Eigenfunctions of Galactic Phase Space Spirals from Dynamic Mode Decomposition

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Abstract

In DMD, the dynamics of nonlinear systems is studied by finding the dominant eigenfunction and eigenvalues of an approximate data driven, linear model for the evolution.

We introduce DMD to the field of galactic dynamics by applying it to the well-worn problem of a 1D plane-symmetric system.

DMD was developed in the field of fluid mechanics to analyze data from experiments and simulations

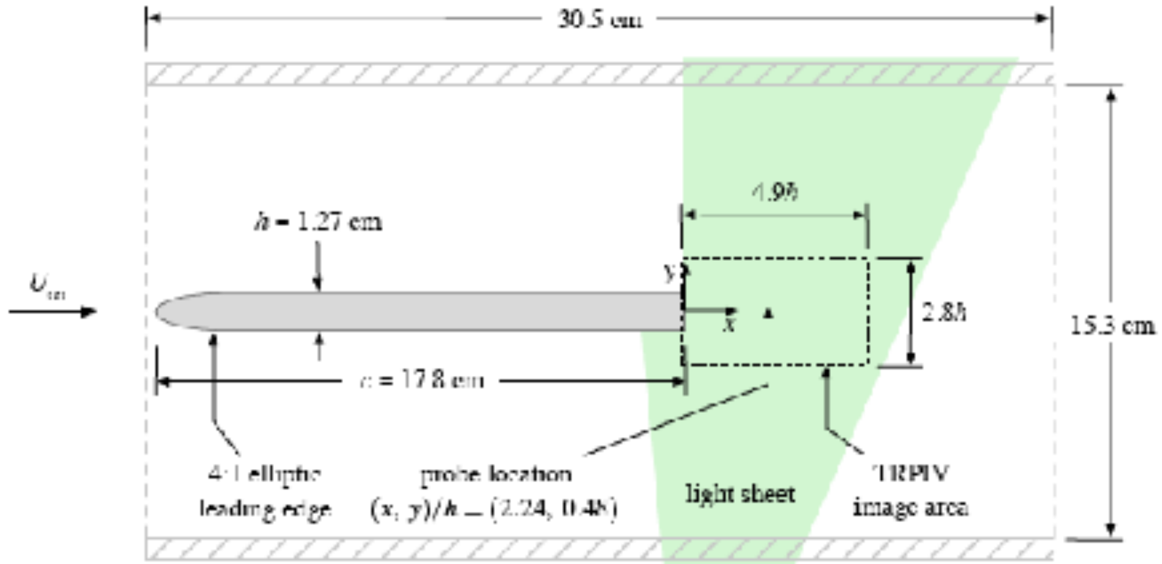
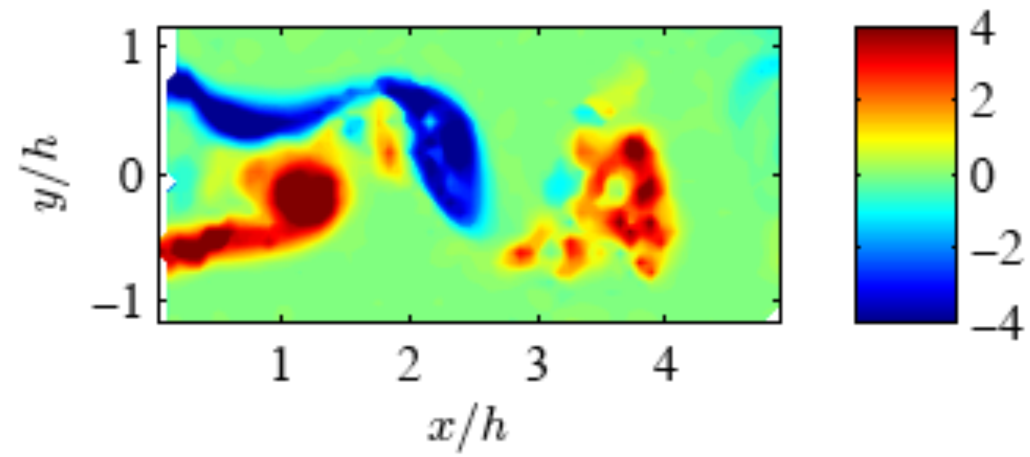


FIGURE 5. Schematic of setup for bluff-body wake experiment.

ON DYNAMIC MODE DECOMPOSITION



Tu+13
See text by Kutz et al on DMD

FIGURE 6. Typical vorticity field from the bluff-body wake experiment depicted in Figure 5. A clear von Kármán vortex street is observed, though the flow field is contaminated by turbulent fluctuations.

Given knowledge of an observable in the form of a series of snapshots (here, map of vorticity at discrete times), what can we say about the evolution of the system?

For background, see papers by Kutz group at UW, Mezic group here at UCSB, and many more

