Fermi-Bose Duality in Ultracold Atomic Vapors in Tight Atom Waveguides

M.D. Girardeau
Optical Sciences Center
University of Arizona

June 1, 2004

Outline

- 1D hard-sphere bosons, Fermi-Bose mapping
- Tight waveguides: 1D dynamics for *real* ultracold atomic vapors
- Statics and dynamics of TG limit
- Atom interferometry
- Generalized Fermi-Bose duality: p-wave fermions $\rightarrow$ s-wave bosons, strong Fermi interactions $\rightarrow$ weak Bose interactions and vice versa

1D hard-sphere bosons, Fermi-Bose mapping

Tonks gave first treatment of statistical mechanics of a 1D hard-sphere gas: L. Tonks, Phys. Rev. **50**, 955 (1936).
Restricted to *classical high-temperature regime*, provided *no information about extreme quantum limit characteristic of ultracold atomic vapors*.
Formula for exact quantum-mechanical ground-state energy of 1D hard-sphere Bose gas appeared without any derivation in a paper of Bijl:
A. Bijl, Physica **4**, 329 (1937), Note II.
The first published derivation seems to be one by Nagamiya:
T. Nagamiya, Proc. Phys. Math. Soc. Japan **22**, 705 (1940).
Then in 1960 Stachowiak and I independently rederived the ground-state energy:
M. Girardeau, J. Math. Phys. **1**, 516 (1960); for the case of nonzero hard core diameter see footnote 7 (a).
H. Stachowiak, Acta Univ. Wratislaviensis **12**, 93 (1960).

The Fermi-Bose mapping method first appeared in my 1960 paper:

Fermi-Bose mapping theorem

$$\hat{H}_{1D} = -(\hbar^2/2m)\sum_{j=1}^{N}\partial_{z_j}^2 + V(z_1,\cdots,z_N)$$

V: All external potentials (e.g., trap potential) + any finite interactions *not including* the hard-core repulsion.
Hard cores treated as a *constraint on allowed wave functions*:
$\psi_B = 0$ if $|z_j - z_k| < a \quad , \quad 1 \le j < k \le N \quad .$
$\psi_F(z_1,\cdots,z_N)$: antisymmetric solution of $\hat{H}_{1D}\psi = E\psi$ satisfying the hard-core constraint.
One can consider ψ_F to be either the wave function of a fictitious system of "spinless fermions", or else that of a system of real, spin-aligned fermions.
Define "unit antisymmetric function"

$$A(z_1,\cdots,z_N) = \prod_{1\le j<k\le N}\mathrm{sgn}(z_k - z_j)$$

where $\mathrm{sgn}(z) = +1(-1)$ if $z > 0(z < 0)$.
Fermi-Bose mapping: $\psi_B(z_1,\cdots,z_N) = A(z_1,\cdots,z_N)\psi_F(z_1,\cdots,z_N)$
Theorem: If $\hat{H}_{1D}\psi_F = E\psi_F$ then $\hat{H}_{1D}\psi_B = E\psi_B$, and ψ_B satisfies impenetrability constraint and same boundary conditions as ψ_F.
Proviso: If N *even* then *periodic* b.c. for ψ_F imply *anti*periodic b.c. for ψ_B and vice versa.

If $V(z_1, \cdots, z_N)$ contains interactions outside the hard cores, then FB mapping transforms a not exactly soluble many-boson problem into a not exactly soluble many-fermion problem, but it can still be useful for approximate calculations.

However, if there are *only* hard core interactions with or without an external longitudinal potential (trap or optical lattice) then FB mapping transforms the strongly interacting many-boson problem into a problem of *non-interacting* fermions, whose *exact* many-body ground and excited states are expressible in terms of *one-particle* orbitals $\varphi_i(z_j$ appropriate to the external conditions:

No external potential: plane waves

Harmonic longitudinal trap potential: Hermite-Gaussians

Optical lattice: Bloch or Wannier orbitals.

Zero hard-core diameter ($a \to 0$): $\psi_F = \det_{i,j=1}^{N} \varphi_i(z_j)$

$a > 0$: $\psi_F = \det_{i,j=1}^{N} \varphi_i(w_j)$ where

$w_1 = z_1, w_2 = z_2 - a, \cdots, w_N = z_N - (N-1)a.$

Bosonic ground state: $\psi_{B0} = A\psi_{F0} = |\psi_{F0}|$

Simplest case: N bosons on ring of circumference L, $a \to 0$, no external potential. Then Van der Monde determinant of plane waves $\to$ Bijl-Jastrow form for exact ground state:

$$\psi_{B0} = \prod_{i>j} |\sin[\pi L^{-1}(z_i - z_j)]|$$

Exact low excited states were found to have phonon character:
Sound speed $c = \pi\hbar n/m$ where $n = N/L$, agrees with thermodynamic value from compressibility of ground state.

"Fermionization" holds only for properties expressible in terms of configurational probability density $|\psi_{B0}(z_1, z_2, \cdots, z_N)|^2$.
Momentum distribution $N(k)$ depends on single-particle correlation function $g_1(z) = \langle\hat{\psi}^\dagger(z)\hat{\psi}(0)\rangle$, very different from that of ideal Fermi gas.
Lenard proved by *tour de force* that $N(0) = \mathcal{O}(\sqrt{N})$, $\gg 1$ but $\ll \mathcal{O}(N)$ value required for BEC:
A. Lenard, J. Math. Phys. **5**, 930 (1964) and **7**, 1268 (1966).
For small k $N(k)$ diverges like $1/\sqrt{k}$:
H.G. Vaidya and C.A. Tracy, Phys. Rev. Lett. **42**, 3 (1979). See also corrected expansions in M. Jimbo, T. Miwa, Y. Mori, and M. Sato, Physica D **1**, 80 (1980).

Tight atom deBroglie waveguides: 1D dynamics for *real* ultracold atomic vapors

Several experimental groups have produced trapped ultracold gases whose dynamics is effectively 1D because longitudinal density and temperature are so low and transverse confinement is so tight that the longitudinal zero-point and thermal energies are small compared with the transverse excitation energy quantum $\hbar\omega_{perp}$.
For bosonic atoms with 1D interactions $g_{1D}\delta(z_i - z_j)$,
1D dimensionless coupling constant $\gamma = \frac{\langle V \rangle}{\langle T \rangle} = mg_{1D}/n\hbar^2$.
Experiments span whole range $\gamma > 0$ from GP (Gross-Pitaevskii, $\gamma \ll 1$) to TG (Tonks-Girardeau, $\gamma \gg 1$).
$\gamma \gg 1 \leftrightarrow$ large g_{1D} or large m or *low* density n.

