Avalanche dynamics on an inclined plane

Thomas C. Halsey
Particle-Laden Flows in Nature
Kavli Institute for Theoretical Physics, December 16, 2013
Granular Flows and Avalanches

**Statistical Mechanics Approach**

- Based on grain-scale theories of grain interaction and instability of avalanches.
- Focus on statistical distributions of avalanche sizes and pattern formation.
- Most developed for highly intermittent flows.
- Now mostly used for problems besides granular flow.

**Fluid Mechanics Approach**

- Based on approximations to rheology and conservation laws.
- Rapid progress since seminal work of Pouliquen (1999).
- Weak connection to underlying particle mechanics, esp. for dense flows.
- Most developed for steady and close-to-steady flows.
Dense Granular Flows

• Quasistatic Flow: Rate independent stress-strain constitutive relations (Critical State Soil Mechanics)

• Dense Granular Flow: dynamic contact network with multi-particle interactions

• Collisional Flow: Constitutive relations based on collision statistics (Kinetic Theory)

• Fluid-dominated flows
  – Wet dense granular flows
  – Turbidity currents
Rheology of Dense Granular Flows

- Well-established phenomenology for dry dense granular flows
  - Campbell, Pouliquen, Silbert et al.
- Pouliquen flow rule on inclined plane
  \[ \frac{u}{\sqrt{gh}} = Fr = \beta \frac{h}{h_s(\theta)} - \gamma \]
- Rheology is established for steady-state, near steady-state conditions
  - Usually for spherical grains

**Can steady-state rheology be used to understand intermittent avalanche regime?**
Experimental Approach (Börsönyi, Ecke)

Copper

Sand

Glass Beads

Salt
Overall Flow Character

- Qualitatively simple “phase diagram” for all materials
- Critical height as function of $\theta$ can be modeled as

$$\frac{h_s}{d} = \frac{a_1}{\tan \theta - \tan \theta_1}$$

- Pouliquen flow rule (or modified Jenkins form) satisfied for sand, glass beads, less robust for copper particles
  - $\beta$ for sand larger than for glass beads
Avalanches

Sand Avalanches

- Particles are added at top of incline
- Avalanches return slope to its critical value
- Avalanches structure and velocity are approximately constant

$\theta = 33.6^\circ$

$\theta = 38.1^\circ$
Weak and Strong Avalanches

- Differing character of avalanches seen
  - Sand avalanches are larger and faster than glass bead avalanches, have a much more dramatic forward profile
Avalanche Structure

- For sand avalanches, front arrives suddenly, with particle velocity at front (at least at surface) exceeding front velocity.
- For glass bead avalanches, particles are gradually accelerated as front arrives.
Depth-Averaged Theory

Pouliquen flow rule

\[
\frac{u(h, \theta)}{\sqrt{gh}} = \beta \frac{h}{h_s(\theta)} - \gamma
\]

Conservation of mass

\[
\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0
\]

Conservation of momentum

\[
\frac{\partial (hu)}{\partial t} + \alpha \frac{\partial (hu^2)}{\partial x} = \left( \tan \theta - \mu(u, h) - K \frac{\partial h}{\partial x} \right) gh \cos \theta
\]

\[
\alpha = \frac{5}{4} \quad \tan \theta = \mu(u(h, \theta), h) \quad K \approx 1
\]
Solution Structure

- Second order hyperbolic (wave) equation with characteristic velocities

\[ c_\pm = u \left( \alpha \pm \sqrt{\alpha(\alpha - 1) + \frac{K}{(Fr^2 \cos \theta)}} \right) \]

\[ Fr = \frac{u}{\sqrt{gh}} \]

- But, for \( Fr \ll 1 \), equations of motion can be directly simplified to give kinematic waves

\[ \frac{\partial h}{\partial t} + a(h) \frac{\partial h}{\partial x} = N \left( h, \frac{\partial h}{\partial x} \right) \]

\[ a(h) = \sqrt{gh} \left( \frac{5}{2} \beta \frac{h}{h_s} - \frac{3}{2} \gamma \right) \]

- Note that it is not automatic that \( a < c_+ \)
Wave Hierarchy

- Kinematic wave cannot move faster than characteristic (maximum velocity of information transport). When $a \geq c_+$, the kinematic wave merges with the forward shock.
Weak Avalanche

- Kinematic waves have a first-order wave, with a diffusive term on the right hand-side (like Burger’s equation)
- Suggests that avalanche should broaden with time—not observed
  - May be too slow to observe in course of experiment
- For glass beads, pure first order theory predicts
  \[ u_f \approx 0.6a(h_m) \quad \ell \approx 6h_s \]
- Acceptable (but not impeccable) agreement
For the shock solution, there will be a jump criterion connecting particle and front velocities with the height of the shock:

\[(u_p - u_f)h_m > u_f h_s\]

• Equivalently

\[\left(\frac{u_p}{u_f} - 1\right) > \left(\frac{h_m}{h_s}\right)^{-1}\]

• So that we must have \(u_p > u_f\) at the shock!
Results for Various Particles

- Note strong correlation between super-critical vs. sub-critical avalanche height (corresponding to which side of the blue or black curves the points occupy) and the particle to front velocity ratio (shown on right).
Instabilities

• This is analogous to result for instabilities in steady flow, analyzed by Forterre and Pouliquen
• Glass beads
  – Flows near critical height were stable
  – Flows away from critical height were unstable
• Sand: the reverse
• Roll waves vs. flood waves
• Criterion for stability of flows:

\[ a < c_+ \]
Reservations

• Both strong and weak avalanches are propagating into static materials; for both types of avalanches the zone behind the avalanche front is settling back into a static state.
  – No modeling of zone of “passive Rankine failure” ahead of front
• Have not addressed lateral structure of avalanches
  – Could be done with straightforward extension of depth-averaged equations
• In practice, $\alpha$ should vary with height
  – Linear velocity profile seen near threshold
  – Bagnold velocity profile seen for deeper flows
Outlook

- Semi-quantitative theory accounts well for transition from weak to strong avalanches
  - Notwithstanding granular complexities, simple depth-averaged fluid mechanical approach is quite successful
- Alas, dry granular flows are limited in their geophysical importance
- “Wet granular flows” (Debris flows)—more complex rheology (although note Marseille group proposal)
- Turbidity currents—simple conceptually (Parker model and its descendants) but large phase space, mathematically more complex

Can steady-state rheology be used to understand intermittent avalanche regime?
Yes! But statistical mechanics may still be needed to underpin fundamental rheology!
Backup
Avalanche Size and Speed
Front and Particle Velocities vs. Angle

Glass beads

Sand