacetime. We take standard coordinates $x^0 =$

$ct, x^1, \dots, x^{n-1}$, where c is the speed of light and t represents time. The metric is

$$(dx^0)^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2$$

and the standard density is

$$|d^n x| = |dx^0\, dx^1 \,\cdots\, dx^{n-1}|.$$

The wave operator on functions is

$$\Box = (-1)^{n-1} * d * d = -d^* d = \frac{\partial^2}{(\partial x^0)^2} - \frac{\partial^2}{(\partial x^1)^2} - \cdots - \frac{\partial^2}{(\partial x^{n-1})^2}.$$

Example 1 (free real scalar field). Let $\phi \colon \check{M} \to \mathbb{R}$ be a real scalar field. We take as lagrangian density

$$L = \big(\frac{1}{2}|d\phi|^2 - \frac{m^2}{2}|\phi|^2\big)\, |d^n x|.$$

Here $m \geq 0$ is the mass of the field. Then a short computation shows that the variational 1-form is

$$\gamma = \delta\phi \wedge *d\phi$$

and the local symplectic form is

$$\omega = *d\delta\phi \wedge \delta\phi. \tag{3.17}$$

The equations of motion which defines the space of classical solutions $\mathcal{M}$ is the linear wave equation

$$(\Box + m^2)\phi(x) = 0. \tag{3.18}$$

Here the space of fields $\mathcal{F}$ is a real vector space, and the space of classical solutions $\mathcal{M}$ is a linear subspace. It carries a symplectic form which we obtain by integrating (3.17) over a codimension one spacelike subspace of Minkowski spacetime.

We remark that for this integral to converge we need to impose some L^2 conditions on the spatial integrals of fields and their variations, but we do not stress that here.

We analyze equation (3.18) by fixing an origin in $\check{M}$ and passing to the Fourier transform $\hat{\phi} \colon V^* \to \mathbb{C}$. It is a complex function which satisfies the reality condition

$$\hat{\phi}(-k) = \overline{\hat{\phi}(k)}, \qquad k \in V^*. \tag{3.19}$$

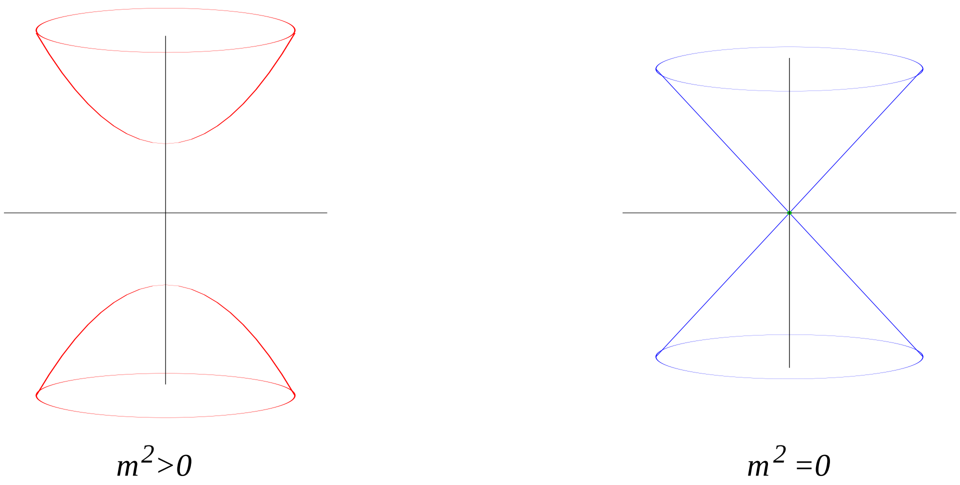

FIGURE 7: THE MASS SHELL FOR $m^2 > 0$ AND $m^2 = 0$

The classical equation of motion (3.18) is

$$\left(-|k|^2 + m^2\right)\hat{\phi}(k) = 0,$$

and so solutions are supported on the mass shell $\{k \in V^* : |k|^2 = m^2\}$ depicted in Figure 7. Note that these solutions form a real representation of the symmetry group $\check{P}^n$.

