Lecture 2: Lagrangians and Symmetries

As we mentioned in Lecture 1 the geometry and physics of a classical field theory—for example, the equations of motion and the symplectic structure—can often be derived from a lagrangian density. That is true of all the theories we will meet. In this lecture we study some of the basic formal structure of *classical* lagrangian field theory, with particular emphasis on symmetries and the associated Noether currents. Although we develop the theory in some generality, our examples in this lecture are from mechanics. Examples with fields will occupy us in the remaining lectures. Our main example here—the superparticle—combines the two systems studied in Lecture 1.

The formal idea of a symmetry in a *quantum* theory is straightforward. Recall that the Hilbert space of the theory, as well as the one particle Hilbert space of the theory, is graded

$$\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1 \tag{2.1}$$

into bosons and fermions. A symmetry of the theory is a unitary operator on $\mathcal{H}$. In ordinary quantum mechanics we demand that a symmetry commute with the Hamiltonian; the group of symmetries in a quantum field theory includes the Poincaré group and perhaps also a group of symmetries which commute with the action of the Poincaré group. These symmetries all preserve the grading in (2.1). We can also talk about the infinitesimal symmetries which generate one parameter groups of unitary transformations—physicists usually take them to be self-adjoint and often mathematicians prefer them to be skew-adjoint. *Supersymmetries* are operators which reverse the grading in (2.1). In quantum mechanics they commute with the Hamiltonian; in fact, in the case we meet in this lecture the Hamiltonian is the square of the supersymmetry. In higher dimensions they combine with ordinary symmetries to form an extension of the Poincaré algebra. We discuss it in Lecture 3. But in this lecture we focus on the classical story.

There are many different notations for calculus and differential geometry in finite dimensions, and even more for calculus on function spaces. I choose to write everything in terms of differential forms and use the few formulas I can remember (Cartan's formula for Lie derivative, Stokes' theorem, etc.). The reader should try the computations in whatever notation is most comfortable.

There are some tricky signs in differential geometry which we encounter, and we comment on them at the end of the lecture. The presence of odd variables in classical theories with fermions is an additional sign danger. Finally, there are some conventions and choices which must be made. We indicate some of these conventions as we go along. (We already met a few in Lecture 1; see (1.???), (1.14), and (1.???). These three choices are not independent.)

Some differential geometric preliminaries

So as not to interrupt the exposition later, we begin with some general facts from differential geometry and comment on some sign conventions.

First, if M is a manifold and $E \to M$ a real or complex vector bundle with connection ∇, then we form the twisted de Rham complex

$$\Omega^0_M(E) \xrightarrow{d_\nabla = \nabla} \Omega^1_M(E) \xrightarrow{d_\nabla} \Omega^2_M(E) \xrightarrow{d_\nabla} \cdots . \tag{2.2}$$

It is not a complex in general, but rather

$$d_\nabla^2 = R_\nabla,$$

where R_∇ is the curvature of ∇.

Next, if $\phi\colon M \to X$ is a map between manifolds, then $d\phi \in \Omega^1_M(\phi^* TX)$. If now TX has a connection ∇, then

$$d_\nabla d\phi = \phi^* T_\nabla, \tag{2.3}$$

where T_∇ is the torsion of the connection ∇. We will apply this to the Levi-Civita connection of a Riemannian manifold; its torsion vanishes.

Recall that a *density* on a manifold M is a tensor field which in a local coordinate system $\{x^\mu\}$ is represented by

$$L(x)\,|dx^1 \cdots dx^n|.$$

It transforms under change of coordinates by the absolute value of the Jacobian of the coordinate change. In other words, it is a twisted n-form—the twisting is by the orientation bundle. We denote the set of densities by '$\Omega^{|0|}(M)$'. Then we define $\Omega^{|-p|}(M)$ to be the set of twisted $(n-p)$-forms. For example, an element of $\Omega^{|-1|}(M)$ in local coordinates looks like

$$j^\mu(x)\,\frac{\partial}{\partial x^\mu} \otimes |dx^1 \cdots dx^n|.$$

On the complex of twisted forms $\Omega^{|\bullet|}(M)$ we have the usual operations of exterior differentiation d, Lie derivative $\mathrm{Lie}(\xi)$ by a vector field ξ, and interior product $\iota(\xi)$, with the usual relationships among them. Just as we can integrate densities over manifolds, we can integrate $|-p|$-forms over codimension p submanifolds with oriented normal bundle. We comment shortly on our motivation for using twisted forms; little is lost by using ordinary forms instead.

Next, we make explicit some sign choices. We define vector fields to act on functions by derivations and denote this action by writing the vector field ξ on the left of the function f in one of two ways: 'ξf' or '$\iota(\xi)df$'. To move the vector field past the function and write '$df(\xi)$' we invoke the sign rule. Thus if we have a local coordinate system $\{x^\mu, \theta^\alpha\}$ on a supermanifold with the x^μ even and the θ^α odd, then

$$\iota(\frac{\partial}{\partial\theta})d\theta = 1, \qquad d\theta(\frac{\partial}{\partial\theta}) = -1.$$

Given a graded vector space $W = W^0 \oplus W^1$, we define the parity-reversed vector space ΠW to be

$$(\Pi W)^0 = W^1, \qquad (\Pi W)^1 = W^0. \tag{2.4}$$

This has sign ambiguities when the coefficients are a general commutative superalgebra, so we are better off defining $\Pi W = W \otimes \mathbb{R}^{\text{odd}}$, where $\mathbb{R}^{\text{odd}}$ are the scalars viewed as an odd vector space. But for most computations the heuristic definition (2.4) is adequate.

Finally, we remind the reader of our convention (1.11) for differential forms.

Lagrangian field theory

A disclaimer: The formal structure we develop here applies to many, but not all, examples. Modifications must be made to accommodate some topological terms (for example, Chern-Simons terms) and other types of terms involving p-form fields for $p \geq 2$. We will not encounter these subtleties.

The stage for Lagrangian field theory is set by a choice of *spacetime* M and a space of *fields* $\mathcal{F}$. We let n denote the dimension of spacetime.

First some remarks about M:

- In physics one often takes the spacetime to be Minkowski spacetime M^n, the affine space discussed following (1.1). In these lectures we work almost exclusively with this choice.
- In mechanics $n = 1$ and Minkowski spacetime M^1 is simply the affine space underlying $\mathbb{R}$; it represents time. This is the case for all examples considered in this lecture.
- For discussions of the *Hamiltonian* theory $M = M^1 \times N$ is time cross space.
- In Euclidean field theory M is Euclidean space or more generally a Riemannian manifold. This is often the case in applications to differential geometry.
- We allow M to be a supermanifold.

Formally the space of fields $\mathcal{F}$ is the space of sections of a fiber bundle $E \to M$. We will not usually write fields in this form, but in all examples considered in these lectures it is possible. For example, a *real scalar field* is a function $M \to \mathbb{R}$, which may be considered a section of the trivial bundle $M \times \mathbb{R} \to \mathbb{R}$. A *connection* on a principal bundle $P \to M$ can be considered to be a section of a certain associated bundle of affine spaces. *Spinor fields* are sections of the vector bundle of spinors. There is a canonical evaluation map

$$\begin{aligned} \mathcal{F} \times M &\longrightarrow E \\ (\phi, x) &\longmapsto \phi(x). \end{aligned} \tag{2.5}$$

Lagrangians are usually defined in terms of this evaluation map and its derivatives, and we will simply use the standard notation '$\phi(x)$' or simply 'ϕ' for it. We allow E (and so $\mathcal{F}$) to be a supermanifold.