Experiments from GP to intermediate $\gamma \sim 1$:
M. Greiner *et al.*, Phys. Rev. Lett. **87**, 160405 (2001)
H. Moritz, *et al.*, Phys. Rev. Lett. **91**, 250402 (2003)
B.L. Tolra *et al*, Phys. Rev. Lett. **92**, 190401 (2004)

TG regime has just been reached in a beautiful recent experiment which achieved $\gamma \gg 1$ by increasing effective mass via a longitudinal lattice potential:
B. Paredes, A. Widera, V. Murg, O. Mandel, S. Fölling, I. Cirac, G.V. Shlyapnikov, T.W. Hänsch, and I. Bloch, Nature **429**, 277 (2004).
See also the news article on this experiment by Murray Holland: M. Holland, Nature **429**, 251 (2004)
and Immanuel Bloch's talk here a few weeks ago:
http://online.kitp.ucsb.edu/online/gases_c04/bloch/

I have stolen a few pages from his talk:

PP. 1, 14, 18, 30, 31 OF Bloch.pdf GO HERE

Theory of reduction from 3D to 1D in this regime has been worked out in detail:
M. Olshanii, Phys. Rev. Lett. **81**, 938 (1998)
D.S. Petrov, G.V. Shlyapnikov, and J.T.M. Walraven, Phys. Rev. Lett. **85**, 3745 (2000)
M.G. Moore, T. Bergman, and M. Olshanii, arXiv:cond-mat/0402149 (2004) (to be published in Proceedings of Euro Summer School on Quantum Gases in Low Dimensions, Les Houches, April 2003)
M.D. Girardeau, Hieu Nguyen, and M. Olshanii, arXiv:cond-mat/0403721 (2004)

Effective 1D Hamiltonian:

$$\hat{H} = \sum_{j=1}^{N} \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_j^2} + V_{long}(x_j, t) \right] + g_{1D} \sum_{1 \le j < k \le N} \delta(x_j - x_k)$$

where g_{1D} is a known function of 3D s-wave scattering length a_s, transverse oscillator level spacing $\hbar\omega_\perp$, and transverse oscillator length $a_\perp = \sqrt{\hbar/\mu\omega_\perp}$ where $\mu = m/2$:

$$g_{1D}^{B} = 2a_s \hbar\omega_\perp \left[1 - \frac{a_s}{a_\perp} |\zeta(1/2)| \right]^{-1}$$

implying the existence of a “confinement-induced resonance”:

Feshbach resonance for longitudinal scattering in the open channel where transverse oscillations are in their ground mode, due to a bound state in a closed, transversely excited channel:

T. Bergeman, M. Moore, and M. Olshanii, Phys. Rev. Lett. **91**, 163201 (2003)

G.E. Astrakharchik, D. Blume, S. Giorgini, and B.E. Granger, Phys. Rev. Lett. **92**, 030402 (2004)

Statics and dynamics of TG limit

Harmonically trapped TG gas:
M.D. Girardeau, E.M. Wright, and J.M. Triscari, Phys. Rev. A **63**, 033601 (2001)

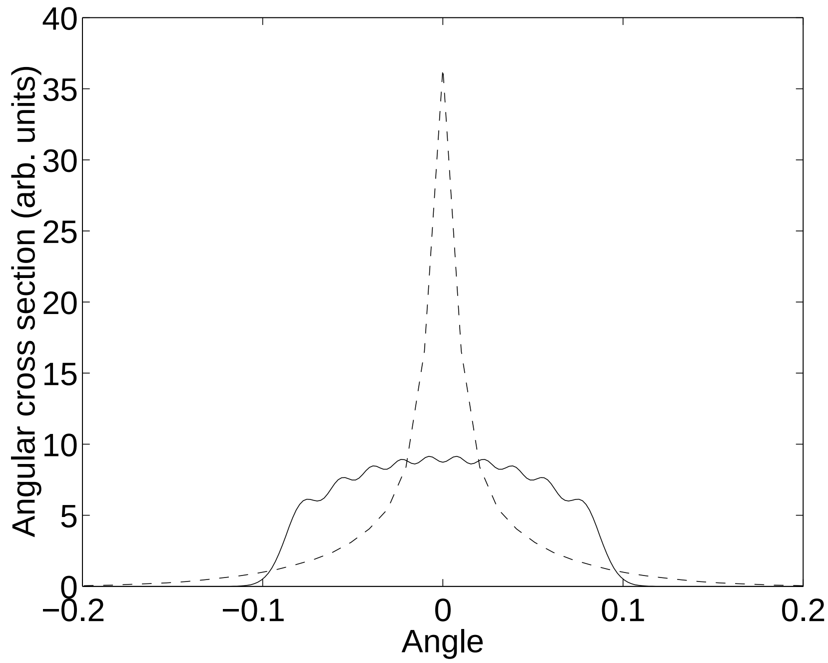
Exact N-particle ground state:

$$\psi_{B0}(x_1, \cdots, x_N) = |\psi_{F0}(x_1, \cdots, x_N)| = |\det_{(n,j)=(0,1)}^{(N-1,N)} \phi_n(x_j)|$$

where the ϕ_n are HO orbitals (Hermite-Gaussians). Elementary row and column operations $\rightarrow$ simple Bijl-Jastrow form:

$$\psi_{B0}(x_1, \cdots, x_N) = C_N \left[\prod_{i=1}^{N} e^{-Q_i^2/2}\right] \prod_{1 \leq j < k \leq N} |x_k - x_j|$$

with $Q_i = x_i/x_{osc}$ and $x_{osc} = \sqrt{\hbar/m\omega_{osc}}$. Note strong resemblance to Laughlin variational wave function of 2D ground state of quantized fractional Hall effect.
Here this form is *exact ground state*, not a variational approximation.
For other properties of ψ_{B0} see our paper. Here I will only show the momentum distribution, quite different from that of the corresponding ideal Bose gas ground state ψ_{F0}:



Bose and Fermi momentum distributions for $N = 10$.
Solid line=Fermi, dashed line=Bose TG gas.
From M.D. Girardeau and E.M. Wright, Phys. Rev. Lett. **87**, 050403 (2001) where we showed that longitudinal momentum distribution can be measured by stimulated Raman scattering. Here angle→momentum, angular cross section→momentum distribution function.
Spatial density is same as that of Fermi gas. See
Erich Mueller, arXiv:cond-mat/0405425 (2004).
For small N there are density ripples from contributions of individual HO orbitals, or equivalently, Friedel oscillations.

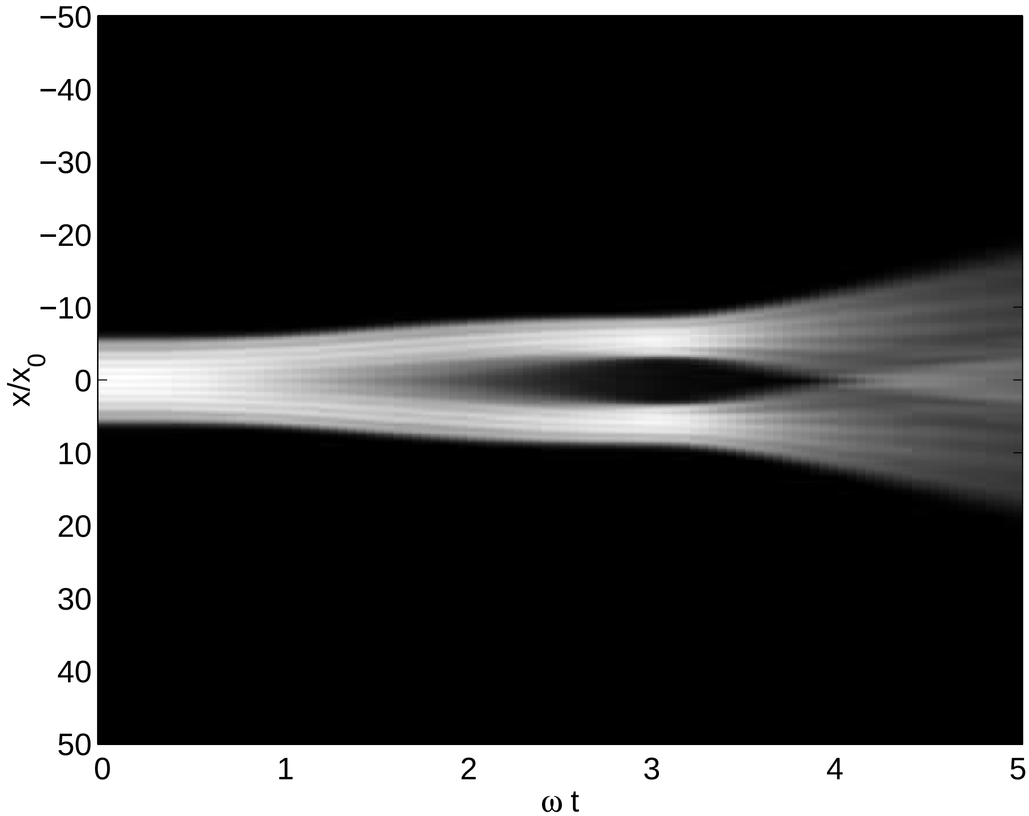
Supression of interference in a dynamically split and recombined TG gas:
M.D. Girardeau and E.M. Wright, Phys. Rev. Lett. **84**, 5239 (2000)