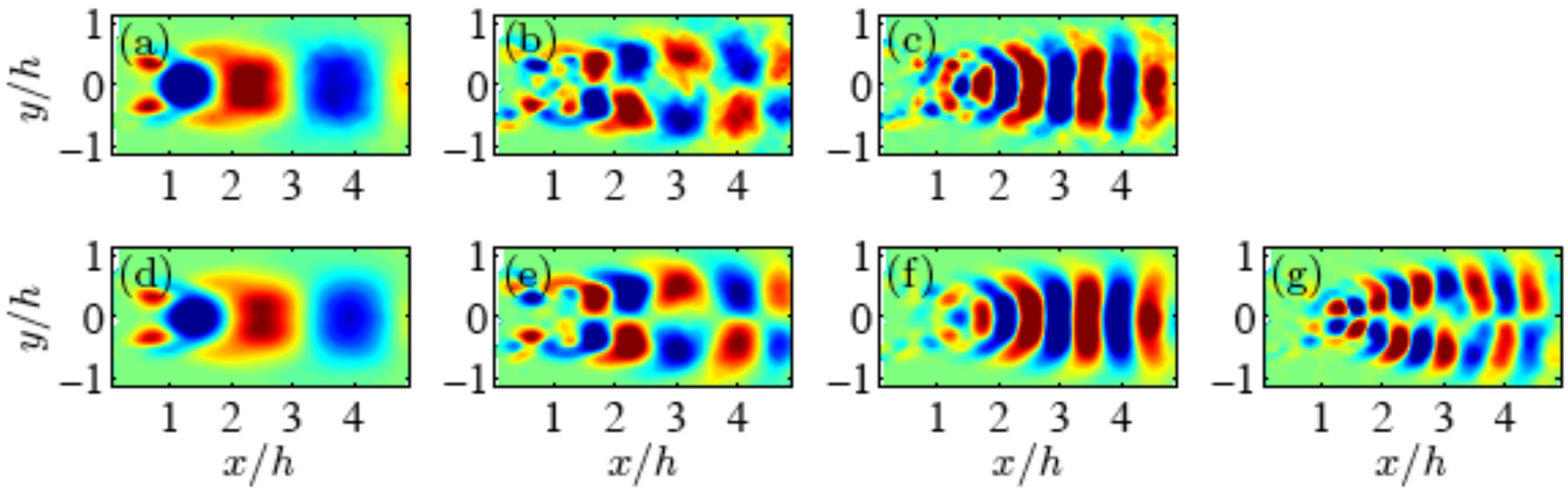
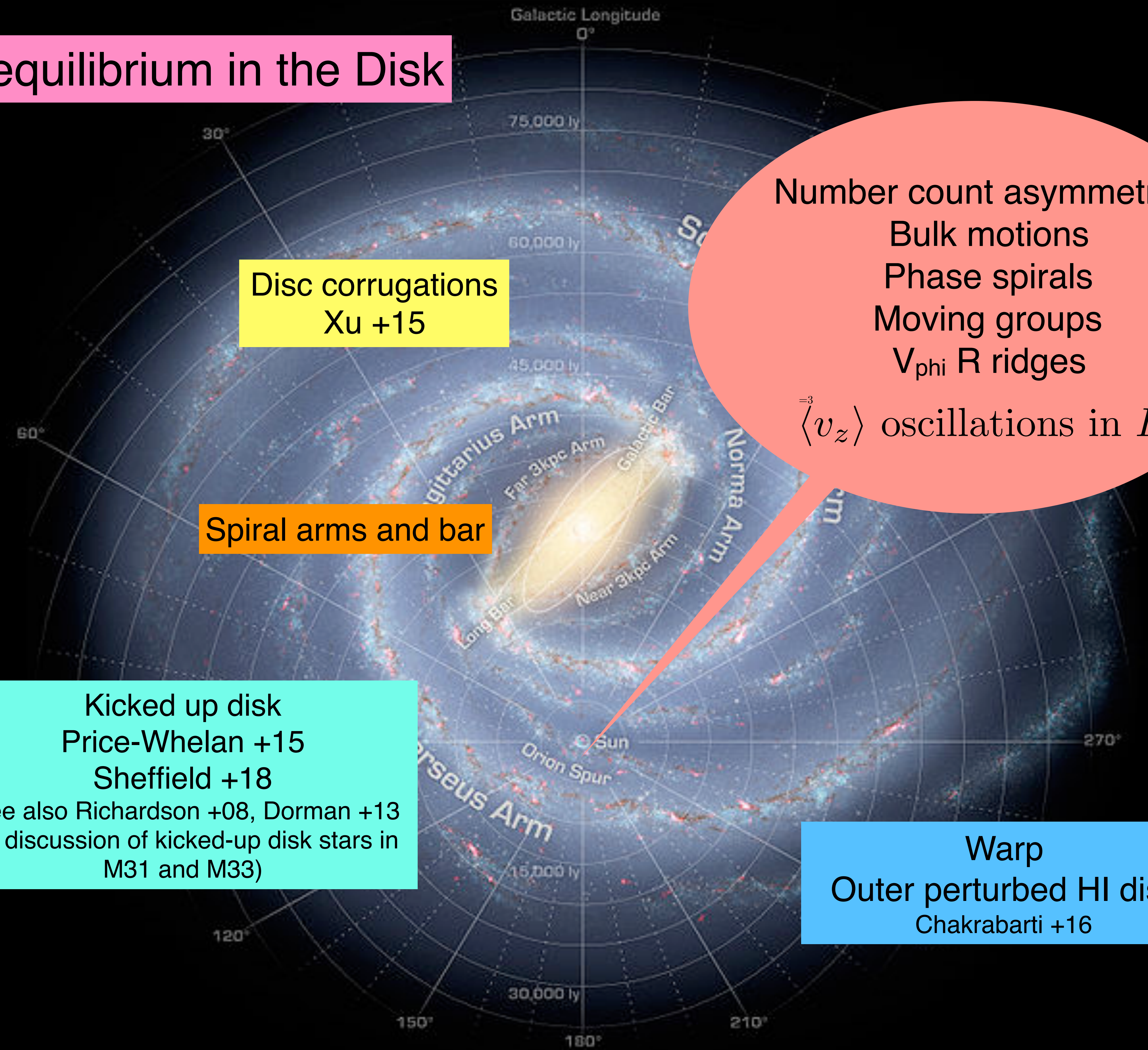


FIGURE 8. Representative DMD modes, illustrated using contours of vorticity. (For brevity, only the real part of each mode is shown.) The modes computed using multiple runs (bottom row) have more exact symmetry/antisymmetry and smoother contours. Furthermore, with multiple runs four dominant mode pairs are identified; the fourth spectral peak is obscured in the single-run computation (see Figure 7). (a) $f = 87.75$ Hz, single run; (b) $f = 172.6$ Hz, single run; (c) $f = 261.2$ Hz, single run; (d) $f = 88.39$ Hz, five runs; (e) $f = 175.6$ Hz, five runs; (f) $f = 264.8$ Hz, five runs; (g) $f = 351.8$ Hz, five runs.

Each DMD mode has a (generally complex) frequency

Disequilibrium in the Disk



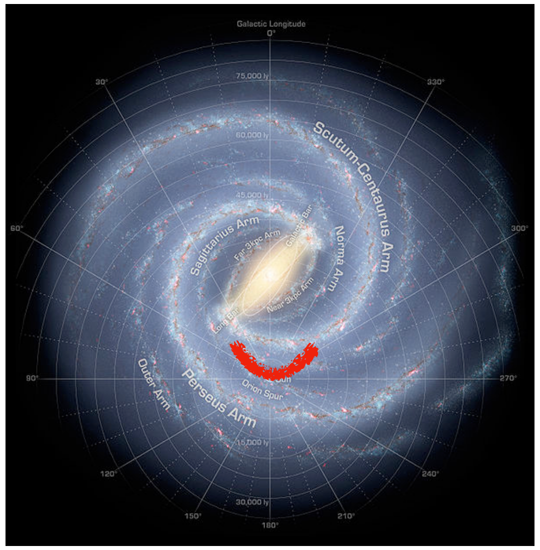
Disc corrugations
Xu +15

Spiral arms and bar

Number count asymmetries
Bulk motions
Phase spirals
Moving groups
V_{phi} R ridges
$\langle v_z \rangle$ oscillations in L_z