Our main examples. We consider two main examples in this lecture. Both examples are in mechanics ($n = 1$) where the spacetime $M = M^1$ is time. In both examples we fix an external Riemannian manifold X which represents space. The reader may wish to consider the case when X is flat Euclidean space. In the first example of a *classical particle* the space of fields

$$\mathcal{F}_1 = \{x\colon M^1 \to X\}$$

is the space of paths on X. The second example is the *superparticle* which in addition to the boson x has a fermionic field ψ which is an odd tangent vector field along the image of x:

$$\mathcal{F}_2 = \{x\colon M^1 \to X, \quad \psi \in \Omega^0_{M^1}(x^*\Pi TX)\}.$$

Notice that $\mathcal{F}_1$ is an ordinary manifold whereas $\mathcal{F}_2$ is a supermanifold. The fermion ψ is interpreted by working over arbitrary commutative superalgebras (auxiliary odd parameters) as explained in Lecture 1.

We do calculus on $\mathcal{F} \times M$ and so study differential forms there. But instead of ordinary forms on the spacetime M we use twisted forms. Thus we work in the bigraded complex $\Omega^{\bullet,|\bullet|}(\mathcal{F} \times M)$. The differential on M is denoted 'd' and the differential on $\mathcal{F}$ is denoted 'δ'. The total differential on $\mathcal{F} \times M$ is

$$D = \delta + d$$

and we have

$$D^2 = d^2 = \delta^2 = 0, \qquad d\delta = -\delta d.$$

We draw pictures of this complex as follows:

$$\begin{array}{c|ccc}
 & 0 & 1 & \cdots \\ \hline
|0| & & & \\
 & d\uparrow & & \\
|-1| & & \underset{\delta}{\longrightarrow} & \\
\cdots & & &
\end{array} \qquad \mathcal{F}$$

$$M$$

Finally we are ready to introduce the main character. A *lagrangian (density)* L is an element

$$L \in \Omega^{0,|0|}(\mathcal{F} \times M).$$

Thus for each field $\phi \in \mathcal{F}$, the lagrangian $L(\phi)$ is a density on M. Usually L is a *local* function of the field ϕ, and in good cases such as we encounter in these lectures L depends only on the *1-jet* of the fields. (More generally, a *local* lagrangian L depends only on the k-jet for some finite k.) This means that the value of the density $L(\phi)$ at a point x of spacetime depends only on $\phi(x)$ and the first derivatives of ϕ at x. Certain features of classical field theory simplify for lagrangians which only depend on the 1-jet of the fields.

There are two reasons we take the lagrangian to be a density, rather than an n-form. First, it is notationally easier to consider it to have total degree 0. More importantly, writing it as a density makes clear the symmetries—there are often orientation-reversing symmetries of a theory. These are not present if writing the lagrangian necessitates a choice of orientation on M. We will see one example of this in Lecture 5.

The *classical action* S is the integral of the lagrangian density:

$$S(\phi) = \int_M L(\phi). \tag{2.6}$$

It is a real-valued function on $\mathcal{F}$. Of course, it is only well-defined on a subspace of fields (with finite action), though we are free to integrate over subsets of M. The action is a function on the space of fields $\mathcal{F}$ and is the main object of study in the calculus of variations. It remains off-stage in our presentation of the formal aspects—we express our computations locally on M in terms of the lagrangian.

The particle lagrangians. For the classical particle the lagrangian is

$$L_1 = \frac{1}{2}|\dot{x}(t)|^2\,dt. \tag{2.7}$$

Usually we omit the argument 't' and simply write $L_1 = \frac{1}{2}|\dot{x}|^2\,dt$. Also note that we have fixed the standard orientation of the line and are not so pedantic as to write the absolute values around dt. (The notation in some of the computations would be awful if we used twisted forms!) However, time-reversal *is* a symmetry of the theory.

The lagrangian of the superparticle is the sum of this lagrangian and a term which couples x to the fermion ψ:

$$L_2 = \left\{\frac{1}{2}|\dot{x}|^2 + \frac{1}{2}\langle\psi, \nabla_{\dot{x}}\psi\rangle\right\}dt. \tag{2.8}$$

Notice that the second term is essentially the lagrangian (1.17) discussed in Lecture 1 when X is flat space.

The supporting character in a classical field theory is the *variational 1-form*

$$\gamma \in \Omega^{1,|-1|}(\mathcal{F} \times M).$$

We define the *total lagrangian* to be the sum of the lagrangian density and the variational 1-form:

$$\mathcal{L} = L + \gamma \in \Omega^0(\mathcal{F} \times M).$$

It has total degree 0. The defining property of a lagrangian field theory is that

$$(D\mathcal{L})^{1,|0|} = \delta L + d\gamma \in \Omega^{1,|0|}(\mathcal{F} \times M)$$

be *linear over functions* (on M). To explain this, note first that the tangent space at ϕ to the space of fields is

$$T_\phi \mathcal{F} = \Omega^0_M\big(\phi^* T(E/M)\big).$$

Here $\phi\colon M \to E$ is a section of the fiber bundle $E \to M$, and $T(E/M)$ is the tangent bundle along the fibers. Now $T_\phi \mathcal{F}$ is a module over Ω^0_M, the algebra of functions on M, and $(D\mathcal{L})^{1,|0|}$ is linear over functions if for all $(\phi, m) \in \mathcal{F} \times M$ we have

$$(D\mathcal{L})^{1,|0|}_{(\phi,m)}(f\xi) = f(m)(D\mathcal{L})^{1,|0|}_{(\phi,m)}(\xi), \qquad f \in \Omega^0_M, \quad \xi \in T_\phi\mathcal{F}.$$

This condition is easier to understand in a computation, after which we make some comments about γ.

Hence we compute the variation of the lagrangian for the classical particle (2.7):

$$\begin{aligned} \delta L_1 &= \langle \delta_\nabla \dot{x}, \dot{x}\rangle \wedge dt \\ &= \langle \delta_\nabla dx, \dot{x}\rangle \\ &= -\langle d_\nabla \delta x, \dot{x}\rangle \\ &= -d\big(\langle \delta x, \dot{x}\rangle\big) - \langle \delta x, d_\nabla \dot{x}\rangle \\ &= -d\big(\langle \dot{x}, \delta x\rangle\big) - \langle \delta x, \nabla_{\dot{x}}\dot{x}\rangle \wedge dt. \end{aligned} \tag{2.9}$$

In the third line we use the fact that the Levi-Civita connection is torsionfree (see (2.3)), and twice we use $dx = \dot{x}\,dt$. Thus we *choose*

$$\gamma_1 = \langle \dot{x}, \delta x\rangle.$$

Then $\delta L_1 + d\gamma_1$ involves only δx and not $\delta_\nabla dx$. So it only involves variations of the fields, and not variations of derivatives of the fields. This is what the linearity over functions means. Notice that γ_1 is also linear over functions.

The manipulations (2.9) are standard and usually done under the integral sign in the classical action (2.6). In that context one only considers tangent vectors to $\mathcal{F}$ which have compact support, so that the integral converges. Then the integral of δL is simply the variation δS of the action, and it vanishes at critical points of S. The exact term in (2.9) integrates to zero. Thus for the classical particle the critical points of the action are the solutions to the geodesic equation

$$\nabla_{\dot x}\dot x = 0.$$

More generally, we define the space of *(classical) solutions* $\mathcal{M} \subset \mathcal{F}$ in our classical field theory to be the largest submanifold on which the *equation of motion* vanishes for all points of M:

$$(D\mathcal{L})^{1,|0|} = 0 \qquad \text{on } \mathcal{M} \times M.$$

Often $\mathcal{M}$ is called the *classical moduli space*. Fields in $\mathcal{M}$ do not necessarily have finite action; rather, they are characterized as fields where the compactly supported variations of the action vanish.