Fermi-Bose mapping theorem still holds if there is a *time-dependent* external potential:

$$\psi_B(z_1, \cdots, z_N; t) = A(z_1, \cdots, z_N)\psi_F(z_1, \cdots, z_N; t)$$

where A is antisymmetric ± 1 function as before, and ψ_F is a Slater determinant of solutions of *single-particle* TDSE in presence of the time-dependent potential $\rightarrow$ time-dependent TG spatial density same as that of ideal Fermi gas in presence of time-dependent potential.

TG gas initially in N-body ground state in a harmonic trap, central Gaussian repulsive potential turned on quasi-adiabatically to split initial state, then both harmonic trap and repulsive potential turned off and two split components allowed to recombine:



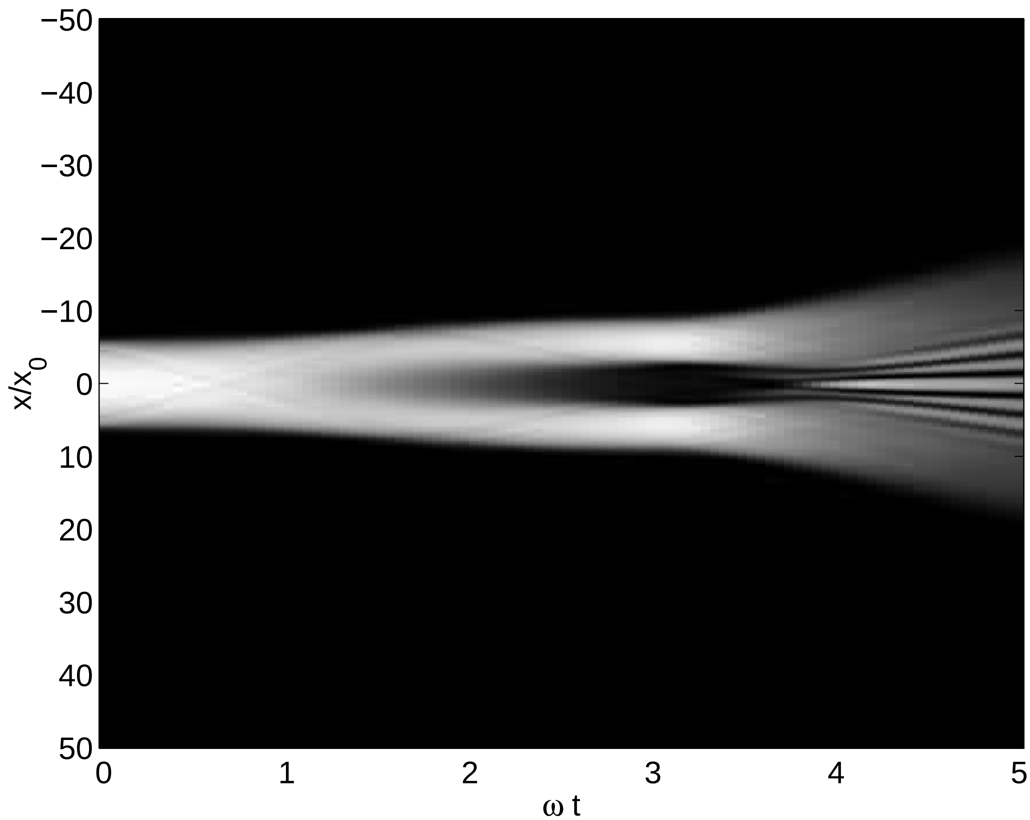
Exact many-body theory simulation of the cool, cut, interfere scenario. The figure shows a gray-scale plot of the particle density $n(x,t)$ as a function of ωt (horizontal axis) and position x/x_0 (vertical axis), with white being the highest density, for $N = 10$.

No noticeable interference fringes in overlap region. Individual orbitals show fringes, but different orbitals have different fringe wavelengths $\rightarrow$ fringes obliterated.

Compare this with a mean field approximation. Usual GP approximation useless here because hard-core potential is infinite.
Kolomeisky has proposed generalized 1D mean field theory adapted to TG gas:
E. B. Kolomeisky *et al.*, Phys. Rev. Lett. **85**, 1146 (2000):

$$i\hbar\frac{\partial\Phi}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t) + \frac{(\pi\hbar)^2}{2m}|\Phi|^4\right]\Phi.$$

Repeating the calculation with this mean field dynamics leads to following result:



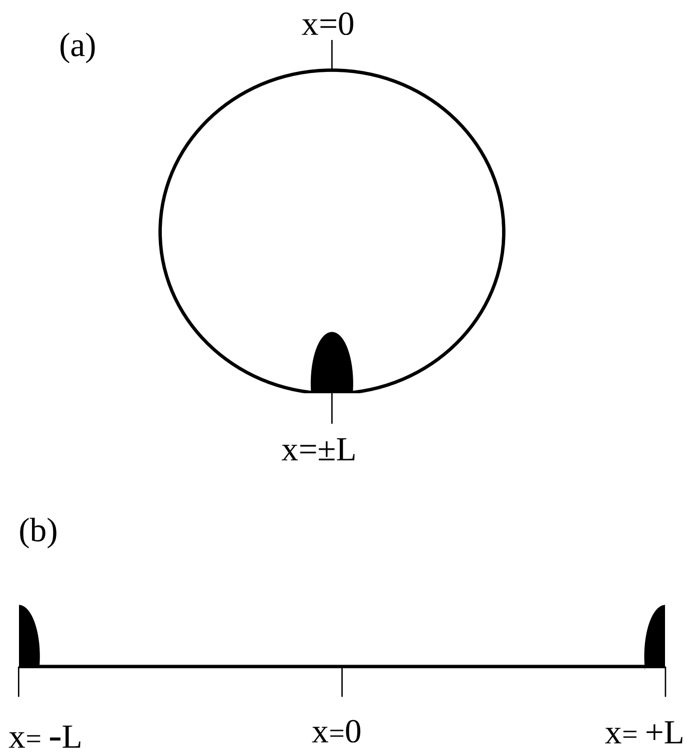
Mean-field theory simulation of the cool, cut, interfere scenario. The figure shows a gray-scale plot of the particle density $n(x,t) = |\Phi(x,t)|^2$ as a function of ωt (horizontal axis) and position x/x_0 (vertical axis).

Note strong fringes in overlap region, showing that effective field theory grossly overestimates coherence here.