Kicked up disk
Price-Whelan +15
Sheffield +18
(see also Richardson +08, Dorman +13 for discussion of kicked-up disk stars in M31 and M33)

Warp
Outer perturbed HI disk
Chakrabarti +16



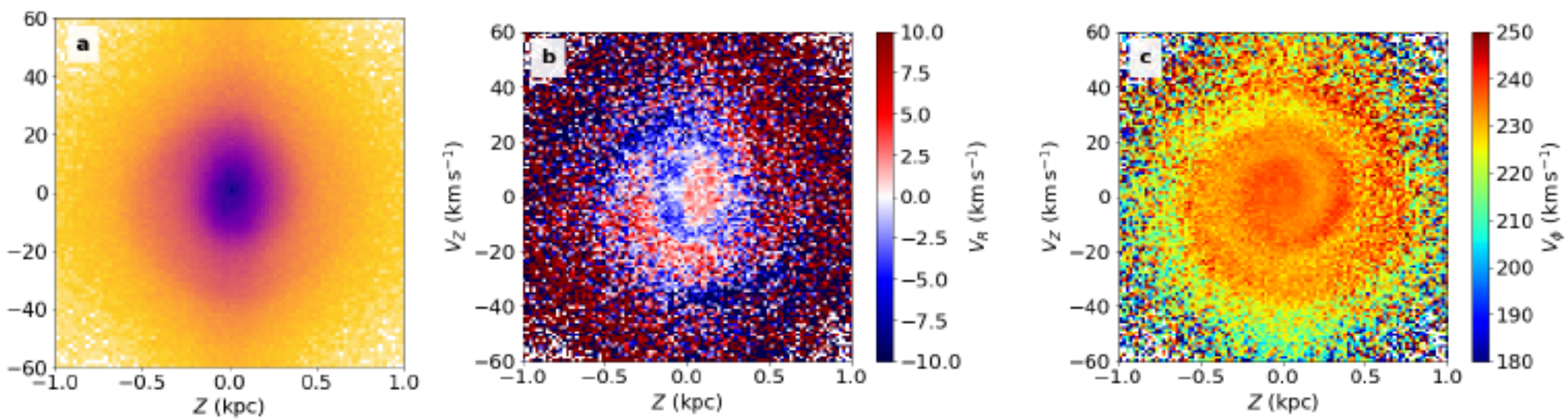
Phase Spirals in z-V_z plane Antoja+18
(GDR2)

Density

V_R

V_{phi}

V_z



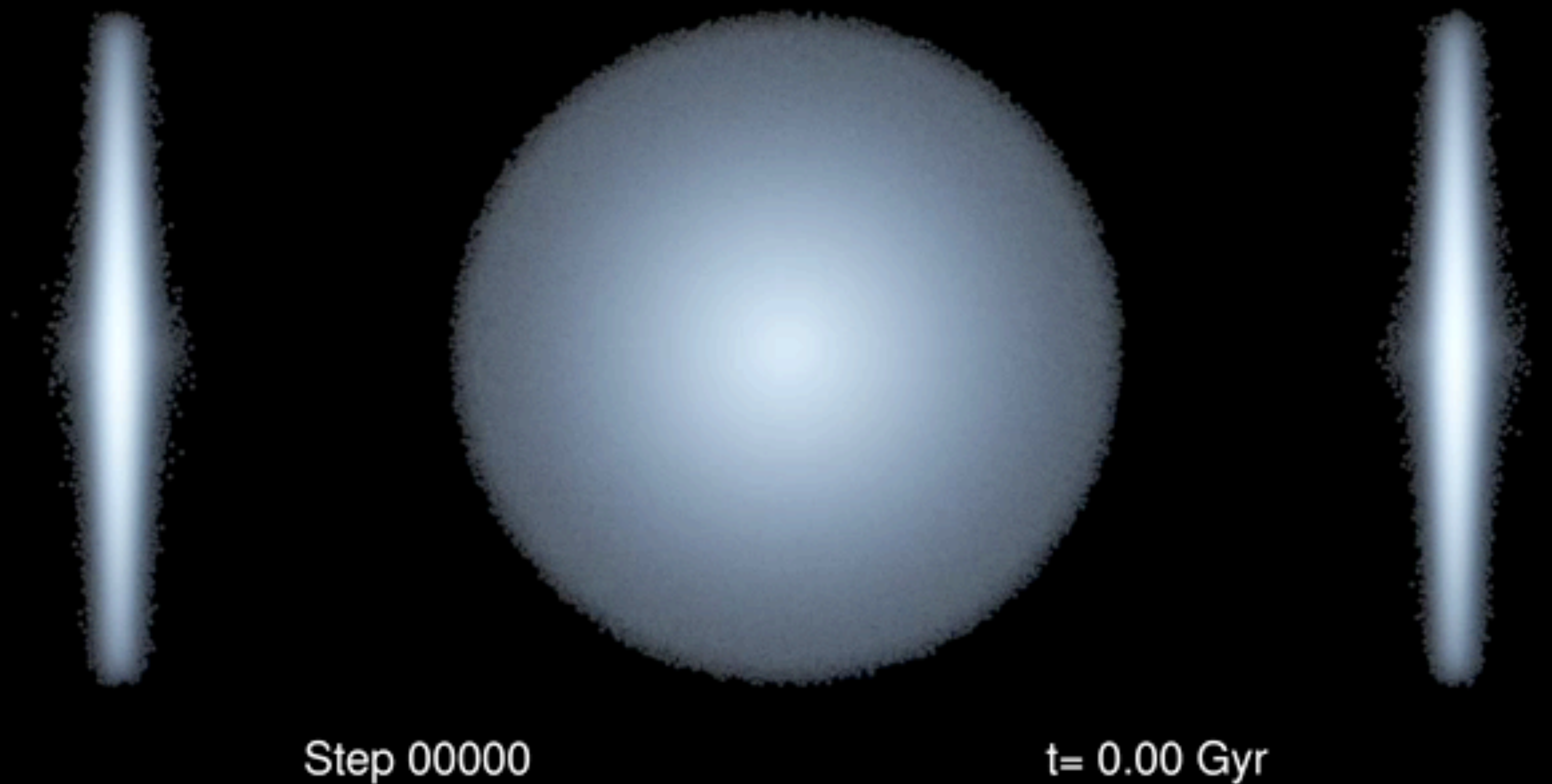
z

Open Questions

Transients vs long-lived structures
Phase-mixing vs self-gravitating waves
Coupling of in-plane and vertical motions

If one of our goals is to determine the gravitational potential and the structure of the Galaxy as well as the amount of dark matter in the Solar Neighbourhood (Oort problem) then does disequilibrium make our job harder or does it present an opportunity?