Some terminology: Physicists refer to the space of classical solutions $\mathcal{M}$ as *on-shell* and the whole space of fields $\mathcal{F}$ as *off-shell*.

Now a few comments about the variational 1-form γ:

- A classical field theory is not defined by just the lagrangian L; we must specify γ as well.
- If the lagrangian L depends only on the 1-jet of the fields, then there is a canonical choice for γ which is characterized as being linear over functions. We always choose this γ.
- In many mechanics texts you will find this canonical γ written as '$p_i dq^i$'.

Next, we compute the variational 1-form and the equation of motion for the superparticle lagrangian (2.8). We already computed the variation of the first term, so now we compute

$$\begin{aligned} \delta\Bigl(\frac{1}{2}\langle\psi,\nabla_{\dot x}\psi\rangle\,dt\Bigr) &= \delta\Bigl(\frac{1}{2}\langle\psi,d_\nabla\psi\rangle\Bigr) \\ &= \frac{1}{2}\langle\delta_\nabla\psi\wedge d_\nabla\psi\rangle + \frac{1}{2}\langle\psi,\delta_\nabla d_\nabla\psi\rangle. \end{aligned} \tag{2.10}$$

The first term is linear over functions; we need to "integrate by parts" in the second:

$$\begin{aligned} \langle\psi,\delta_\nabla d_\nabla\psi\rangle &= -\langle\psi,d_\nabla\delta_\nabla\psi\rangle + \langle\psi,R(\delta x\wedge dx)\psi\rangle \\ &= -d\bigl(\langle\psi,\delta_\nabla\psi\rangle\bigr) + \langle d_\nabla\psi\wedge\delta_\nabla\psi\rangle + \langle\psi,R(\delta x\wedge dx)\psi\rangle \\ &= -d\bigl(\langle\psi,\delta_\nabla\psi\rangle\bigr) + \langle\delta_\nabla\psi\wedge d_\nabla\psi\rangle + \langle\delta x\wedge R(\psi,\psi)dx\rangle. \end{aligned} \tag{2.11}$$

Here R is the Riemann curvature tensor of X and we have used its symmetries in passing to the last equation. In the first step we used (2.2). Combining (2.9), (2.10), and (2.11) we find

$$\delta L_2 = -d\gamma_2 + \langle \delta x, -\nabla_{\dot{x}}\dot{x} + \frac{1}{2}R(\psi,\psi)\dot{x}\rangle \wedge dt + \langle \delta_{\nabla}\psi, \nabla_{\dot{x}}\psi\rangle \wedge dt, \tag{2.12}$$

where

$$\gamma_2 = \langle \dot{x}, \delta x\rangle + \frac{1}{2}\langle \psi, \delta_{\nabla}\psi\rangle.$$

We also read off from (2.12) the equations of motion:

$$\begin{aligned} \nabla_{\dot{x}}\dot{x} &= \frac{1}{2}R(\psi,\psi)\dot{x} \\ \nabla_{\dot{x}}\psi &= 0. \end{aligned} \tag{2.13}$$

Some remarks about these computations and results:

- The reader may want to put in some auxiliary odd parameters to understand the manipulations better. Also, there are many alternative ways to do this computation (e.g. in local coordinates) and there is profit in trying alternative methods.
- Recall that with no auxiliary parameters the classical fermion field ψ vanishes, and so the superparticle reduces to the ordinary bosonic classical particle. In particular, the equations of motion reduce to the geodesic equation. I do not know any cogent geometric interpretation to the modification which occurs in (2.13) when we put in auxiliary parameters other than the general picture for such equations presented in Lecture 1.
- The lagrangian L_2 involves the square of the first derivative of the boson x but only the first power of the first derivative of the fermion ψ. This is typical for kinetic terms in lagrangians. If we introduce scaling dimensions $[x], [\psi]$ for the fields x, ψ, then the bosons and fermions will be weighted differently so that both terms have the same total weight. For example, we can assign
$$\begin{aligned} [x] &= -1/2 \\ [\psi] &= 0 \\ [t] &= -1. \end{aligned}$$
Then both terms have total weight 0. Notice that no derivatives of ψ enter into the variational 1-form γ, and that the equation of motion for ψ is first order whereas the equation of motion of x is second order.
- The second term in the lagrangian couples x to ψ, and so the equations of motion also couple the two fields. By this argument alone we know that the geodesic equation gets modified. This is a version of Newton's law that for every action there is an equal and opposite

reaction: The equation of motion for ψ depends on x, hence the equation of motion for x must depend on ψ.

- We assume that X is a *complete* Riemannian manifold. Then if we fix a time $t = t_0$ then a point in the space $\mathcal{M}_2$ of classical solutions is determined by the triple $\big(x(t_0), \dot{x}(t_0), \psi(t_0)\big)$. This is a point in the supermanifold described as the vector bundle

$$\pi^*\Pi TX \longrightarrow TX, \tag{2.14}$$

where $\pi\colon TX \to X$ is the projection. In this supermanifold the ring of functions is the space of sections of the exterior algebra on the dual of the bundle (2.14). The space of classical solutions $\mathcal{M}_2$ is a subsupermanifold of the supermanifold of fields $\mathcal{F}_2$, but the explicit description (2.14) depends on the choice of time.

- Below (2.15) we introduce a symplectic form on $\mathcal{M}_2$. We discuss the quantization of $\mathcal{M}_2$ briefly towards the end of the lecture.

Returning to a general classical field theory $\mathcal{L} = L + \gamma$, on-shell we define the *local symplectic form* ω to be

$$\omega = D\mathcal{L} = \delta\gamma \in \Omega^{2,|-1|}(\mathcal{M} \times M).$$

Our data on-shell fits into the diagram:

$$\begin{array}{c|ccccc} & 0 & & 1 & & 2 \\ \hline |0| & L & \to & 0 & & \\ & & & \uparrow & & \\ |-1| & & & \gamma & \to & \omega \end{array} \quad \mathcal{M}$$

$$M$$

It is simple to see that

$$D\omega = 0.$$

In mechanics ($n = 1$) for each time t the form $\omega(t)$ is a closed 2-form on the space of classical solutions $\mathcal{M}$ since $\delta\omega = 0$, and it is independent of t since $d\omega = 0$. A theory is *nondegenerate* if ω is a symplectic form.

In higher dimensions we must work in the Hamiltonian framework with a spacetime $M = M^1 \times N$ to see a symplectic structure. Then for each time $t \in M^1$ we integrate the local symplectic form to obtain a 2-form

$$\Omega(t) := \int_{\{t\}\times N} \omega$$

on the space of classical solutions. For simplicity we assume N compact so as not to have convergence issues. Then again $\Omega(t)$ is independent of t and in good cases is a symplectic form.

In classical mechanics texts the symplectic form ω is usually denoted '$dp_i \wedge dq^i$'.