However, it *is* possible to design TG interferometers with strong fringes by more intelligent state preparation:

Atom interferometry

TG gas on a ring:
K.K. Das, G.J. Lapeyre, and E.M. Wright, Phys. Rev. A **65**, 063603 (2002)



(a): N hard core bosons trapped on a ring of circumference $2L$
(b): Ring unfolded to describe system by 1D coordinate $x \in [-L, L]$

Initial condition:
For times $t < 0$ the N atoms are confined to a narrow segment of the ring by a harmonic trapping potential. Initial ideal Fermi gas ground state ψ_B is filled Fermi sea of N lowest harmonic oscillator (HO) orbitals in trap potential, of form

$$\phi_n(x, t \leq 0) = [u_n(x+L) + u_n(x-L)]$$

where the u_n are HO orbitals with origin 0

Evolution:
Trap potential turned off at $t = 0$ and optical lattice Bragg pulse

$$-\delta(t)\eta\cos[k(x-L)]$$

applied, producing periodically phase-imprinted HO orbitals:

$$\begin{aligned}
\phi_n(x, 0+) &= u_n(x+L, 0+) + u_n(x-L, 0+) \\
u_n(x, 0+) &= e^{i\eta\cos[k(x-L)]}u_n(x, 0) \\
&= \sum_{m=-\infty}^{\infty} i^m J_m(\eta) e^{imk(x-L)} u_n(x, 0)
\end{aligned}$$

For $|\eta| < 1/2$ higher order diffraction $< 1\%$
$\Rightarrow$ pulse kicks out two daughter wavepackets of momenta $\pm\hbar k$ with same shape as original HO orbital:

$$u_n(x, 0+) \approx \left[1 + iJ_1(\eta)\left(e^{ik(x-L)} + e^{-ik(x-L)}\right)\right] u_n(x, 0)$$

Interference of daughter wavepackets:
Daughters propagate as wavepackets in free space so long as they have not wrapped all the way around the ring.

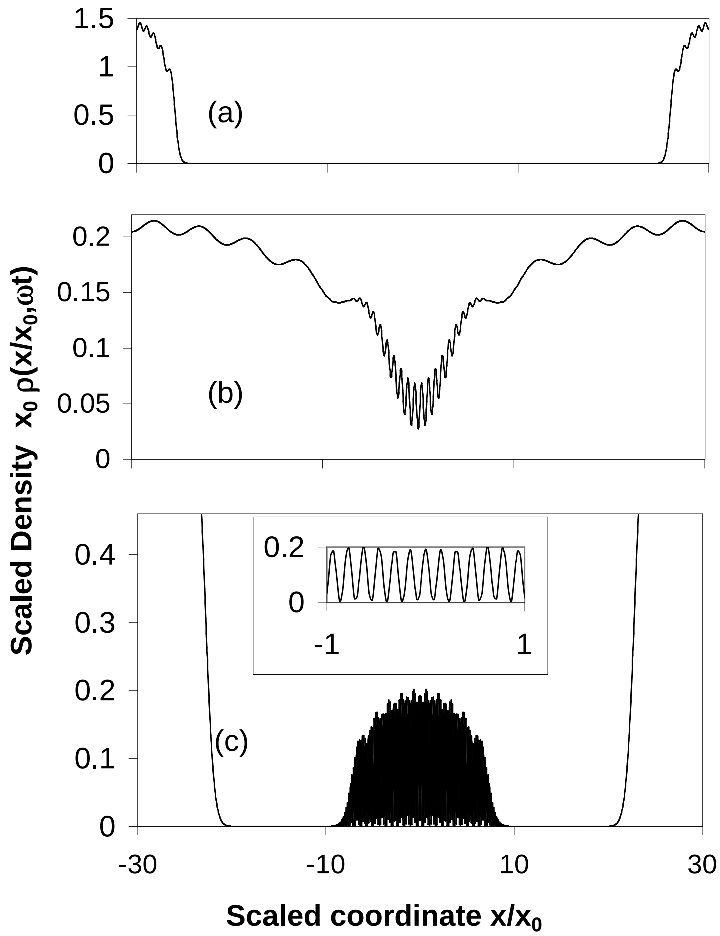
Centers propagate with velocity $\hbar k/m$.

Each broadens with velocity $\hbar k_F/m$ where $k_F = \sqrt{2N}/x_0$ is wavevector at top of initial Fermi sea and $x_0 = \sqrt{\hbar/m\omega}$ is HO width of initial trapped cloud before release.
Note: k_F is from asymptotic form of highest Hermite polynomial $u_N(x, 0) \propto \cos(\sqrt{2N}x/x_0)$.

Initial wavepacket wraps around ring in time

$$t_{wrap} \approx \frac{L}{(\hbar k_F/m)} = \frac{1}{\omega\sqrt{2N}}\left(\frac{L}{x_0}\right)$$

Daughters meet and interfere at opposite side of ring $x \approx 0$ at time $t_r \approx Lm/\hbar k = (k_F/k)t_{wrap}$.



Scaled density profiles for $N = 10, \eta = 0.5$, $L/x_0 = 30$ and (a) the initial condition $t = 0$, (b) $x_0 k = 0, \omega t = \omega t_{wrap} = 6.7$, and (c) $x_0 k = 20, \omega t = \omega t_r = 1.5$

(b): low-visibility fringes from self-interference of mother;

(c): high-visibility fringes because daughters have $k \gg k_F$.

Generalization to nonzero temperature:
M.D. Girardeau, K.K. Das, and E.M. Wright, Phys. Rev. Lett. **89**, 170404 (2002)

- Initial state: Bose Tonks gas at nonzero temperature, statistical density operator
$$\hat{\rho}_0 = Z^{-1} \sum_\alpha e^{-\beta E_\alpha} |\Psi_{B\alpha}(0)\rangle\langle\Psi_{B\alpha}(0)|$$
with
$$Z = \sum_\alpha e^{-\beta E_\alpha}$$
- Subsequent evolution of statistical average of an observable O:
$$\langle\hat{O}(t)\rangle_B = Z^{-1} \sum_\alpha e^{-\beta E_\alpha} \langle\Psi_{B\alpha}(t)|\hat{O}|\Psi_{B\alpha}(t)\rangle$$
- Statistical Fermi-Bose mapping: $|\Psi_{B\alpha}\rangle = \hat{A}|\Psi_{F\alpha}\rangle \Rightarrow$
$$\langle\hat{O}(t)\rangle_B = Z^{-1} \sum_\alpha e^{-\beta E_\alpha} \langle\Psi_{F\alpha}(t)|\hat{A}^{-1}\hat{O}\hat{A}|\Psi_{F\alpha}(t)\rangle$$
where
$$\hat{A} = \prod_{1\le j<k\le N} sgn(x_k - x_j)$$
in Schrödinger representation $\Rightarrow \langle\hat{O}(t)\rangle_B = \langle\hat{O}(t)\rangle_F$ if O commutes with A
Note: $E_\alpha = E_{B\alpha} = E_{F\alpha}$ are ideal Fermi gas N-particle energy levels, by pure-state mapping theorem