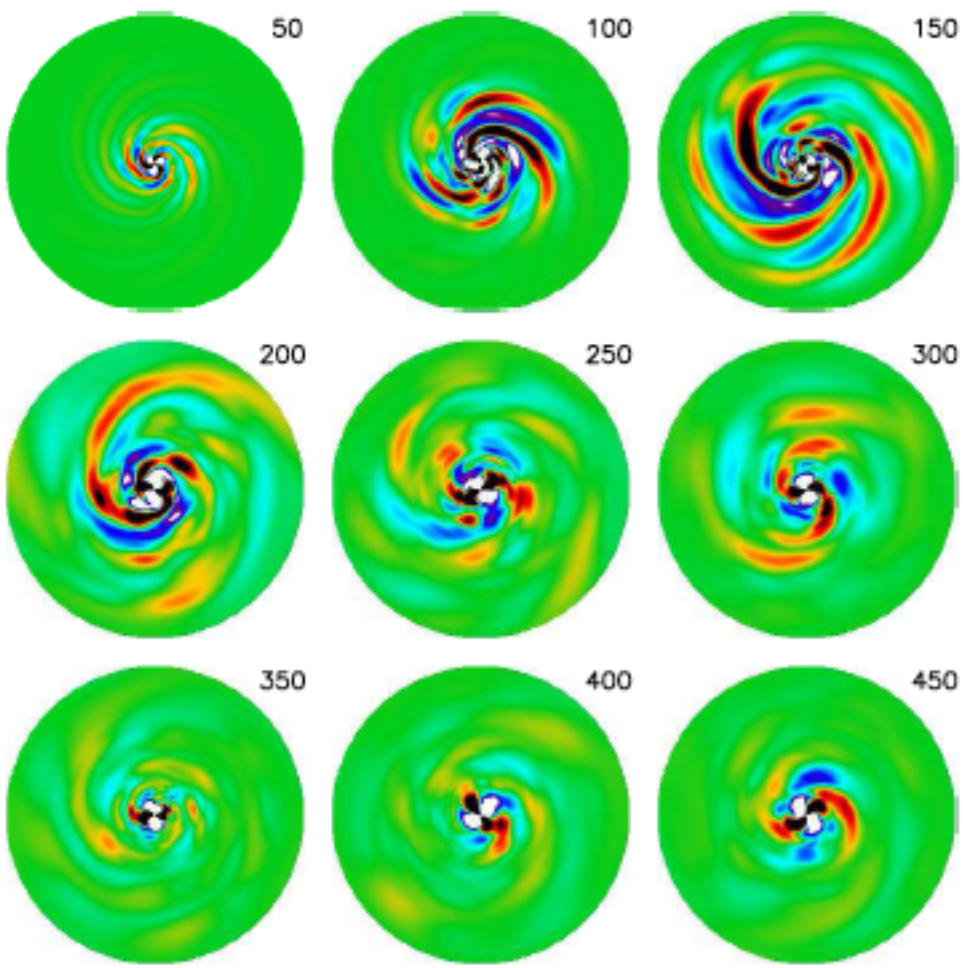
How can we best use simulations to (a) understand the physics of disk disequilibrium and (b) interpret observations?



Animation by J. Dubinski

Spectral analysis has proved extremely useful in the study of bars and spiral structure

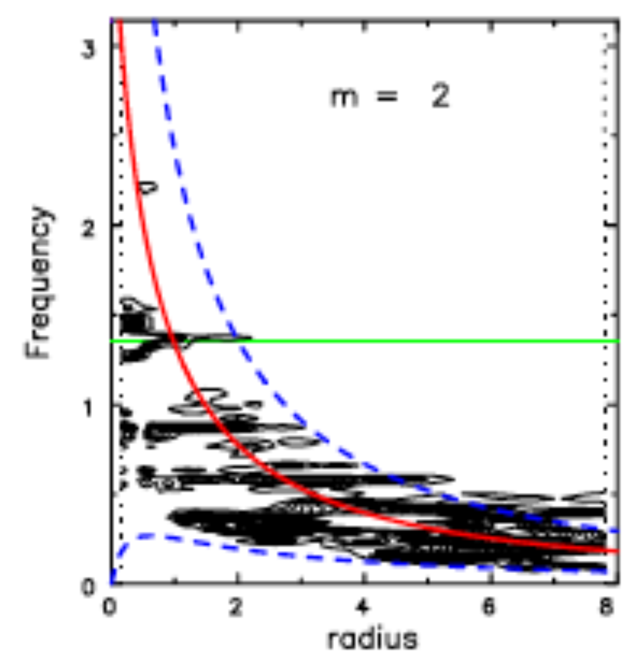
(see Sellwood & Athanassoula 1986)



Sellwood & Carlberg

$$\Sigma(R,\, z,\, t) = \sum_{m=0}^{\infty} A_m(R,\, t) e^{im\phi}$$

$$= \sum_{m=0}^{\infty} \int_0^{\infty} \tilde{A}_m(R,\, \Omega) e^{I(m\phi - \Omega t)}\, d\Omega$$