We leave the reader to compute the symplectic forms

$$\omega_1 = \langle \delta_\nabla \dot{x} \wedge \delta x \rangle$$

for the classical particle and

$$\omega_2 = \langle \delta_\nabla \dot{x} \wedge \delta x \rangle + \frac{1}{2}\langle \delta_\nabla \psi \wedge \delta_\nabla \psi \rangle + \frac{1}{2}\langle \psi, R(\delta x \wedge \delta x)\psi \rangle \tag{2.15}$$

for the superparticle. (Here again we use the fact that the Levi-Civita connection is torsionfree: $\delta_\nabla \delta x = 0$.)

Symmetries and Noether currents

As with any mathematical structure we want to consider symmetries. We begin by recalling the notion of symmetry in symplectic geometry. Thus suppose that $(\mathcal{M}, \Omega)$ is a symplectic manifold. A symmetry (often called a *symplectomorphism*) is simply a diffeomorphism $\varphi\colon \mathcal{M} \to \mathcal{M}$ which preserves the symplectic form Ω. An infinitesimal symmetry, or Hamiltonian vector field, is a vector field ξ with $\mathrm{Lie}(\xi)\Omega = 0$. In that case $-\iota(\xi)\Omega$ is a closed 1-form and if it is exact it is the differential of a function which is the associated *momentum* or *charge* Q_ξ:

$$\delta Q_\xi = -\iota(\xi)\Omega. \tag{2.16}$$

(We use 'δ' for the differential on $\mathcal{M}$.) The charge is detemined up to a locally constant function. If ξ_1, ξ_2 are infinitesimal symmetries with associated charges Q_1, Q_2, then the *Poisson bracket* $\{Q_1, Q_2\}$ is the function

$$\{Q_1, Q_2\} = \xi_1 Q_2 = \iota(\xi_2)\iota(\xi_1)\Omega,$$

and it is a charge for the infinitesimal symmetry $[\xi_1, \xi_2]$. If $\mathfrak{h}$ is a Lie algebra of infinitesimal symmetries, then the Poisson algebra of all associated charges is a central extension of $\mathfrak{h}$ by locally constant functions.

In lagrangian field theory there is a local version of this story. We first define a symmetry, and it turns out there are two types: manifest and nonmanifest. To each symmetry we attach a local Noether current, and the global Noether charge is obtained in a Hamiltonian picture by integrating the Noether current over a spacelike codimension one submanifold. There is a Poisson bracket on Noether currents which is a local version of the usual Poisson bracket on Noether charges. In mechanics there is no integration over space and the Noether current is the Noether charge. On-shell the global charge is related to the symmetry by (2.16)—it is the charge as defined in symplectic geometry. There is a local version relating the Noether current to the local symplectic form. In the lagrangian framework we have explicit formulas for the currents and charges in terms of the lagrangian. In particular, to a manifest symmetry a lagrangian field theory attaches a *canonical* Noether charge, whereas the symplectic structure only determines a charge up to a locally constant function.

So we begin by defining a symmetry in a lagrangian field theory. We work in our general setup with spacetime M, space of fields $\mathcal{F}$, total lagrangian $\mathcal{L} = L + \gamma$, local symplectic form ω, and space of classical solutions $\mathcal{M}$. For simplicity we only give the definitions for infinitesimal symmetries, which are vector fields on $\mathcal{F} \times M$.

Definition 2.40. A vector field ξ on $\mathcal{F} \times M$ is a *manifest infinitesimal symmetry* if

$$\mathrm{Lie}(\xi)\mathcal{L} = 0 \qquad \text{on } \mathcal{F} \times M. \tag{2.17}$$

A vector field ξ on $\mathcal{F} \times M$ is a *nonmanifest infinitesimal symmetry* if there exists

$$\alpha_\xi \in \Omega^{-1}(\mathcal{F} \times M)$$

such that

$$\bigl(\mathrm{Lie}(\xi)L\bigr)^{0,|0|} = (d\alpha_\xi)^{0,|0|} \qquad \text{on } \mathcal{F} \times M; \tag{2.18}$$

$$\bigl(\mathrm{Lie}(\xi)\mathcal{L}\bigr)^{1,|-1|} = \bigl(D\alpha_\xi\bigr)^{1,|-1|} \qquad \text{on } \mathcal{M} \times M. \tag{2.19}$$

The definition of a manifest symmetry is the usual functorial notion—a symmetry of a mathematical structure is an automorphism which preserves all the data. Roughly speaking, a nonmanifest symmetry preserves the data up to exact terms. But be careful that (2.18) holds off-shell and (2.19) holds on-shell.

We illustrate the definition in examples soon, but first make several remarks:

- For manifest symmetries $\alpha_\xi = 0$. Note that the condition (2.17) to be a manifest symmetry is stronger than the conditions (2.18), (2.19) with $\alpha_\xi = 0$.
- There is ambiguity in the definition of α_ξ for a nonmanifest symmetry—we can add any D-closed form. In fact, it is only the D-cohomology class which affects the global Noether charge, but the local Noether current is changed by D-exact forms as well. By contrast, the Noether current of a manifest symmetry is canonically determined.

- The vector fields in examples are sums of a vector field on $\mathcal{F}$ and a vector field on M. Such *decomposable* vector fields preserve the bigrading on differential forms, and the Lie derivative by such vector fields commutes with both d and δ.
- The usual definition of a symmetry in lagrangian field theory only specifies conditions on the Lie derivative of L. Thus it is natural to ask to what extent the conditions on γ follow from the conditions on L. For manifest symmetries there is nothing to worry about. A manifest symmetry is usually induced from a bundle automorphism of $E \to M$ which covers a diffeomorphism of M. (Recall that fields are sections of $E \to M$.) If the lagrangian L depends only on the 1-jet of the fields, and if it is preserved by the diffeomorphism of $\mathcal{F} \times M$, then it follows that γ is also preserved and we have a manifest symmetry. For nonmanifest symmetries I don't know the full answer. Here the usual condition required is (2.18)—that there be some
$$a_\xi \in \Omega^{0,|-1|}(\mathcal{F} \times M)$$
such that
$$\mathrm{Lie}(\xi)L = da_\xi \qquad \text{on } \mathcal{F} \times M.$$
In any theory this implies that if ξ is decomposable, then
$$\mathrm{Lie}(\xi)\gamma - \delta a_\xi$$
is d-closed, but to have a nonmanifest symmetry we need to know that it is d-exact. I do not know conditions which guarantee this.

We summarize the conditions for a nonmanifest symmetry in the diagrams:

$$\begin{array}{c|c} & 0 \\ \hline |0| & \left(\mathrm{Lie}(\xi)\mathcal{L}\right)^{0,|0|} \\ & \uparrow \\ |-1| & a_\xi \end{array} \quad \mathcal{F}$$
$$M$$

$$\begin{array}{c|ccc} & 0 & & 1 \\ \hline |0| & \left(\mathrm{Lie}(\xi)\mathcal{L}\right)^{0,|0|} & & \\ & \uparrow & & \\ |-1| & a_\xi & \to & \left(\mathrm{Lie}(\xi)\mathcal{L}\right)^{1,|-1|} \\ & & & \uparrow \\ |-2| & & & b_\xi \end{array} \quad \mathcal{M}$$
$$M$$

It is easy to check that for any infinitesimal symmetry ξ, we have
$$\mathrm{Lie}(\xi)\omega = d(\text{something}) \qquad \text{in } \Omega^{2,|-1|}(\mathcal{M} \times M).$$
This is a local version of the statement that ξ is a Hamiltonian vector field.

