Phonon Excitations of a Bose Gas within an Optical Lattice

Experimental Background

1. Burger et al., PRL 86, 4447 (2001)
   - observation of dissipative flow ⇒ breakdown of superfluidity: Landau instability?

   - similar to Burger et al., but with deeper potentials ⇒ tight-binding regime
   - qualitative agreement with DNLSE results of Smorch et al. ⇒ dynamical instability

   - effects of thermal component

4. Felloni et al., cond-mat/0404045
   - adiabatic loading into a moving optical lattice
**Theoretical Work**

1. Wu + Niu, PRA 64, 061603 (2001)
   - 1D optical lattice $\rightarrow$ Bogoliubov excitations
   - emphasize distinction between energetic (Landau) and dynamic instabilities
   - suggest that dynamic instability is origin of dissipative in Burger et al. expr.

   - detailed numerical study of 1D optical lattices - band structure (swallowtails)
   - excitations $\rightarrow$ dynamic + energetic instabilities

   - numerical study of 1D optical lattices in 'weak' interaction limit (no swallowtails)
   - excitations - emphasize long wavelength behaviour.

4. Modugno et al., cond-mat/0405653
   - excitation spectrum for cylindrical geometry
   - generate stability diagram (similar to Wu Niu)
   - first full 3D simulation of Burger et al. expr.
   $\rightarrow$ dynamical instabilities

**FIG. 1.** Density distribution of a BEC in a harmonic trap with a superimposed optical lattice, from a numerical simulation of the 3D GPE for $N = 3 \times 10^5$ and $V_0/k_B = 270$ nK. The inset shows an enlargement of the central region of the BEC. The envelope of the modulated density distribution follows the parabolic distribution in the harmonic trap.

**FIG. 2.** Superfluid oscillations of a BEC in the presence of an optical lattice potential of height $V_0/k_B = 270$ nK (squares) and in a purely magnetic trap (triangles), for initial displacement $\Delta x = (31 \pm 3)$ μm. The lines give results from a numerical simulation of the 1D GPE at the experimental parameters.
FIG. 3. Ratio of the first-peak amplitude of the oscillation of the ensemble to the free-oscillation amplitude, $A_1/A$, as a function of initial displacement $\Delta x$, for the potential $V_0/k_B = 270$ nK and atom number $N = 3 \times 10^6$.

FIG. 4. The fraction of atoms remaining in the undistorted part of the BEC, $N_f/N$, as a function of the velocity reached during the evolution in the periodic potential.

FIG. 1. Inhomogeneous BEC Bloch waves after evolving $t = 1.1$ ms. (a) Bloch wave number $k = 0.2k_L$; (b) $k = 0.8k_L$. The distorted and fragmented wave function signals the onset of dynamical instability.

B. Wu & Q. Niu, PRL (Comment) 84, 088901 (2002)
FIG. 1: Lowest Bloch bands at $v = 0.1$ for $c = 0.0$, $c = 0.05$, $c = 0.1$, and $c = 0.2$ (from bottom to top). As $c$ increases, the tip of the Bloch band turns from round to sharp at the critical value $c = v$, followed by the emergence of a loop.

B. Wu & Q. Niu, cond-mat/0306411


FIG. 3. Energy per particle in the first Brillouin zone as in Fig. 2. The results are obtained by a variational method with the trial function given in Eq. (10). (a) In the absence of interaction the band structure (bold curves) exhibits the usual band gaps at $k = 0$ and $k = \pi / d$. The band gap is $V_0$ at $k = \pi / d$ and $V_0^2 / 8E_0$ at $k = 0$ for small $V_0$. The thin curves show the energies for $V_0 \to 0$, i.e., for the free noninteracting system. (b) In the presence of interaction the swallow tails appear for $U_0$ larger than a critical value, which depends on $V_0$ and is different for two band gaps (bold curves). The thin curves illustrate the limit $V_0 \to 0$. 
Condensate Dynamics

- at low temperatures, the dynamics of the condensate is governed by the time-dependent Gross-Pitaevskii (GP) equation

\[ i\hbar \frac{\partial \Psi}{\partial t} = \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{opt}} + g|\Psi|^2 \right) \Psi \]

where \( g|\Phi|^2 \) is a mean-field interaction

- the GP equation provides a description of the possible collective excitations of the condensate