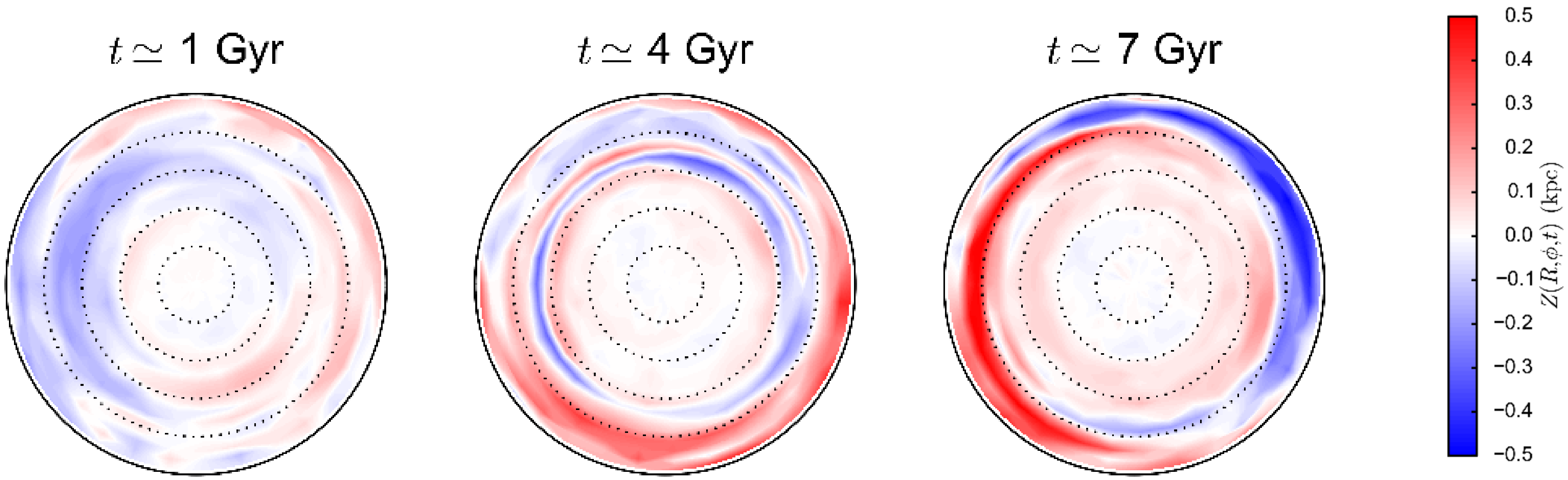


We can apply the same methods to the vertical moments

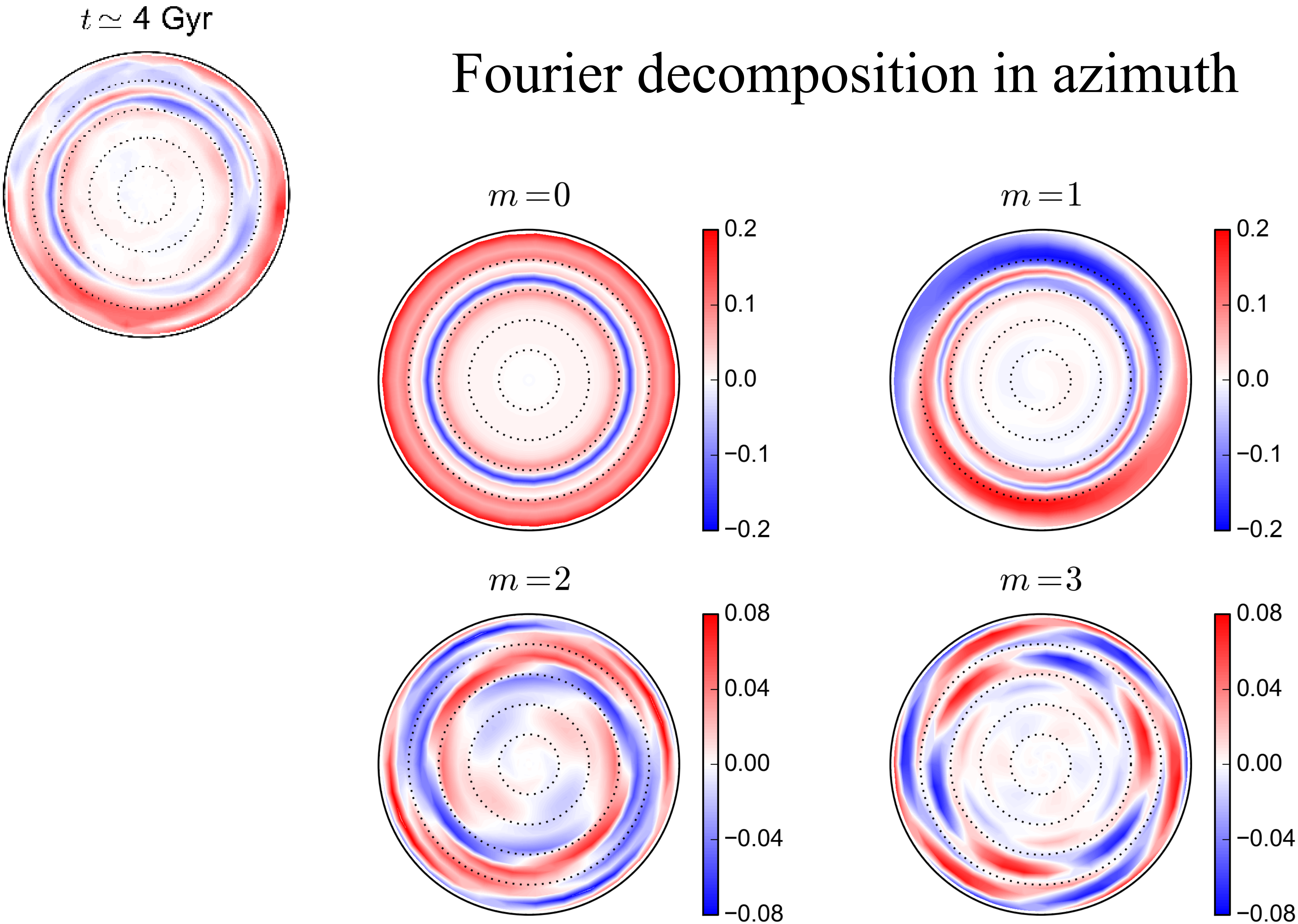
$$\langle z \rangle (R,\, \phi) \text{ and } \langle v_z \rangle (R,\, z)$$