We quantize $\mathcal{M}$ just as we did the free particle in Lecture 1. That is, we choose a polarization and take the quantum Hilbert space to be the space of functions on one of the lagrangian subspaces in the polarization. In this example we choose a *complex* polarization

$$\mathcal{M} \otimes \mathbb{C} \cong L \oplus L'.$$

The complexification of $\mathcal{M}$ consists of complex functions on the mass shell with no reality condition, and we take L to be the subspace of functions supported on the set of points with positive time. Then the quantum Hilbert space is the Fock space

$$\mathcal{H} = L^2(L) \supset \mathrm{Sym}^\bullet(L^*).$$

The *one particle Hilbert space* is

$$\mathcal{H}_1 = L^*.$$

It is the irreducible representation of $\check{P}^n$ with the trivial spin 0 representation of the little group. This is the quantum particle which corresponds to the free field ϕ.

Observe that in the massless case there is a global symmetry group of translations in the codomain $\mathbb{R}$ of the scalar field.

Example 2 (free spinor field). Let S be a real spin representation of $\mathrm{Spin}(1, n-1)$. A spinor field is a map $\psi\colon \check{M} \to \Pi S$. In this case the space of fields $\mathcal{F}$ is an odd real vector space. We use the symmetric pairing

$$\tilde{\Gamma}\colon S \otimes S \longrightarrow V$$

(see (3.2)) to construct the kinetic term. A mass term exists only if we have a nonzero *skew-symmetric* pairing

$$M\colon S \otimes S \longrightarrow \mathbb{R}.$$

If there is a nonzero mass pairing M, then there exists a *normalized* skew-symmetric pairing

$$\epsilon\colon S \otimes S \longrightarrow \mathbb{R};$$

the normalization condition is (3.5). For simplicity assume that ϵ is irreducible. Then we can write

$$M = m\epsilon, \qquad m \in \mathbb{R}. \tag{3.20}$$

The free lagrangian density for the spinor field is

$$L = \Big(\frac{1}{2}\psi \not{D}\psi - \frac{1}{2}\psi M \psi\Big)\,|d^n x|.$$

The kinetic term is the *Dirac* form

$$\psi \not{D}\psi = \tilde{\Gamma}^{\mu}(\psi, \partial_\mu \psi) = \tilde{\Gamma}^{\mu\alpha\beta}\psi_\alpha\,\partial_\mu \psi_\beta.$$

It is a symmetric (in the graded sense) bilinear form up to an exact term, so it is exactly symmetric after integrating over M, assuming no contribution at infinity. We use the convention that the name of a bilinear form is written between the arguments. The mass pairing is

$$\psi M \psi = M^{\alpha\beta}\psi_\alpha \psi_\beta.$$

Now the equation of motion asserts that ψ is in the kernel of $\not{D} - M$. As for the scalar field, we analyze this condition using the Fourier transform. Then ψ is a classical solution if and only if

$$\sqrt{-1}\,\tilde{\Gamma}^{\mu\alpha\beta}k_\mu \hat{\psi}_\beta(k) = M^{\alpha\beta}\hat{\psi}_\beta(k), \qquad \text{for all } \alpha. \tag{3.21}$$

Multiplying both sides by $\sqrt{-1}\,\Gamma^{\nu}_{\gamma\alpha}k_\nu$, we deduce from the Clifford relation (3.3) and also (3.20), (3.5) that $\hat{\psi}$ is supported on the mass shell $|k|^2 = m^2$. Furthermore, (3.21) imposes a constraint on $\hat{\psi}$ restricted to the mass shell; it cuts down by a factor of 2 the number of degrees of freedom. Therefore, the space of classical solutions $\mathcal{M}$ to the Dirac equation is the space of Real sections of an odd complex vector bundle E of rank $\dim S/2$ over the mass shell; the Reality condition (3.19) is that coming from the Fourier transform. As before we quantize using a complex polarization. The one particle (odd) Hilbert space is the space of sections of E^* on the "forward half" of the mass shell. It is a spin 1/2 representation of $\check{P}^n$ with $\dim S/2$ degrees of freedom. The (graded) Hilbert space of the free quantum theory is the completed symmetric algebra on the one particle Hilbert space; it contains both fermions and bosons.