We define the *Noether current* to be

$$J_\xi = \iota(\xi)\mathcal{L} - \alpha_\xi \in \Omega^{-1}(\mathcal{F}\times M), \tag{2.20}$$

and the $(0,|-1|)$-component to be

$$j_\xi = J_\xi^{0,|-1|}.$$

Note for manifest symmetries $\alpha_\xi = 0$ and the right hand side of (2.20) reduces to just the first term. It is easy to verify from the Cartan formula that in all cases

$$DJ_\xi = -\iota(\xi)\omega + \beta^{2,|-2|} \qquad \text{on } \mathcal{M}\times M \tag{2.21}$$

for some $\beta^{2,|-2|}$. In particular,

$$dj_\xi = 0 \qquad \text{on } \mathcal{M}\times M. \tag{2.22}$$

This is the statement that the Noether current j_ξ is *conserved*. Also, (2.21) implies

$$\delta j_\xi = \bigl(-\iota(\xi)\omega\bigr)^{1,|-1|}, \tag{2.23}$$

which is the local version of (2.16). In a Hamiltonian situation $M = M^1\times N$ we define the Noether charge to be the function on $\mathcal{M}$ given by

$$Q_\xi = \int_{\{t\}\times N} j_\xi.$$

The local conservation law (2.22) implies that Q_ξ is well-defined independent of t, i.e., that it is a constant of motion. The integral of (2.23) is exactly the relation (2.16) between the charge, the symmetry, and the symplectic form. Note: The Noether charge is only defined for a vector field on $\mathcal{F}\times M$ which projects to a vector field on $\mathcal{F}$.

In mechanics the manifold N is a point and there is no integration. So $Q_\xi = j_\xi$.

Examples of symmetries in mechanics

We give a series of examples culminating in an example of an odd symmetry—a supersymmetry—for the superparticle.

Example 1 (linear momentum). Consider a classical free particle $x\colon M^1 \to X$ moving in flat space $X = \mathbb{R}^m$. We include the mass m in the lagrangian

$$L = \frac{m}{2} \sum_i (\dot{x}^i)^2 \, dt.$$

Here we use standard coordinates on $\mathbb{R}^m$. As we computed before

$$\gamma = m \sum_i \dot{x}^i \, \delta x^i.$$

We take as symmetry ξ_j translation in X in the j^{th} coordinate direction:

$$\begin{aligned} \iota(\xi_j)dt &= 0 \\ \iota(\xi_j)\delta x^i &= \delta^i_j, \end{aligned} \tag{2.24}$$

where δ^i_j has its usual meaning. The reader should check that this is a manifest symmetry. This can be done by computation or by a simple geometrical argument. The associated current, or charge, is a component of the linear momentum (mass times velocity):

$$j_{\xi_j} = m\delta_{jk}\dot{x}^k.$$

Angular momentum can be treated similarly; it is the conserved charge associated to rotations.

Note that in this example ξ_j is a vector field on $\mathcal{F} \times M$ which points only in the $\mathcal{F}$ direction, and it projects to a vector field on $\mathcal{F}$. Such manifest symmetries are "internal" in the sense that they leave the points of spacetime invariant and only act on the codomain of the fields. For theories formulated on Minkowski spacetime we usually have the entire Poincaré group acting as manifest symmetries of the theory, but those symmetries move the points of spacetime. In one dimension (mechanics) the Poincaré group reduces to time translation, which we consider in the next example.

We remark that the notation in (2.24) is not the usual one in the physics literature, nor in the mathematics literature for that matter! Usually one writes simply '$\delta x^i = \delta^i_j$' or perhaps '$\delta_{\xi_j} x^i = \delta^i_j$'. In our notation δ is the differential on the space of fields, whereas in these other notations δ is a vector field on that space. We choose our notation simply to be able to use the usual formulas for differential forms, and we live with the somewhat burdensome (2.24). We are happy here to use the alternative equations $\xi_j t = 0$, $\xi_j x^i = \delta^i_j$ here, but in other examples (e.g. Example 2 below) this notation is not adequate.

Example 2 (time translation as a manifest symmetry). Again we consider the classical particle $x\colon M^1 \to X$ and now we allow an arbitrary Riemannian manifold X. We also introduce a (time-independent) potential function $V\colon X \to \mathbb{R}$ and the lagrangian

$$L = \Bigl\{ \frac{m}{2}|\dot{x}|^2 - V(x) \Bigr\}\, dt. \tag{2.25}$$

This lagrangian is kinetic energy *minus* potential energy, the standard form for a particle lagrangian. The potential term does not affect the computation of γ since it involves no derivatives of the fields so does not enter into any integration by parts. Time translation is a manifest symmetry of this theory: we translate in time and simultaneously pull back the field x by the inverse. This transformation fixes the evaluation map (2.5) and also the density dt, and so L is manifestly invariant. The corresponding infinitesimal symmetry is the vector field ξ on $\mathcal{F}\times M$ given as

$$\begin{aligned} \iota(\xi)dt &= 1 \\ \iota(\xi)\delta x &= -\dot{x}. \end{aligned} \tag{2.26}$$

The associated charge has contributions from both $\iota(\xi)\gamma$ and $\iota(\xi)L$. It turns out to be

$$j_\xi = -\Bigl\{ \frac{m}{2}|\dot{x}|^2 + V(x) \Bigr\},$$

which is *minus* the *hamiltonian* or *energy*

$$H = -j_\xi = \frac{m}{2}|\dot{x}|^2 + V(x).$$

Notice the sign change from the lagrangian: The hamiltonian is kinetic energy *plus* potential energy. It is unfortunate that the hamiltonian is *minus* the Noether charge of time translation, but there is nothing we can do about it. The definition is chosen so that the kinetic energy is positive.

Example 2′ (time translation as a nonmanifest symmetry). Time translation is manifest because the action on fields is induced from the action on spacetime. We can make it nonmanifest simply by forgetting the action on spacetime. So define the vector field $\hat{\xi}$ by

$$\begin{aligned} \iota(\hat{\xi})dt &= 0 \\ \iota(\hat{\xi})\delta x &= -\dot{x}. \end{aligned}$$

It is a vector field on $\mathcal{F}$ which has been lifted to $\mathcal{F}\times M$. Now $\hat{\xi}$ is a nonmanifest symmetry of the lagrangian (2.25) with

$$\alpha_{\hat{\xi}} = -\bigl(\frac{m}{2}|\dot{x}|^2 - V(x)\bigr) + C.$$

We put in a constant C to emphasize the indeterminacy which exists for nonmanifest symmetries. Then the Noether current is

$$j_{\hat{\xi}} = \iota(\hat{\xi})\gamma - \alpha_{\hat{\xi}} = -\big(\frac{m}{2}|\dot{x}|^2 + V(x) + C\big).$$

Of course we have the same result as before, but now there is an undetermined constant in the hamiltonian.