- Apply to single-particle density:

$$n(x,t) = \sum_{n=0}^{\infty} f_n |\phi_n(x,t)|^2$$

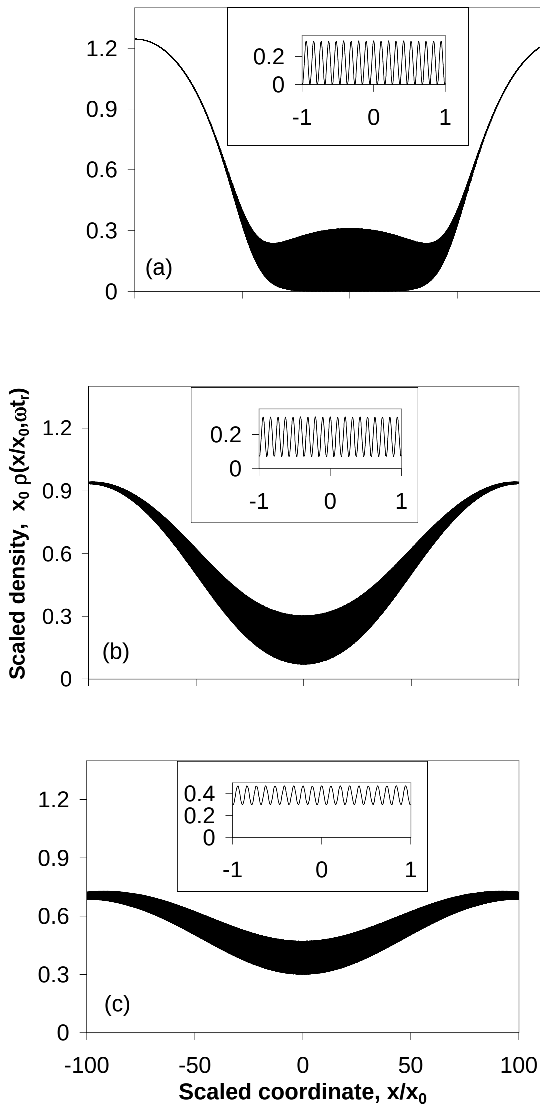
  where

$$f_n = \frac{1}{1 + e^{\beta(\epsilon_n - \mu)}}$$

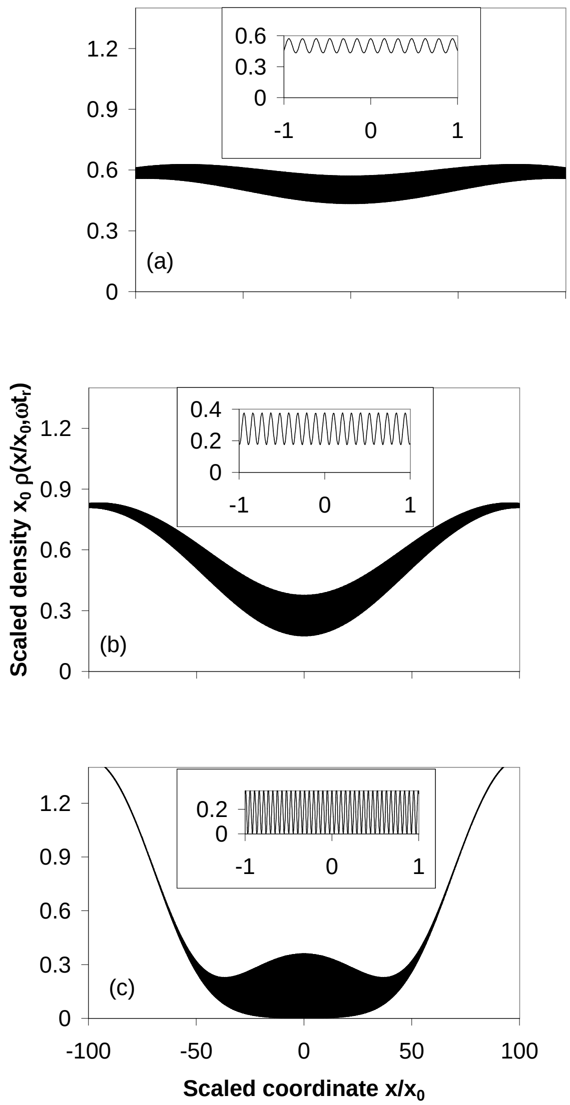
  is Fermi-Dirac distribution of initial trapped Tonks gas and $\epsilon_n = (n + \frac{1}{2})\hbar\omega$ are its HO energy levels

- Trap turned off at $t = 0$ and Bragg pulse applied, as before

- Orbitals $\phi_n(x,t)$ are solutions of single-particle TDSE. Bragg pulse kicks out daughter wavepackets which interfere on opposite side of ring as before.
  Only difference is that now n runs from 0 to ∞ and density contributions of individual orbitals are weighted by f_n, with chemical potential chosen so that $\Sigma_n f_n = N$.



Density profiles for N=100 atoms, $\eta = 0.5$, $L/x_0 = 100$ with momentum of daughter packets fixed, $kx_0 = 30 \approx 2.1\ k_F x_0$, for (a) $T = 5$ nK $\ll T_F$, (b) 20 nK $\approx T_F$, and (c) 45 nK $> T_F$. Insets show details of fringes.

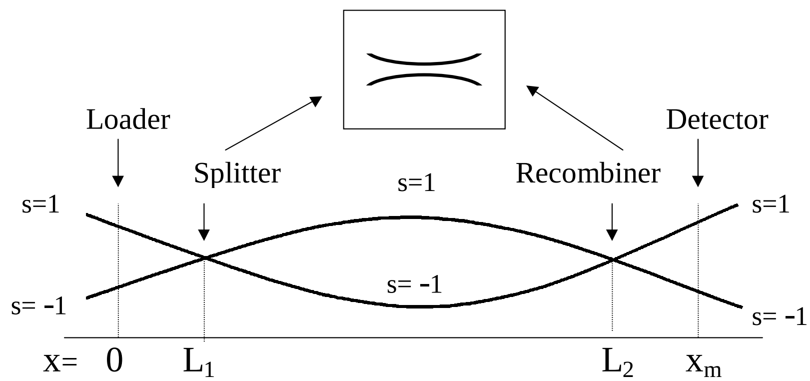


Scaled density profiles for N=100 atoms, $\eta = 0.5$, $L/x_0 = 100$ with temperature fixed, $T = 30\,\mathrm{nK} > T_F$ for different momenta of daughters: (a) $kx_0 = 20 \approx 1.4\ k_F x_0$, (b) 30 and (c) 60. Insets show details of fringes.

- For fixed daughter momentum $> k_F$, fringe visibility is high for $T \ll T_F$ and low for $T \gg T_F$.
- For fixed temperature $\sim T_F$, fringe visibility is high for $k \gg k_F$ and low for $k \approx k_F$.
- Conclusions: High fringe visibility for $k \gg k_F$ even if $T \sim T_F$, and also for $T \ll T_F$ even if $k \sim k_F$, i.e., for high visibility it is not necessary to have *both* $k \gg k_F$ *and* $T \ll T_F$.

DOUBLE-X TONKS GAS INTERFEROMETER

M.D. Girardeau, Kunal K. Das, and E.M. Wright, Phys. Rev. A **66**, 023604 (2002)



- Both arms satisfy necessary and sufficient conditions for Tonks regime (Olshanii 1998, Petrov *et al.* 2000)
- Junctions are actually avoided crossings with tunnelling between waveguides
- Upper and lower arms have same length, so can denote coordinates of each particle by (x, s) where x =longitudinal position, $s = \pm 1$ = pseudospin quantum number:
  $s = 1 \rightarrow$ upper arm, $s = -1 \rightarrow$ lower arm
- Tunneling regions idealized to points where tunneling is treated by boundary conditions

Generalized mapping theorem

N-atom wave functions: $\Psi(x_1, s_1; \cdots; x_N, s_N; t)$

Point hard cores $\Rightarrow$ wave functions vanish if two particles have same longitudinal coordinate *and* are in same arm:
$\Psi = 0$ if $s_j = s_k$ and $x_j = x_k$, $1 \leq j < k \leq N$