See Chequers & LW 17,18

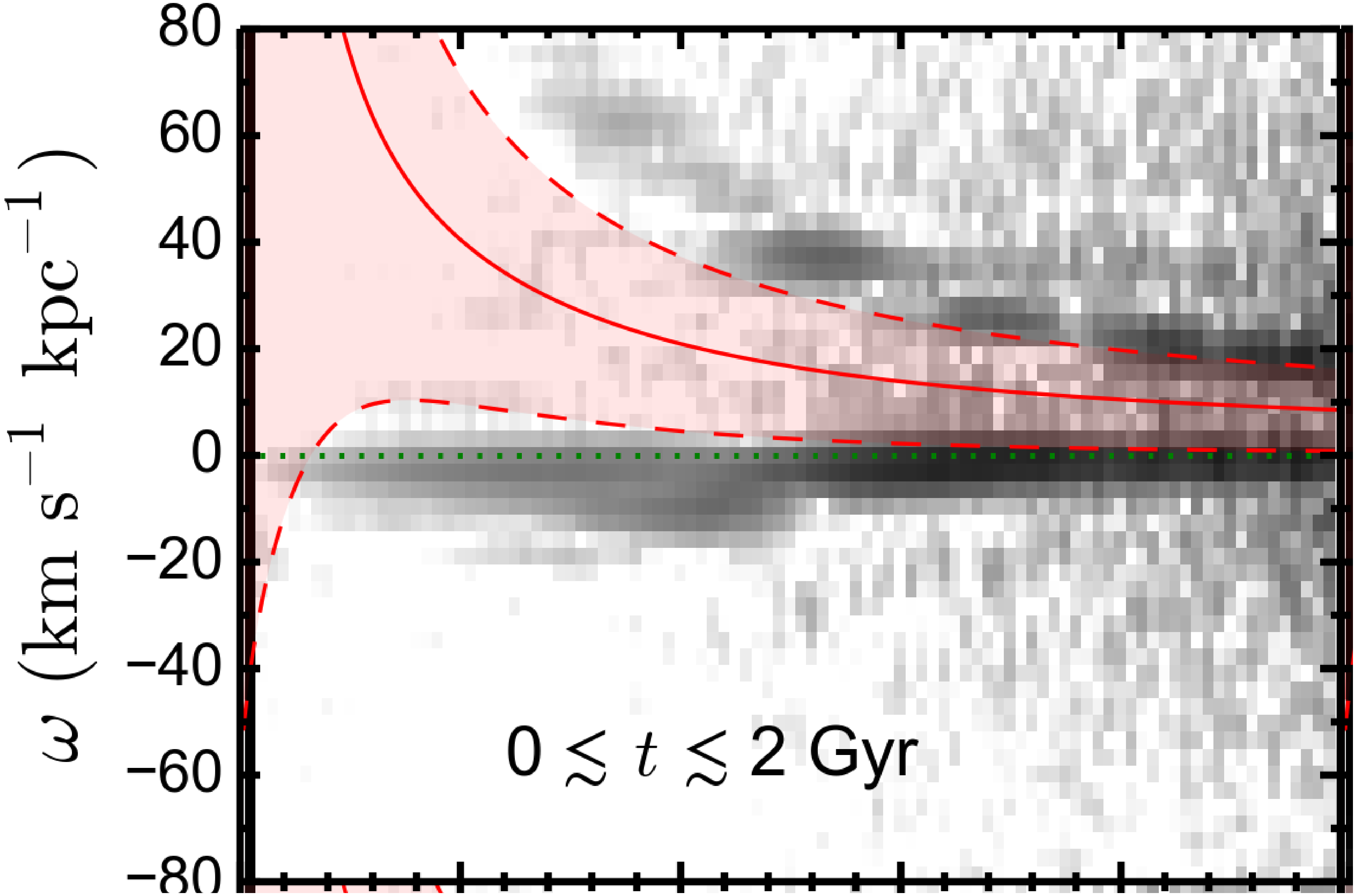
map of vertical displacement (colour range is +- 500 pc)



Fourier decomposition in azimuth



Spectral analysis of simulations show good agreement with predictions from eigenmode/WKB



Possible shortcomings of spectral decomposition

1) Will not capture non-linear dynamics such as mode coupling

Astron. Astrophys. 318, 747–767 (1997)

Non-linear generation of warps by spiral waves in galactic disks

F. Masset and M. Tagger

2) Is this the right approach when we have both phase mixing and long-lived waves/oscillations?

The goal of DMD is to find something akin to the normal modes of a nonlinear system

Normally, we think of eigenfunction methods as being applicable to Hamiltonian systems in the limit of small (linear) oscillations

$$\mathbf{x} = \begin{pmatrix} q \\ p \end{pmatrix}$$

$$\begin{aligned} H\left(q,\, p\right) &= \mathrm{kin} + \mathrm{pot} \\ &= \frac{1}{2} p^T C p \;+\; \frac{1}{2} q^T B q \end{aligned}$$

$$\frac{d\mathbf{x}}{dt} = \mathcal{A}\mathbf{x}$$

$$\mathcal{A} = \begin{pmatrix} 0 & C \\ -B & 0 \end{pmatrix}$$

By finding the eigenfunctions and eigenvalues of A, we can immediately write down the state of the system at some time t given the state of the system at t=0.

$$\mathcal{A}\,\psi_j = \omega_j\,\psi_j$$

$$x(t) \;=\; \sum_j b_j\,\psi_j\,e^{\omega_j t}$$

Dynamic Mode Decomposition (DMD)

For background on method, see book by Kutz et al

Method uses simulation data to learn the “modes” of a nonlinear system

State space

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x})$$

→

space of observables

$$x_{j+1} = A x_j$$

Sample system at discrete times

$$\mathbf{X} = \begin{pmatrix} | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \ldots & \mathbf{x}_{m-1} \\ | & | & & | \end{pmatrix}$$

$$\mathbf{X}' = \begin{pmatrix} | & | & & | \\ \mathbf{x}_2 & \mathbf{x}_3 & \ldots & \mathbf{x}_{m} \\ | & | & & | \end{pmatrix}$$

Columns are snapshots of data

$$\mathbf{A} = \mathbf{X}'\mathbf{X}^{+}$$

Find dominant eigenvalues of A via SVD

Our goal is to find the dominant eigenfunctions/eigenvalues of

$$A = X'X^+$$

$$X^+ \text{ is the Moore} - \text{Penrose inverse}$$

The rest is linear algebra

$$X \simeq U\Sigma V^\dagger \qquad \text{SVD}$$

$$X^+ \simeq V\Sigma^{-1}U^\dagger$$

$$A \simeq X'V\Sigma^{-1}U^\dagger$$

We next project A on to an r-dimensional sub matrix corresponding to the r largest singular values from Sigma