Example 3 (system of harmonic oscillators). Here we take space X to be a real inner product space of dimension m and the potential to be quadratic:

$$L = \big(\frac{1}{2}|\dot{x}|^2 - \frac{1}{2}|x|^2\big)\,dt. \tag{2.27}$$

The equation of motion is $\ddot{x} = -x$ which leads to oscillatory motion. Consider the vector field ξ on $\mathcal{F} \times M$ (really on $\mathcal{F}$) defined by

$$\begin{aligned} \iota(\xi)dt &= 0 \\ \iota(\xi)\delta x &= Ax + B\dot{x}, \end{aligned}$$

where A is skew-symmetric and B is symmetric. Then ξ is a nonmanifest symmetry with

$$\alpha_\xi = \frac{1}{2}\langle \dot{x}, B\dot{x}\rangle - \frac{1}{2}\langle x, Bx\rangle.$$

The associated charge is

$$j_\xi = \langle \dot{x}, Ax\rangle + \frac{1}{2}\langle \dot{x}, B\dot{x}\rangle + \frac{1}{2}\langle x, Bx\rangle.$$

We make several remarks:

- For $B = 0$ the symmetry is manifest. These infinitesimal symmetries generate the action of the orthogonal group on X, and the lagrangian (2.27) is manifestly invariant under orthogonal transformations of the target. The associated charges in this case are angular momenta.
- For $A = 0$, $B = -\mathrm{id}$ this is the (nonmanifest) infinitesimal time translation considered in the previous example; the associated charge is minus the hamiltonian.
- For all other cases the symmetry is nonmanifest. Physically, the additional conserved charges are clear. This is a system of m coupled harmonic oscillators and if we choose a basis which diagonalizes the metric we have m oscillators moving independently. They each have a conserved energy, and these are the charges associated to special choices of A

and B. Other charges are generated from these and the angular momenta by Poisson bracket.

- We have specified a finite dimensional vector space of vector fields on $\mathcal{F}$. It is not closed under Lie bracket, but rather generates an infinite dimensional Lie algebra of vector fields. However, if we look on-shell—that is, on the space of classical solutions $\mathcal{M}$—then this vector space of vector fields *is* closed under Lie bracket. In fact, the resulting Lie algebra is isomorphic to the Lie algebra of the unitary group $U(k)$. Note that it is the elements of the Cartan subalgebra whose associated charges are the independently conserved energies referred to in the previous remark.

The reader may wish to return to this example after learning more about supersymmetry. For some supersymmetric theories there is a formulation in which the supersymmetry is manifest (superspace formulation), and then another formulation (component formulation) where the supersymmetry is not manifest but still the supersymmetry transformations "close off-shell". (In the harmonic oscillator example the vector fields with $B = 0$ close off-shell; they form a Lie algebra isomorphic to the Lie algebra of $O(k)$.) The superparticle we consider next is an example of such a supersymmetric theory, and we will discuss the superspace formulation in the next section. For other supersymmetric theories, however, there is no known formulation which makes the supersymmetry manifest and the supersymmetry transformations only close on-shell. (In the harmonic oscillator example the entire vector space of symmetries only closes on-shell.)

I do not know if there is an alternative formulation of this system of harmonic oscillators which makes the $U(k)$ symmetry manifest. By "alternative formulation" we mean the following. There are often different lagrangians with different spaces of fields which describe the same classical field theory—the same space of classical solutions, symplectic structure, Noether currents, etc. For example, instead of (2.27) we can consider a theory with two fields $x, y\colon M^1 \to X$ and lagrangian

$$L = \Big\{ \langle y, \dot{x}\rangle - \frac{1}{2}\big(|x|^2 + |y|^2\big) \Big\}\, dt.$$

Notice that y only enters algebraically—no derivatives—and it can be eliminated using its equation of motion $y = \dot{x}$ to recover (2.27). This is simply the lagrangian version of the device which converts a second order differential equation into a system of first order equations (which is what happens to the equations of motion for this system). But in this formulation still only the $O(k)$ symmetry is manifest.

Example 4 (a supersymmetry at last!). For convenience we collate the formulas for the superparticle:

$$\begin{aligned} L_2 &= \left\{\frac{1}{2}|\dot{x}|^2 + \frac{1}{2}\langle\psi, \nabla_{\dot{x}}\psi\rangle\right\} dt \\ \gamma_2 &= \langle\dot{x}, \delta x\rangle + \frac{1}{2}\langle\psi, \delta_\nabla\psi\rangle \\ \omega_2 &= \langle\delta_\nabla\dot{x}\wedge\delta x\rangle + \frac{1}{2}\langle\delta_\nabla\psi\wedge\delta_\nabla\psi\rangle + \frac{1}{2}\langle\psi, R(\delta x\wedge\delta x)\psi\rangle. \end{aligned}$$

We also recall

$$\delta L_2 = -d\gamma_2 + \langle\delta x, -\nabla_{\dot{x}}\dot{x} + \frac{1}{2}R(\psi,\psi)\dot{x}\rangle\wedge dt + \langle\delta_\nabla\psi, \nabla_{\dot{x}}\psi\rangle\wedge dt.$$

As for the classical particle, infinitesimal time translation is both a manifest symmetry ξ and a nonmanifest symmetry $\hat{\xi}$. The nonmanifest symmetry is the vector field

$$\begin{aligned} \iota(\hat{\xi})dt &= 0 \\ \iota(\hat{\xi})\delta x &= -\dot{x} \\ \iota(\hat{\xi})\delta\psi &= -\nabla_{\dot{x}}\psi. \end{aligned}$$

The direct computation that $\hat{\xi}$ is a nonmanifest infinitesimal symmetry is a good exercise in the sign rules of calculus with odd variables. The associated Noether charge is

$$j_{\hat{\xi}} = -\frac{1}{2}|\dot{x}|^2$$

and so the hamiltonian is

$$H = \frac{1}{2}|\dot{x}|^2.$$

Notice that the fermion does not enter into the formula for the hamiltonian. Quite generally, *kinetic energy terms in hamiltonians do not involve fermions*, whereas fermions may enter into potential energy terms.

Now we introduce an odd vector field $\hat{\zeta}$ which is a nonmanifest symmetry. It is our first example of a classical supersymmetry. Let η be an auxiliary odd parameter; then $\eta\hat{\zeta}$ is even. The definition of $\hat{\zeta}$ is

$$\begin{aligned} \iota(\eta\hat{\zeta})dt &= 0 \\ \iota(\eta\hat{\zeta})\delta x &= -\eta\psi \\ \iota(\eta\hat{\zeta})\delta_\nabla\psi &= +\eta\dot{x}. \end{aligned} \tag{2.28}$$

There is no component in the M^1 direction; $\hat{\zeta}$ projects to a vector field on $\mathcal{F}$. We leave the reader to compute that

$$[\hat{\zeta}, \hat{\zeta}] = 2\hat{\xi}. \tag{2.29}$$

Thus the two dimensional vector space generated by the vector fields $\hat{\zeta}, \hat{\xi}$ closes off-shell. It is the simplest example of a supersymmetry algebra.