Start from N-fermion solutions $\Psi_F(x_1, s_1; \cdots; x_N, s_N; t)$
of $\hat{H}\Psi = i\hbar\partial\Psi/\partial t$

Ψ_F antisymmetric under space-pseudospin exchange $(x_j, s_j) \leftrightarrow (x_k, s_k)$
$\Rightarrow \Psi_F$ satisfies impenetrability constraint automatically

Define "unit antisymmetric function"

$$A(x_1, s_1; \cdots; x_N, s_N) = \prod_{1 \leq j < k \leq N} \alpha(x_j, s_j; x_k, s_k)$$

$$\alpha(x_j, s_j; x_k, s_k) = \delta_{s_k, s_j} \mathrm{sgn}(x_k - x_j)$$

$$+ \delta_{s_j 1}\delta_{s_k, -1} - \delta_{s_j, -1}\delta_{s_k 1}$$

Define bosonic wave function by $\Psi_B = A\Psi_F$
$\Rightarrow$ Ψ_B satisfies impenetrability constraint and satisfies $\hat{H}\Psi_B = i\hbar\partial\Psi_B/\partial t$

X-splitter boundary conditions

- Ψ_F is a determinant of N orbitals each of which is a Pauli pseudospinor:

$$\Phi_n(x,t) = \begin{bmatrix} \phi_n(x,+1;t) \\ \phi_n(x,-1;t) \end{bmatrix}$$

- Each such pseudospinor satisfies *single*-particle Schrödinger equation in potential $V_s(x,t)$ which can differ between upper and lower arms

- Tunneling implemented by boundary condition at each X-junction:

$$\Phi_n(L+,t) = T\Phi_n(L-,t)$$

  where $L-$ and $L+$ denote left and right sides of an X-junction at $x = L$ and T is a unitary matrix

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

  chosen so that a wavebacket entering in the upper arm splits equally into both arms at the junction.

Dynamics

- Interferometer loaded with harmonically trapped Tonks gas in upper arm, which is moving to the right (towards first X-junction), then trap potential turned off

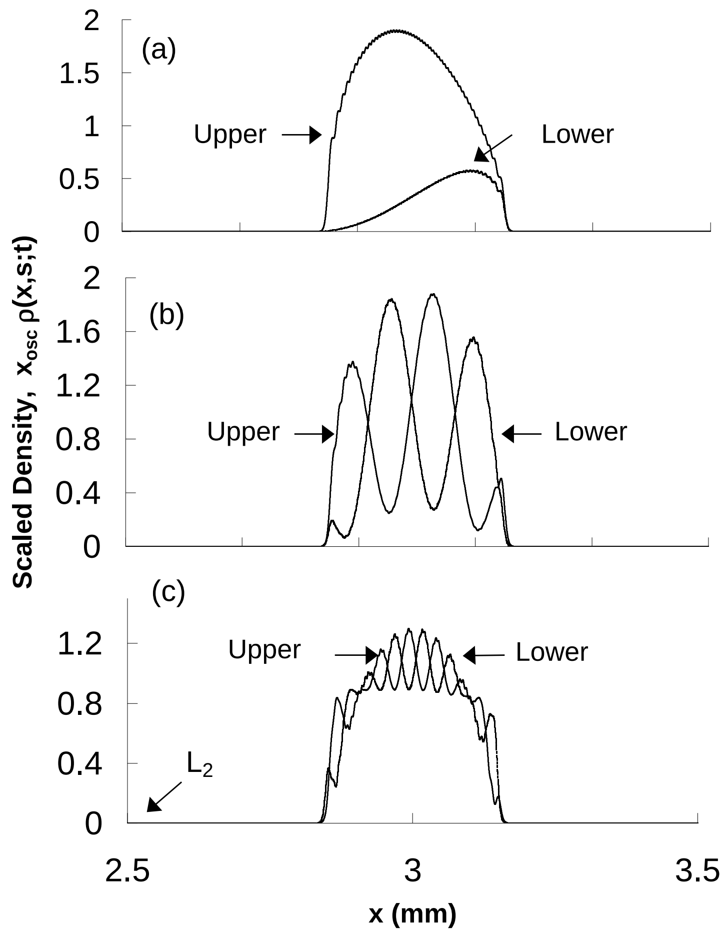
- Wavepacket splits at first junction according to T, propagates to second junction in presence of pulsed potential difference $V_s(x,t) = \hbar\Lambda_s(x)\delta(t-t_0)$ between the arms, where

$$\Lambda_s(x,t) = -s(k_p x + Qx^2)\delta(t-t_0)$$

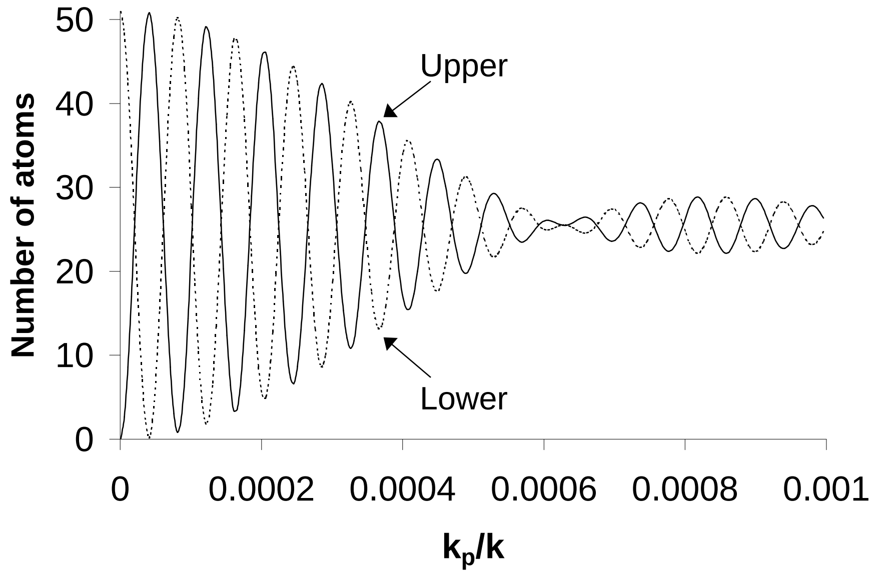
  $k_p x \Rightarrow$ spatially uniform force
  $Qx^2 \Rightarrow$ spatial gradient of force

- Wavepacket splits/recombines at second junction according to same T

- Density in both arms observed to right of second junction to exhibit interference fringes



Density profile of recombined wavepackets for 51 atoms, for pulsed potentials $V_s(x,t)$ applied at center of arms:
(a): $k_p/k = 10^{-4}$ and $Q = 0$
(b): $k_p/k = 10^{-3}$ and $Q = 0$
(c): $k_p/k = 10^{-3}$ and $Q/k = 5 \times 10^{-4}$ mm^{-1}
($k_p \propto$ pulse strength, $Q \propto$ pulse gradient)



Number of atoms in upper and lower exit arms as function of linear pulse strength k_p, with $Q = 0$ and time of application of pulses fixed.
Initial wavepacket has 51 atoms moving with velocity $v_x = 70$ mm/s.