$$A \longrightarrow \tilde{A}$$

Eigenfunctions of $\tilde{A}$ are the (dominant) DMD modes

$$\tilde{A}\,\psi_k = e^{\lambda_k t}\,\psi_k$$

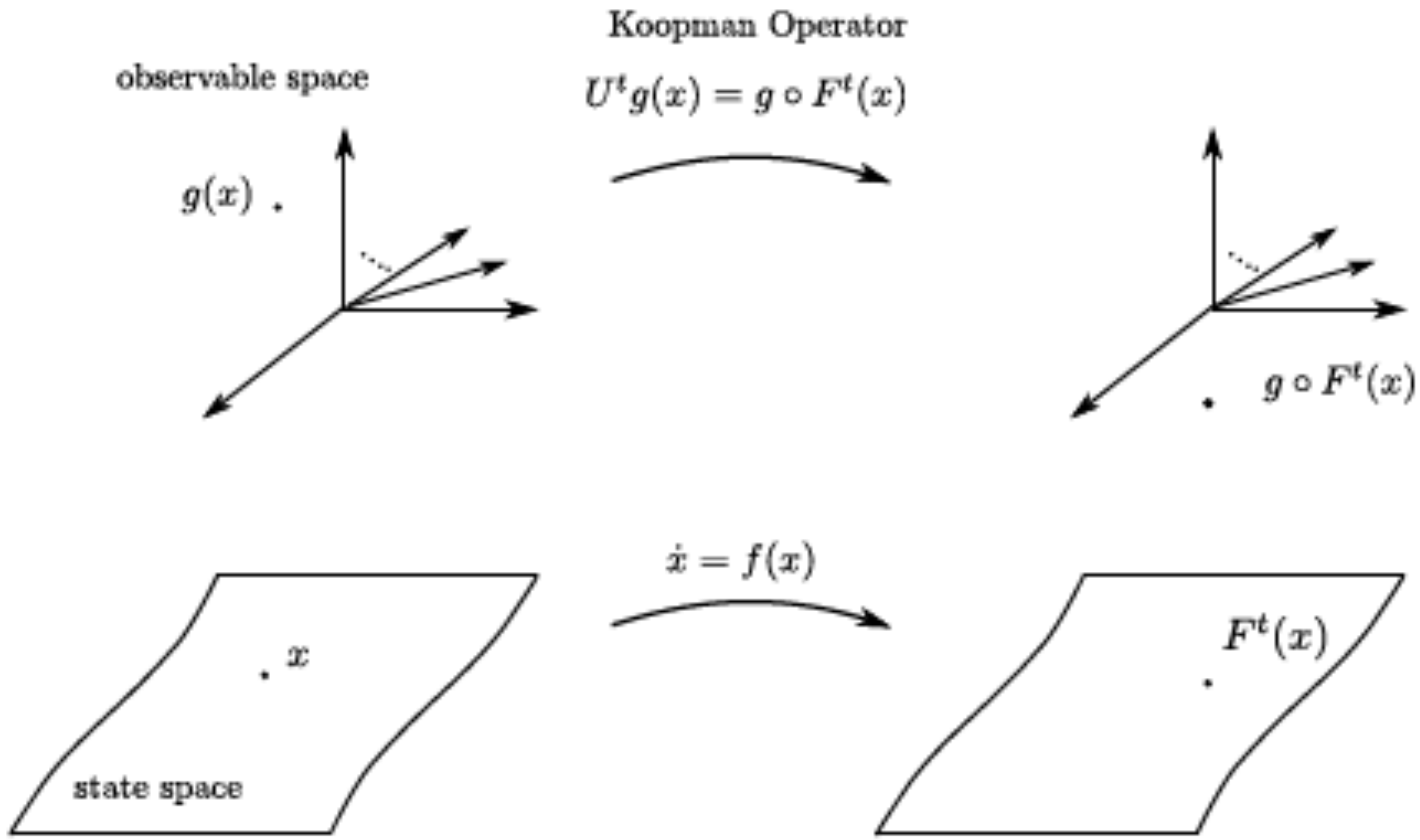


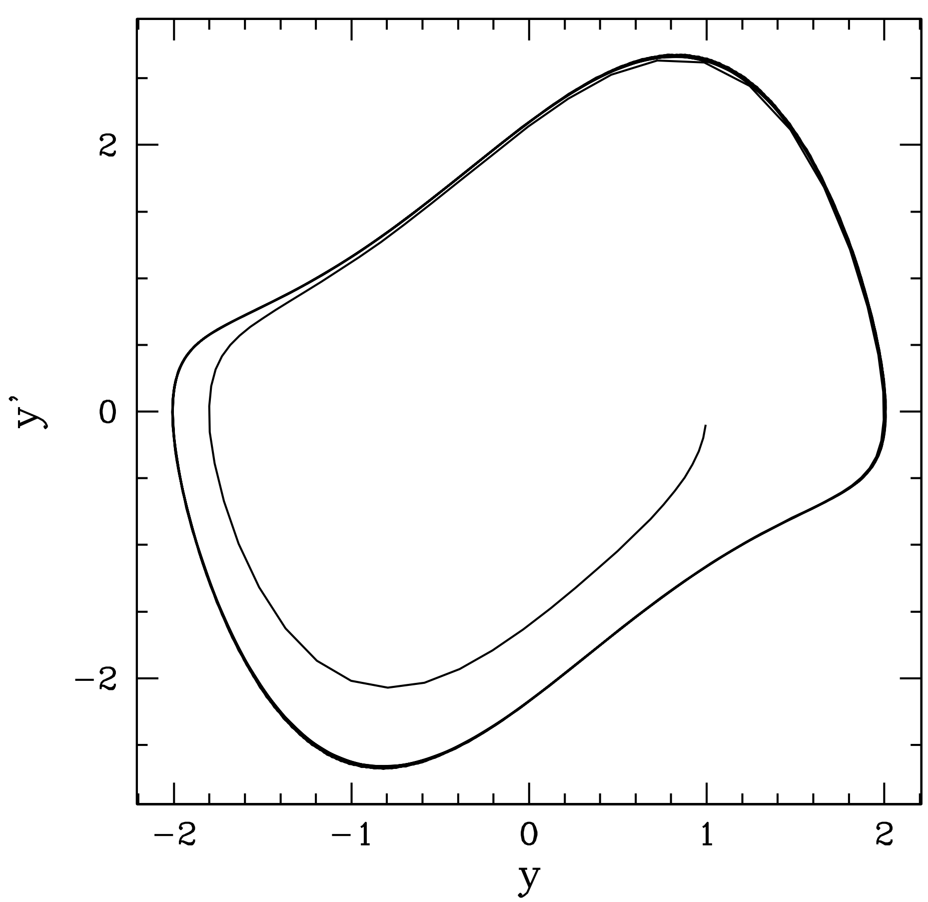
Figure 0.1: Koopman viewpoint lifts the dynamics from state space to the observable space, where the dynamics is linear but infinite dimensional.