- we consider an extended condensate in a 3D optical potential: \( V_{\text{opt}}(r + R) = V_{\text{opt}}(r) \)

- of particular interest are the long wavelength phonon excitations of the optical lattice

Stationary States

\[ \Psi(r, t) = \Phi_0(r)e^{-i\mu t/\hbar}, \quad \int_V d^3r |\Phi_0|^2 = N \]

\[ -\frac{\hbar^2}{2m} \nabla^2 \Phi_0 + V_{\text{opt}} \Phi_0 + g|\Phi_0|^2 \Phi_0 = \mu \Phi_0 \]

- because of lattice periodicity, the GP equation admits Bloch state solutions of the form

\[ \Phi_0(r) = \sqrt{n}e^{i\mathbf{k}\cdot \mathbf{r}}w(r) \]

where \( w(r + R) = w(r) \)

- for \( k \neq 0 \), the Bloch state is an excited state in which the condensate has the superfluid current density

\[ j_s(r) = \frac{\hbar}{2mi} [\Phi_0^* \nabla \Phi_0 - (\nabla \Phi_0)^* \Phi_0] \]
Collective Excitations

- the condensate supports small amplitude oscillations about the stationary state:

\[ \Psi(\mathbf{r}, t) = [\Phi_0(\mathbf{r}) + \delta\Phi(\mathbf{r}, t)] e^{-i\mu t/\hbar} \]

- the fluctuation \( \delta\Phi(\mathbf{r}, t) \) is a solution of the linearized TDGP equation and has the form

\[ \delta\Phi(\mathbf{r}, t) = u(\mathbf{r})e^{-iEt/\hbar} - v^*(\mathbf{r})e^{iEt/\hbar} \]

- the quasiparticle amplitudes \((u, v)\) and the quasiparticle energy \(E\) are determined by the Bogoliubov equations

\[ \hat{L}u(\mathbf{r}) - g\Phi_0^2(\mathbf{r})v(\mathbf{r}) = Eu(\mathbf{r}) \]
\[ \hat{L}v(\mathbf{r}) - g\Phi_0^2(\mathbf{r})u(\mathbf{r}) = -Ev(\mathbf{r}) \]

where

\[ \hat{L} \equiv -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{opt}} + 2g|\Phi_0|^2 - \mu \]
the Bogoliubov equations admit Bloch-like solutions

\[ u(r) = e^{i(q-k) \cdot r} \bar{u}(r) \]
\[ v(r) = e^{i(q-k) \cdot r} \bar{v}(r) \]

where \( \bar{u}(r + R) = \bar{u}(r) \), etc. \( k \) is the Bloch wave vector of the stationary state, while \( q \) is the Bloch wave vector of the excitation.

- the excitations form bands, labelled by the band index \( m \) and the wave vector \( q \)
- of particular interest is the lowest band \( (m = 0) \) in the long wavelength limit \( (q \to 0) \)
- solutions of the Bogoliubov equations can be developed by means of a systematic expansion in \( q \)

**Phonon Excitations \( (k = 0) \)**

- for the lowest band,

\[ E(q) = \hbar sq + \cdots \]

- the sound speed is given by

\[ s = \sqrt{\frac{\bar{n}}{m_0}} \frac{\partial \mu_0}{\partial \bar{n}} \]

- \( \mu_0 \) is the GP eigenvalue of

\[ \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{opt}} + g\bar{n}w_0^2 \right) w_0 = \mu_0 w_0 \]

- \( w_0 \) and \( \mu_0 \) depend parametrically on \( \bar{n} \), the mean density
The effective mass $m_0$ of the lowest band is determined by the equation

$$\left(-\frac{\hbar^2 \nabla^2}{2m} + V_{\text{opt}} + g_0 \bar{n} \right) \phi_q = \varepsilon_q \phi_q$$

where

$$\phi_q = e^{iq \cdot r} w_q, \quad \varepsilon_q = \mu_0 + \frac{\hbar^2 q^2}{2m_0} + \ldots$$

NB: one does not have to solve the GP equation self-consistently to determine $\varepsilon_q$ and hence $m_0$

- $m_0$ increases with the strength of the lattice potential and the sound speed decreases from the uniform gas limit $s_0 = \sqrt{g_0 n}/m$.