Generalized Fermi-Bose duality: p-wave fermions $\rightarrow$ s-wave bosons, strong Fermi interactions $\rightarrow$ weak Bose interactions and vice versa

M.D. Girardeau and M. Olshanii, arXiv:cond-mat/0401402 (2004)

M.D. Girardeau, Hieu Nguyen, and M. Olshanii, arXiv:cond-mat/0403721 (2004)

Consider first 3D two-body scattering for spin-$\frac{1}{2}$ fermions.
s-wave scattering states: Space symmetric and spin antisymmetric, singlet spin eigenfunctions.
p-wave scattering states: Space antisymmetric and spin symmetric, triplet spin eigenfunctions.
s-wave scattering impossible in spin-polarized Fermi gas, but usually dominant in spinor Fermi gas since p-wave spatial antisymmetry suppresses short-range interactions. However, both s and p-wave interactions can be greatly enhanced by Feshbach resonances:
J.L. Roberts, N.R. Claussen, S.L. Cornish, E.A. Donley, E.A. Cornell, and C.E. Wieman, Phys. Rev. Lett. **86**, 4211 (2001)
C.A. Regal, C. Ticknor, J.L. Bohn, and D.S. Jin, Phys. Rev. Lett. **90**, 053201 (2003)

Assume that Hamiltonian does not depend on spin. Then spin dependence of 2-particle wave functions need not be indicated explicitly and they can be written as the sum of spatially even and odd parts ψ_e and ψ_o.

In a tight wave guide with effectively 1D dynamics (as before),

3D s-wave scattering $\rightarrow$ even-wave scattering, 1D scattering length a_{1D}^e

3D p-wave scattering $\rightarrow$ odd-wave scattering, 1D scattering length a_{1D}^o

Let $z = z_1 - z_2$ be relative coordinate for 1D scattering and z_0 =range of interaction.

Even and odd-wave scattering lengths defined in terms of ratio of derivative to value of wave functions just outside the range z_0 of the interaction:

$$\begin{aligned} \psi_e^{'}(z_0) &= -\psi_e^{'}(-z_0) = -(a_{1D}^e - z_0)^{-1}\psi_e(\pm z_0) \\ \psi_o(z_0) &= -\psi_o(-z_0) = -(a_{1D}^o - z_0)\psi_o^{'}(\pm z_0) \quad . \end{aligned}$$

In limit $z_0 \rightarrow 0$ a_{1D}^e is a known function of 3D s-wave scattering length a_s and a_{1D}^o is a known function of 3D p-wave scattering volume

$V_p = a_p^3 = -\lim_{k\rightarrow 0} \tan\delta_p(k)/k^3$

Olshanii '98:

$$(a_{1D}^e)^{-1} = \frac{-2a_s}{a_\perp^2}[1 - (a_s/a_\perp)|\zeta(1/2)|]^{-1}$$

B.E. Granger and D. Blume, Phys. Rev. Lett. **92**, 133202 (2004):

$$a_{1D}^o = \frac{6V_p}{a_\perp^2}[1 + 12(V_p/a_\perp^3)|\zeta(-1/2,1)|]^{-1}$$

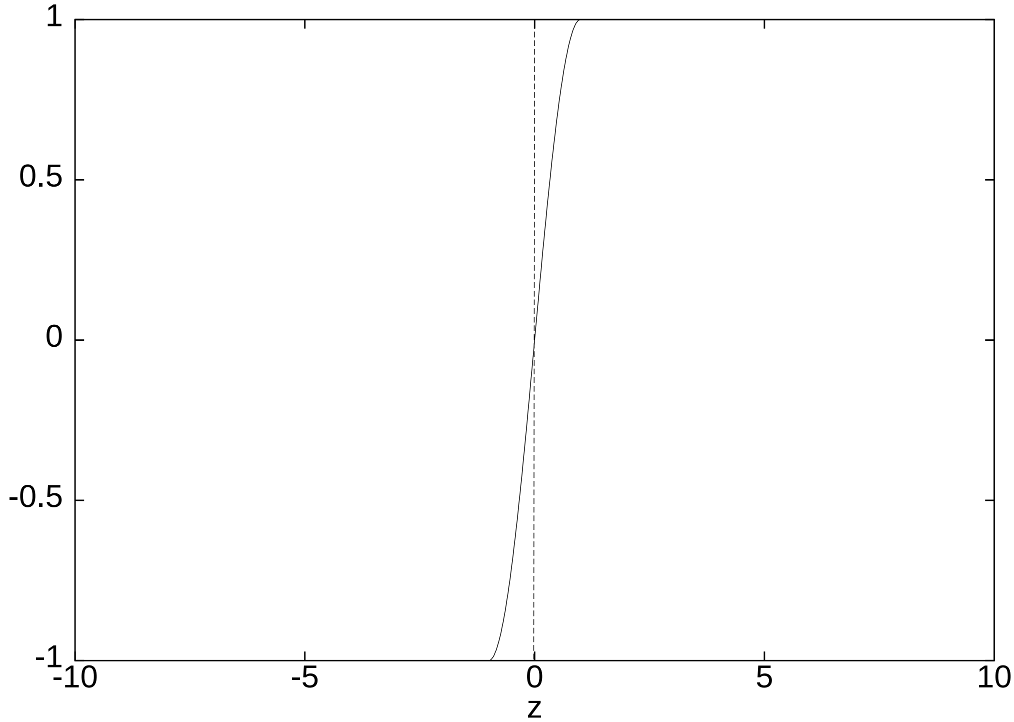
where $a_\perp = \sqrt{\hbar/\mu\omega_\perp}$ = transverse oscillator length
μ = effective mass
$\zeta(1/2) = -1.460\cdots$ = Riemann zeta function
$\zeta(-1/2,1) = -\zeta(3/2)/4\pi = -0.2079\cdots$ = Hurwitz zeta function.
In zero-range limit $z_0 \to 0$ these reduce to "contact conditions" relating a discontinuity in ψ_e' at contact to $\psi_e(0)$, and a discontinuity of ψ_o at contact to $\psi_o'(0)$.

Discontinuity in derivative is well-known consequence of zero-range delta function pseudopotential and plays a crucial role in solution of the Lieb-Liniger model of bosons in 1D with interaction $c\delta(z_i - z_j)$:
E.H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963)

Discontinuities of ψ itself have received little attention, although they have been discussed previously by Cheon and Shigehara:

T. Cheon and T. Shigehara, Phys. Lett. A **243**, 111 (1998) and Phys. Rev. Lett. **82**, 2536 (1999)

and are implicit in the recent work of Granger and Blume. For odd wave ψ_o the discontinuity $2\psi(0+)$ is a trivial consequence of antisymmetry together with the fact that a nonzero odd-wave scattering length cannot be obtained in the limit $z_0 \to 0$ unless $\psi_o(0\pm) \neq 0$. These discontinuities are rounded off when $z_0 > 0$, since interior wave function interpolates smoothly between the values at $z = -z_0$ and $z = z_0$:



$N = 2$ untrapped fermionic TG gas ground state (dashed line) compared with zero-energy scattering solution for a square well with range z_0 and depth V_0 corresponding to the boundary between no bound state and one bound state, a zero energy resonance (solid line), as function of relative coordinate z. Units are such that $z_0 = 1$.