Now we do part of the computation that $\hat{\zeta}$ is a nonmanifest infinitesimal symmetry. We keep the auxiliary odd parameter η in the computation:

$$\begin{aligned}
\mathrm{Lie}(\eta\hat{\zeta})L_2 &= \iota(\eta\hat{\zeta})\delta L_2 \\
&= d\iota(\eta\hat{\zeta})\gamma_2 + \langle \eta\psi, \nabla_{\dot{x}}\dot{x}\rangle \, dt - \frac{1}{2}\langle \psi, R(\psi,\psi)\dot{x}\rangle \, dt + \langle \eta\dot{x}, \nabla_{\dot{x}}\psi\rangle \, dt \\
&= d\iota(\eta\hat{\zeta})\gamma_2 + \eta\big(\langle \psi, \nabla_{\dot{x}}\dot{x}\rangle + \langle \nabla_{\dot{x}}\psi, \dot{x}\rangle\big) \, dt \\
&= d\big(\iota(\eta\hat{\zeta})\gamma_2 + \eta\langle \psi, \dot{x}\rangle\big).
\end{aligned}$$

We used the Bianchi identity $R(\psi,\psi)\psi = 0$. The reader should also compute $\mathrm{Lie}(\eta\hat{\zeta})\gamma_2$ to complete the check that $\eta\hat{\zeta}$ is a symmetry. The conserved Noether charge is

$$\begin{aligned}
j_{\eta\hat{\zeta}} &= \iota(\eta\hat{\zeta})\gamma_2 - \big(\iota(\eta\hat{\zeta})\gamma_2 + \eta\langle \psi, \dot{x}\rangle\big) \\
&= -\eta\langle \psi, \dot{x}\rangle.
\end{aligned}$$

Now just as the hamiltonian H is *minus* the charge associated to $\hat{\xi}$, we define the *supercharge* Q to be *minus* the current associated to $\hat{\zeta}$:

$$Q = -j_{\hat{\zeta}} = \langle \psi, \dot{x}\rangle.$$

We leave the reader to check the Poisson bracket

$$\{Q, Q\} = -2H. \tag{2.30}$$

The supercharge Q does not have an immediate physical interpretation in terms of energy, momentum, or the like. We might not expect that here since our model is not physical, but even in more physical field theory models there is no immediate physical significance to the classical supercharge simply because there are in some physical sense no classical fermions. There are important physical consequences of the fermions on the bosons, however. For example, the energy H is nonnegative since it is a square. (The sign conventions work out so that (2.30) does give positive energy—see (1.???).) We will see a more direct geometric meaning for the supercharge when we quantize.

Superspace and manifest supersymmetry

Equation (2.29) can be interpreted loosely as saying that $\hat{\zeta}$ is a square root of $\hat{\xi}$. Since $\hat{\xi}$ is the effect on fields of infinitesimal time translation, we are led to wonder if there is a square root of infinitesimal time translation whose action on fields induces $\hat{\zeta}$. Of course, this cannot happen on the real line, nor on any ordinary manifold. The "square" of a vector field—one-half its bracket with itself—vanishes for even vector fields, since the Lie bracket is skew-symmetric. So we replace M^1 by a supermanifold and seek an odd vector field as the square root. Then we write the fields in our theory as functions on that supermanifold so that such an odd vector field induces an action on fields.

Such a construction exists and it leads to a formulation of the superparticle in which the supersymmetry is manifest. The supermanifold we seek goes under the name *superspace*. The superspace formulation of supersymmetric theories goes back to the mid '70s when it was introduced by Salaam and Strathdee. Be warned that not all supersymmetric theories have a superspace formulation, as mentioned earlier. We give a general construction of superspace in the next lecture. Here we simply introduce the superspace relevant to the superparticle.

Let $M^{1|1}$ denote the affine space whose ring of functions is

$$C^\infty(M^{1|1}) = C^\infty(M^1)[\theta]$$

for an odd variable θ. Note:

- θ is *not* an *auxiliary* odd variable—it is a bonafide odd function on $M^{1|1}$.
- In our *component* formulation of the superparticle we had fields x, ψ which map an ordinary manifold into a supermanifold. Now we consider a *superfield* formulation in which the field Φ maps a supermanifold into an ordinary manifold. (See Figure 6.)

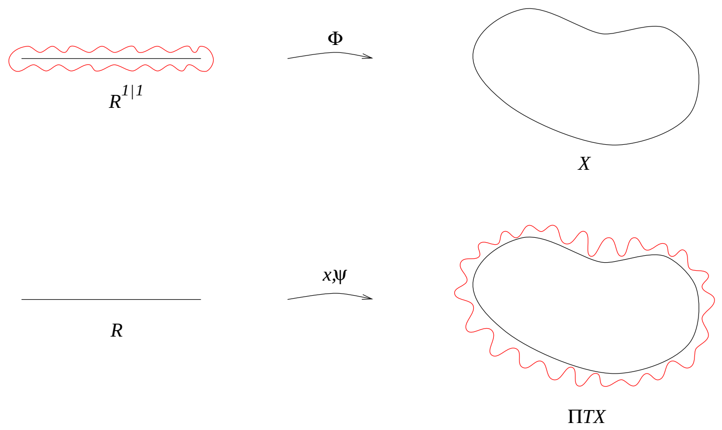

FIGURE 6: THE SUPERFIELD Φ AND THE COMPONENT FIELDS x, ψ

Let t be a standard affine coordinate on M^1, so that t, θ are global coordinates on $M^{1|1}$. Let

$$i\colon M^1 \hookrightarrow M^{1|1}$$

be the inclusion defined by

$$\begin{aligned} i^*t &= t \\ i^*\theta &= 0. \end{aligned}$$

We introduce a global framing of $M^{1|1}$ by the vector fields

$$\begin{aligned} \partial_t &= \partial/\partial t \\ D &= \partial_\theta - \theta\partial_t. \end{aligned}$$

Here ∂_t is even and D is odd. It is unfortunate that the vector field 'D' has the same symbol as the total differential considered in earlier parts of the lecture. *Tant pis*, there just aren't enough Ds in the world! We also introduce the odd vector field

$$\tau_Q = \partial_\theta + \theta\partial_t.$$

We will see in the next lecture that $M^{1|1}$ is the supermanifold underlying a Lie group on which $\{\partial_t, D\}$ is a basis of left invariant vector fields and $\{\partial_t, \tau_Q\}$ a basis of right invariant vector fields. They satisfy the bracket relations

$$\begin{aligned} [D, D] &= -2\partial_t \\ [\tau_Q, \tau_Q] &= +2\partial_t \\ [D, \tau_Q] &= \quad 0. \end{aligned}$$

The vector field ∂_t commutes with both D and τ_Q.

We now formulate a field theory with spacetime $M^{1|1}$ and space of fields

$$\mathcal{F}_2' = \{\Phi\colon M^{1|1} \longrightarrow X\},$$

where X is our fixed Riemannian manifold. In Lecture 1 we discussed what such a map looks like in case we replace $M^{1|1}$ by $M^{0|1}$, a one dimensional odd vector space. From that discussion we see that Φ is intuitively a path in X and an odd tangent vector field along the path. In other words, it is exactly the same data encoded by the fields x, ψ in our previous formulation. We recover these *component fields* from Φ by restricting D-derivatives of the superfield to the underlying Minkowski spacetime:

$$\begin{aligned} x &:= i^*\Phi \\ \psi &:= i^*D\Phi. \end{aligned} \tag{2.31}$$

The lagrangian density in superspace is

$$\mathcal{L} = -\frac{1}{2}\langle D\Phi, \partial_t\Phi\rangle \, dt\, d\theta = \ell \, dt\, d\theta.$$

Here $dt\, d\theta$ is a bi-invariant density on $M^{1|1}$—it is invariant under ∂_t, D, τ_Q. We will not give a detailed development of densities and integration on supermanifolds in these lectures. We finesse the issue below.

Before that we stop to point out that we have achieved our goal of making the supersymmetry manifest. Namely, the lagrangian $\mathcal{L}$ is invariant under the vector fields $\{\zeta, \xi\}$ on $\mathcal{F}_2' \times M^{1|1}$ induced by $\{\tau_Q, \partial_t\}$. Explicitly, we have

$$\begin{aligned} \iota(\zeta)dt &= \theta & \iota(\xi)dt &= 1 \\ \iota(\zeta)d\theta &= 1 & \iota(\xi)d\theta &= 0 \\ \iota(\zeta)\delta\Phi &= -\tau_Q\Phi & \iota(\xi)\delta\Phi &= -\partial_t\Phi. \end{aligned}$$

The invariance of $\mathcal{L}$ follows *a priori* from the fact that τ_Q, ∂_t commute with D, ∂_t and the fact that the density $dt\, d\theta$ is invariant.