**Phonon Excitations ($k \neq 0$)**

- for the case of a current-carrying condensate

$$E(q) = e_{n,i} q_i + \sqrt{e_{n,ij} q_i q_j}$$

- $e(n,k) \equiv \bar{n} \varepsilon(n,k)$ is the mean energy density $E_{\text{tot}}/V$ and

$$e_{n,n} = \frac{\partial^2 e}{\partial \bar{n}^2}, \quad e_{n,i} = \frac{\partial^2 e}{\partial \bar{n} \partial k_i}, \quad e_{i,j} = \frac{\partial^2 e}{\partial k_i \partial k_j}$$

- the above result can be derived from the pair of hydrodynamic equations (*Machholm et al., 2003*)

$$\frac{\partial \bar{n}}{\partial t} = -\nabla \mu, \quad \frac{\partial \bar{n}}{\partial t} + \nabla \cdot j_s = 0$$

with

$$\mu = \frac{\partial e}{\partial \bar{n}}, \quad j_s = \frac{1}{\hbar} \nabla_k e$$
for small \( k \),

\[
E(q) = \hbar^2 k \cdot q \frac{\partial}{\partial \tilde{n}} \left( \frac{\tilde{n}}{m_0} \right) + \hbar sq
\]

- for \( V_{opt} \to 0 \), \( q = \pm q \hat{k} \),

\[
E(q) = \hbar (s_0 \pm v)q
\]

where \( v = \hbar k/m \)

- if \( v > s_0 \), \( E \) becomes negative for \( q = -q \hat{k} \); this signals an energetic instability given by the Landau criterion

\[
\delta G = \frac{1}{2} \int d^3r \delta \Phi \hat{A} \delta \Phi
\]

where

\[
\delta \Phi = \begin{pmatrix} \delta \phi \\ \delta \phi^* \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} \hat{L} & g\Phi_0^2 \\ g\Phi_0^* & \hat{L} \end{pmatrix}
\]

- if the operator \( \hat{A} \) has negative eigenvalues, then the state \( \Phi_0 \) is energetically unstable; for a homogeneous system, a zero eigenvalue occurs when \( v \cdot q = \pm s_0 q \), the Landau criterion for the spontaneous emission of phonons
Energetic Instability

\[ \delta s = \frac{i}{\hbar} \int d^3\rho \, \delta \hat{A} \, \delta \hat{A}^* \]
\[ \hat{A} = \begin{pmatrix} \hat{L} & \hat{S}^x \\ \hat{S}^{-x} & \hat{L}^* \end{pmatrix} \]
\[ \hat{A} \, \hat{X}_\lambda = \lambda \hat{X}_\lambda \]

\[ \lambda > 0 \Rightarrow \delta s \text{ is a minimum} \]
\[ \lambda = 0 \text{ signals an energetic instability} \]

Bogoliubov Equations

\[ \begin{pmatrix} \hat{L} & \hat{S}^x \\ \hat{S}^{-x} & \hat{L}^* \end{pmatrix} \begin{pmatrix} \psi \\ \psi^* \end{pmatrix} = E \begin{pmatrix} \psi \\ \psi^* \end{pmatrix} \]
\[ \gamma \hat{A} \begin{pmatrix} \psi \\ \psi^* \end{pmatrix} = E \begin{pmatrix} \psi \\ \psi^* \end{pmatrix} \]

\[ \lambda = 0 \text{ eigenvalue eigenvector of } \hat{A} \text{ is also a solution of this equation with } E = 0. \]

\[ \lambda < 0 \Rightarrow \text{a Bogoliubov eigenvalue becomes negative} \]
FIG. 5: Stability phase diagram of BDC Bloch waves. $k$ is the wave number of BDC Bloch waves; $\alpha$ denotes the wave number of perturbation modes. In the shaded (light or dark) area, the perturbation mode has negative excitation energy; in the dark shaded area, the mode grows or decays exponentially in time. The triangles in (a.1-a.4) represent the boundary, $q^2/4 + c = k^2$, of saddle point regions at $v = 0$. The solid dots in the first column are from the analytical results of Eq.(5.13). The circles in (b.1) and (c.1) are based on the analytical expression (5.14). The dashed lines indicate the most unstable modes for each Bloch wave $\phi_\alpha$.