Example 3 (massless abelian gauge field). For definiteness take the gauge group to be the circle group $\mathbb{T}$ of unit norm complex numbers. On the Lie algebra $\mathrm{Lie}(\mathbb{T}) \cong \sqrt{-1}\,\mathbb{R}$ we have the positive definite inner product $\langle a, b\rangle = -ab$. The field in a gauge theory is a connection A on $\check{M}$ with structure group $\mathbb{T}$, and we work with fields up to gauge equivalence. The space of equivalence classes is a real affine space. The lagrangian is

$$L = -\frac{1}{2}\langle F_A \wedge *F_A\rangle = -\frac{1}{2}|F_A|^2\,|d^n x|. \tag{3.22}$$

The factor $-\frac{1}{2}$ is chosen so that the kinetic term is properly normalized—write (3.22) out in coordinates to check! The classical equation of motion is the linear equation

$$d * F_A = 0 \tag{3.23}$$

on the curvature, and the classical moduli space of connections modulo equivalence which satisfy (3.23) is an affine subspace of the space of fields mod equivalence. We can quantize this symplectic affine space as usual.

Of course, the detailed analysis involves choosing an origin, so we may as well from the beginning work with the vector space of translations associated to connections. Hence we take the field to be a *real* 1-form α up to addition of an exact real function. The lagrangian is

$$L = -\frac{1}{2}d\alpha \wedge *d\alpha = -\frac{1}{2}|d\alpha|^2\,|d^n x|,$$

and the equation of motion is

$$d * d\alpha = 0.$$

As usual, we consider the Fourier transform $\hat{\alpha} = \hat{\alpha}_\mu dx^\mu$ on V^*. The vector space of classical solutions is the first cohomology space of the complex

$$\Omega^0(\check{M}) \xrightarrow{d} \Omega^1(\check{M}) \xrightarrow{*d*d} \Omega^1(\check{M}).$$

On the Fourier transforms we have

$$\begin{aligned} d\hat{f}(k) &= \sqrt{-1}\, k_\mu \hat{f}(k)\, dx^\mu \\ * \, d * d\hat{\alpha}(k) &= |k|^2 \hat{\alpha}(k) - \langle \hat{\alpha}(k), k \rangle k. \end{aligned}$$

An easy argument shows that the first cohomology is isomorphic to the collection of Real functions $\{\hat{\alpha}\colon \text{lightcone} \to V^* \otimes \mathbb{C}\}$ on the lightcone which satisfy $\langle \hat{\alpha}(k), k\rangle = 0$ modulo the set of functions $\{\hat{\alpha}(k) = \hat{f}(k)k\}$, where $\hat{f}$ ranges over the complex functions on the lightcone. (The Reality condition is that of the Fourier transform.) Again we polarize by choosing the lagrangian subspace of complex-valued functions supported on the forward lightcone. Its dual, the one-particle Hilbert space, is the irreducible representation of $\check{P}^n$ induced from the $n-2$ dimensional representation of the massless little group $\mathrm{Spin}(n-2)$, the so-called "vector representation". It has $n-2$ degrees of freedom. The Hilbert space of the free theory is the completed symmetric algebra on this representation.

Finally, we answer the question raised earlier: Why is there no massless vector particle in $n = 3$ dimensions? Recall that the little group is $\mathrm{Spin}(1) \cong \mathbb{Z}/2$ and there is only one bosonic representation—the trivial representation. It is the quantum particle which corresponds to the free massless scalar field on $\check{M}^3$. But clearly we can also consider a free massless abelian gauge field on $\check{M}^3$. From Example 3 we see that the corresponding quantum particle is the same as for the massless scalar field. This is the first example of a general and important phenomenon called *duality*. A single quantum theory—here the free field theory of a massless boson in $n = 3$ dimensions—has distinct realizations in terms of fields.