Now to the integration. We define the *component lagrangian* L from the superspace lagrangian $\mathcal{L}$ by "integrating out" the odd variable θ. This is the famous *Berezin* integral which in this case amounts to

$$L = (i^* D\ell)\, dt.$$

(This is our finesse: We simply introduce this definition without more explanation. We will say a bit more in Lecture 4.) So we compute

$$\begin{aligned} i^* D\ell &= -\frac{1}{2} i^* D\langle D\Phi, \partial_t\Phi\rangle \\ &= -\frac{1}{2} i^* \Big\{ \langle \nabla_D D\Phi, \partial_t\Phi\rangle - \langle D\Phi, \nabla_D \partial_t\Phi\rangle \Big\} \\ &= -\frac{1}{2} i^* \Big\{ -|\partial_t\Phi|^2 - \langle D\Phi, \nabla_{\partial_t} D\Phi\rangle \Big\} \\ &= \frac{1}{2}|\dot{x}|^2 + \frac{1}{2}\langle \psi, \nabla_{\dot{x}}\psi\rangle. \end{aligned}$$

So we recover the superparticle component lagrangian L_2 which we introduced in (2.8).

The reader would do well to compute that other features of the superspace model match the component formulation. For example, compute the action of τ_Q on the component fields using the definition (2.31) of the component fields to recover (2.28). This computation is a bit tricky, but can be viewed as a problem in ordinary differential geometry—the odd variables cause no additional difficulties. Also, we can do the classical mechanics directly on $M^{1|1}$: Compute γ, ω, the equations of motion, the supercharge, etc. This is a good exercise in calculus on supermanifolds and the formulas in components in the text are there as a check.

Quantization of the superparticle

We make only brief heuristic remarks about the quantization. After all, we promised in Lecture 1 that we would only quantize symplectic affine spaces and here we are with a curved example! A more complete discussion is contained in [D]. We work in the component formulation. Quite generally, it is easier to see the physics—in particular, the quantization—in components, whereas the formal properties due to supersymmetry are manifest in the superspace formulation.

We remarked in (2.14) that if we fix a time then we can identify the supermanifold of classical solutions with the symplectic supermanifold

$$\pi^*\Pi TX \longrightarrow TX; \tag{2.32}$$

the symplectic form is (2.15). For $X = \mathbb{R}^m$ this combines the two examples we studied in Lecture 1. We quantize in stages, first quantizing the fibers of (2.32) and then the base. Now the fiber over a fixed element of T_xX is the odd vector space ΠT_xX with the symplectic form induced from the metric. We found previously that if $\dim X$ is even its quantization is the spin space $S_x = S_x^0 \oplus S_x^1$. In fact, only the projective space is well-defined. Thus we obtain the projective spin bundle of X, pulled back over TX. To go further we need to assume that this can be lifted to an honest graded vector bundle, i.e., we need to assume that the manifold X has a Spin^c structure. The quantization of the base TX we take to be the space of L^2 functions on X. Therefore the quantization of the fibered space is the "twisted tensor product" of the L^2 functions on X and the spinor bundle on X, i.e., it is the Hilbert space

$$\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$$

of L^2 spinor fields on X.

Now we ask for the self-adjoint operators $\hat{Q}, \hat{H}$ which quantize the supercharge Q and the hamiltonian H. Roughly speaking, the quantization $\hat{\psi}$ of ψ is Clifford multiplication and the quantization $\hat{\dot{x}}$ of $\dot{x}$ is the covariant derivative. Hence the quantization of $Q = \langle \psi, \dot{x} \rangle$ is the Dirac operator $\hat{Q}$ and the quantization of $H = \frac{1}{2}|\dot{x}|^2$ is the Dirac laplacian . Notice that (2.30) at the classical level is reflected by the operator equation which asserts that the Dirac laplacian is the square of the Dirac operator. In fact, there is some indeterminacy in quantizing $\hat{H}$; we fix it by requiring (2.30) to hold at the level of operators. There is much more to say here, but these brief remarks will have to suffice.

As promised, the supercharge has more immediate geometric meaning in the quantum story ($\hat{Q}$) than in the classical story (Q). In fact, this quantum mechanical model was used in the early '80s to give a physicists' proof of the Atiyah-Singer index theorem for Dirac operators.

A word about signs

The definition of Lie bracket on a manifold is in terms of the action of vector fields on functions rather than points. This leads to a sign when we consider *left* group actions. Namely, if G is a

Lie group which acts on the left on a manifold N, then the induced map $\mathfrak{g} \to \mathcal{X}(N)$ from the Lie algebra of G to the Lie algebra of vector fields on N is an *anti*homomorphism: it reverses the sign of brackets. If N is symplectic and we lift this antihomomorphism to a moment map $\mathfrak{g} \to \Omega^0(N)$, then the moment map is also an *anti*homomorphism.

We follow standard mathematical usage which has symmetries acting on the left, and so in a lagrangian field theory if we have a group G of symmetries we take it to act on the left of $\mathcal{F}\times M$. This explains the choice of sign in (2.26), for example. This choice—between left and right actions—is often referred to as the choice between the passive point of view and the active point of view. Now in classical lagrangian field theory we pass from an infinitesimal symmetry λ to a vector field ξ_λ on $\mathcal{F} \times M$ to a Noether current J_λ to a Noether charge Q_λ. When we quantize we then pass from the Noether charge Q_λ to a self-adjoint operator $\hat{Q}_\lambda$ on a Hilbert space $\mathcal{H}$, but since they do not form a Lie algebra we pass to a skew-adjoint operator $-\sqrt{-1}\hat{Q}_\lambda$. We indicate these maps in the following sequence:

$$\begin{array}{ccccccccccc}
\mathfrak{g} & \longrightarrow & \mathcal{X}(\mathcal{F}\times M) & \longrightarrow & \Omega^{-1}(\mathcal{F}\times M) & \longrightarrow & \Omega^0(\mathcal{M}) & \longrightarrow & \text{Self-Adj}(\mathcal{H}) & \longrightarrow & \text{Skew-Adj}(\mathcal{H}) \\
\lambda & \longmapsto & \xi_\lambda & \longmapsto & J_\lambda & \longmapsto & Q_\lambda & \longmapsto & \hat{Q}_\lambda & \longmapsto & -i\hat{Q}_\lambda \\
[\lambda_1,\lambda_2] & \longmapsto & -[\xi_{\lambda_1},\xi_{\lambda_2}] & \longmapsto & -\{J_{\lambda_1},J_{\lambda_2}\} & \longmapsto & -\{Q_{\lambda_1},Q_{\lambda_2}\} & & \longmapsto & & +[-i\hat{Q}_{\lambda_1},-i\hat{Q}_{\lambda_2}]
\end{array}$$

The bottom line tells which maps are homomorphisms and which are antihomomorphisms. In particular, the map from infinitesimal symmetries to skew-adjoint operators in the quantum theory is a homomorphism. Note that we used (1.14) here.

These signs follow from the fact that we chose a left group action and from our choice of sign in (1.14). We remind the reader of a convention regarding the hamiltonian: it is *minus* the conserved current associated to time translation. The same holds for the supercharge in a supersymmetric theory and for the entire *energy-momentum tensor* and *supercurrent* in a field theory.