Even and odd-wave pseudopotentials:

Take 2-particle Hamiltonian to be

$$\hat{H}_{1D} = -(\hbar^2/2\mu)\partial_z^2 + v_{1D}^e + v_{1D}^o$$

where v_{1D}^e and v_{1D}^o are even- and odd-wave pseudopotentials to be determined.
Define two linear operators $\hat{\delta}_\pm$ and $\hat{\partial}_\pm$ by

$$\begin{aligned} \hat{\delta}_\pm \psi(z) &= (1/2)[\psi(0+) + \psi(0-)]\delta(z) \\ \hat{\partial}_\pm \psi(z) &= (1/2)[\psi^{'}(0+) + \psi^{'}(0-)] \end{aligned}$$

where $\delta(z)$ is the usual Dirac delta function. The even and odd-wave pseudopotential operators are then

$$v_{1D}^e = g_{1D}^e \hat{\delta}_\pm \quad , \quad v_{1D}^o = g_{1D}^o \delta^{'}(z)\hat{\partial}_\pm \ .$$

For details see previously referenced recent work by Maxim and me.
Terms in $\delta(z)$ and $\delta^{'}(z)$ cancel from $\hat{H}_{1D}$ if

$$g_{1D}^e = -\frac{\hbar^2}{\mu a_{1D}^e} \quad , \quad g_{1D}^o = -\hbar^2 a_{1D}^o/\mu$$

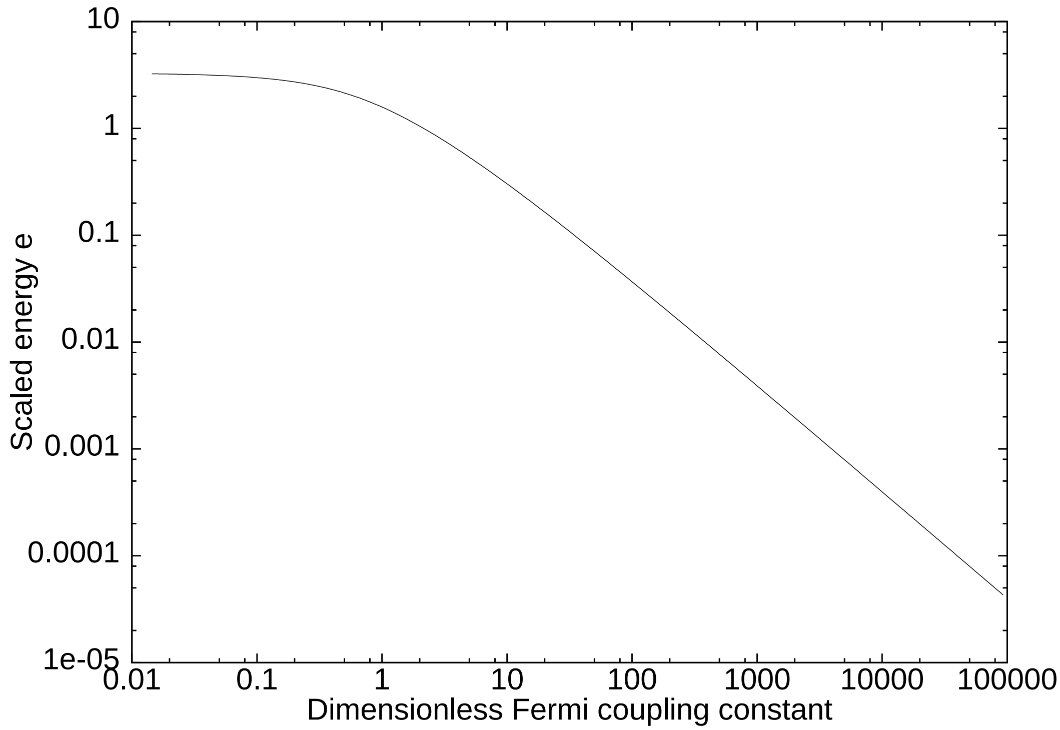
Fermi-Bose mapping:

Two-body states $\psi(z)$ considered so far are fermionic, i.e., spatially even part $\psi_e(z)$ contains an implicit spin-odd singlet spin factor, and spatially odd part $\psi_o(z)$ contains implicit spin-even triplet spin factors.
To emphasize combined space-spin fermionic antisymmetry, denote these by $\psi_F(z) = \psi_F^e(z) + \psi_F^o(z)$.
States of combined space-spin bosonic symmetry can be defined by mapping $\psi_B(z) = \mathrm{sgn}(z)\psi_F(z)$ where $\mathrm{sgn}(z)$ is $+1$ if $z > 0$ and -1 if $z < 0$.
This maps spatially even fermionic function ψ_F^e to spatially odd bosonic function ψ_B^o and spatially odd fermionic function ψ_F^o to spatially even bosonic function ψ_B^e while leaving the spin dependence unchanged, and corresponding scattering lengths are also unchanged:
$a_{1D,B}^o = a_{1D,F}^e$ and $a_{1D,B}^e = a_{1D,F}^o$.
Odd-wave contact conditions for $a_{1D,F}^o \rightarrow$ even-wave contact conditions for $a_{1D,B}^e$
Even-wave contact conditions for $a_{1D,F}^e \rightarrow$ odd-wave contact conditions for $a_{1D,B}^o$.
Since kinetic energy contributions from $z \neq 0$ also agree, one has a mapping from fermionic to bosonic problem which preserves energy eigenvalues and dynamics.

The bosonic Hamiltonian is of same form as fermionic one but with mapped coupling constants

$$g_{1D,B}^{e} = \frac{\hbar^4}{\mu^2 g_{1D,F}^{o}} \quad , \quad g_{1D,B}^{o} = \frac{\hbar^4}{\mu^2 g_{1D,F}^{e}}$$

Spin-aligned ground state:
If spins aligned (total spin $S = N/2$) then ground state is totally spin-symmetric and space-antisymmetric $\rightarrow \psi_{F0}$ maps to space-symmetric Bose ground state ψ_{B0}.
This is the Lieb-Liniger (LL) model (bosons with delta function interaction in 1D), whose solution in untrapped case was found by LL by Bethe Ansatz method.
Then we get spin-aligned Fermi gas ground state energy by inverting the coupling constant:



Log-log plot of scaled ground state energy per particle $e = 2m\epsilon/\hbar^2 n^2$ for the spatially antisymmetric spinor Fermi gas, versus dimensionless fermionic coupling constant γ_F.

Define dimensionless bosonic and fermionic coupling constants $\gamma_B = mg^e_{1D,B}/n\hbar^2$ and $\gamma_F = mg^o_{1D,F}n/\hbar^2$ where n is longitudinal particle number density. They satisfy $\gamma_B\gamma_F = 4$.

"Fermionic TG gas" $\gamma_F \to \infty$ can be realized by tuning $g^o_{1D,F}$ to large positive values by a Feshbach resonance. It maps to *ideal Bose gas.*

There is a region of (γ_B, γ_F) plane where ground state has $S = 0$ instead of $S = N/2$. It is more complicated because it cannot be factorized into space$\times$spin, because for $N > 2$ there are no totally spin-antisymmetric states. Solved analytically in spatially uniform case by Yang and numerically in longitudinally trapped case by Astrakharchik *et al.*:
C.N. Yang, Phys. Rev. Lett. **19**, 1312 (1967)
G.E. Astrakharchik, D. Blume, S. Giorgini, and L.P. Pitaevskii, arXiv:cond-mat/0312538 (2003).

Define dimensionless even-wave and odd-wave fermionic coupling constants by $\gamma_e = mg^e_{1D,F}/n\hbar^2$ and $\gamma_o = mg^o_{1D,F}n/\hbar^2$. Maxim and I have determined the *exact* phase boundary between the $S = 0$ (paramagnetic) and $S = N/2$ ferromagnetic regions of the (γ_e, γ_o) plane. This can be found by a symmetry argument even though exact ground state is *not* known for general values of γ_e and γ_o. We find:
$\gamma_e\gamma_o < 4 \rightarrow$ paramagnetic, $S = 0$
$\gamma_e\gamma_o > 4 \rightarrow$ ferromagnetic, $S = N/2$
Thus a *quantum phase transition can be induced by either a p-wave or s-wave Feshbach resonance.*
Proof is complicated; for details see our preprint cond-mat/0403721.