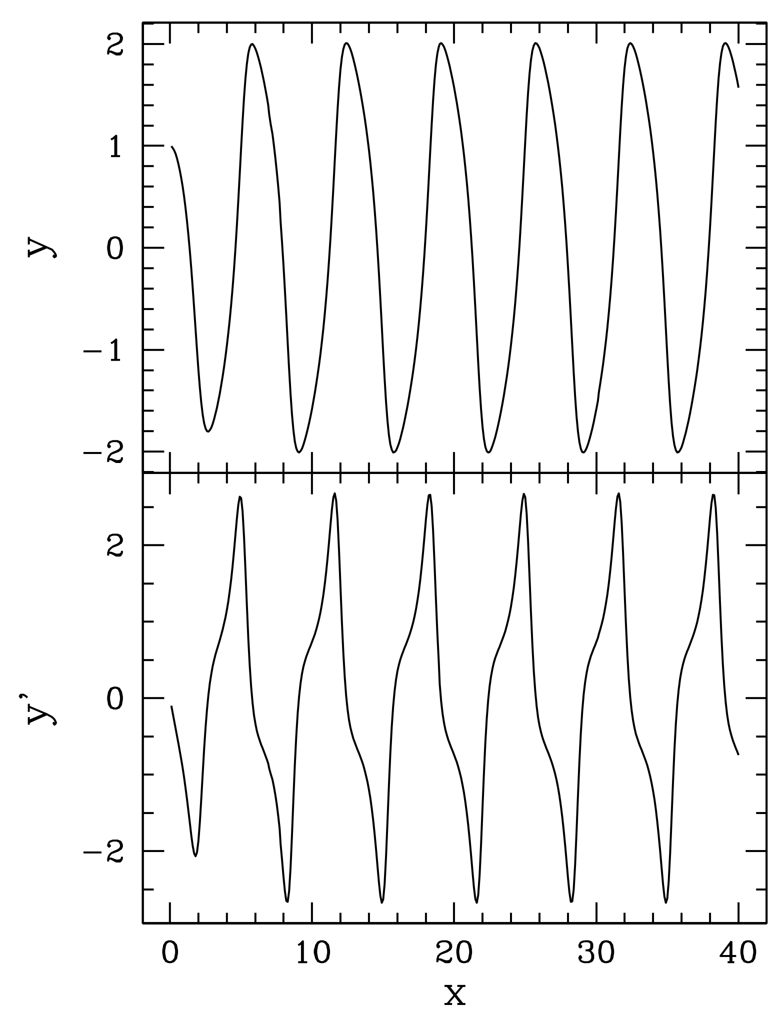
From Arbabi 2019

An embarrassingly simple example

$$\frac{d^2x}{dt} \; - \; a\left(1 - x^2\right)\frac{dx}{dt} \; + \; x \; = \; 0$$

van der Pol equation





This is a two-dimensional non-linear system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = a\left(1 - x^2\right) y - x$$

$$\mathbf{w}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Consider any observable of the state vector **w**
g(**w**(t))

Our goal is to describe this in terms of DMD modes

That is, our goal is to decompose

g(**w**(t))

Into DMD modes

$$
\begin{aligned}
g(\mathbf{w}) &= \sum_{k=0}^{\infty} g_k \, \phi_k(\mathbf{w}) \\
U^t \, \phi_k &= e^{\lambda_k t} \phi_k \\
U^t \, g(\mathbf{w}) &= \sum_{k=0}^{\infty} g_k \, e^{\lambda_k t} \phi_k
\end{aligned}
$$

Since the system reaches a limit cycle, the nonlinear behaviour is periodic with some period T. So for any observable we can construct a Fourier series:

$$g(\mathbf{w}(t)) \;=\; \sum_k g_k \, e^{2\pi i k t/T}$$

but this is precisely the desired form for DMD modes with

$$\phi_k \;=\; e^{2\pi i k t/T} \qquad\qquad \lambda_k \;=\; 2\pi i k/T$$

Of course, most nonlinear systems are not periodic and we won't have the luxury of simply writing down a Fourier series. The idea is to use the data to discover the analog of Fourier modes for a more general nonlinear system.

Let's apply to a more complicated (but still toy model) problem namely, 1D planar dynamics with gravity

(Spitzer 1942, Camm 1950)

$$f_{\rm eq}\left(z,\, v_z\right) = \frac{\rho_0}{\left(2\pi\sigma_z^2\right)^{1/2}} e^{-E_z/\sigma_z^2}$$

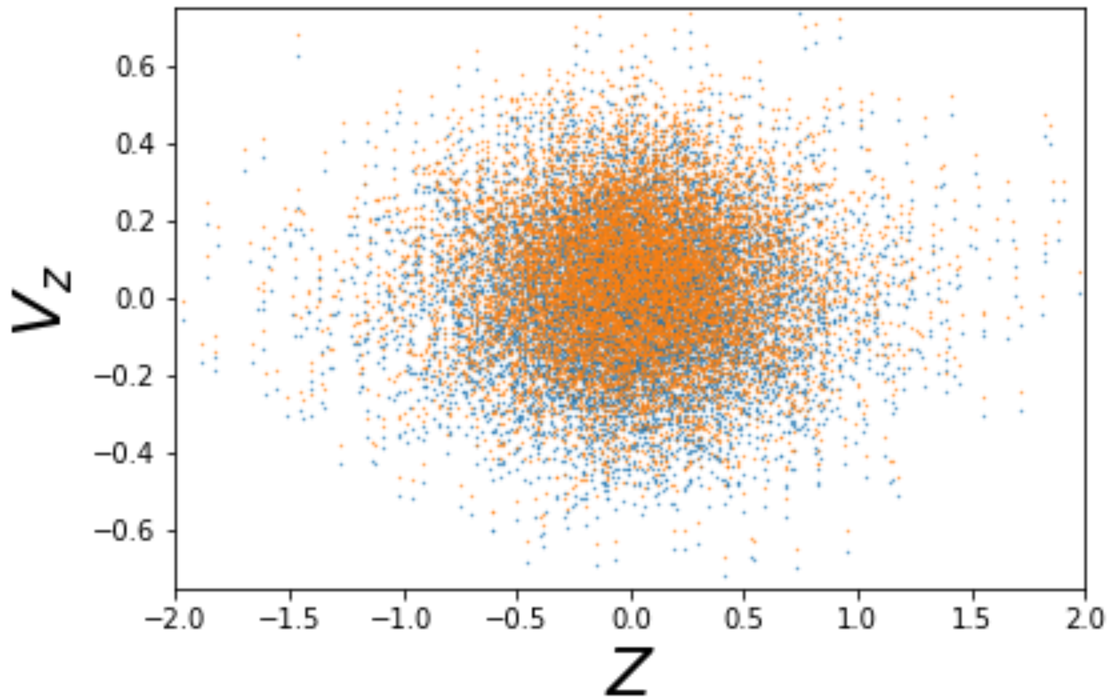
$$\rho_{\rm eq}(z) = \rho_0 e^{-\psi(z)/\sigma_z^2}$$

$$\psi_{\rm eq}(z) = 2\sigma_z^2 \ln\cosh\left(z/z_0\right),$$

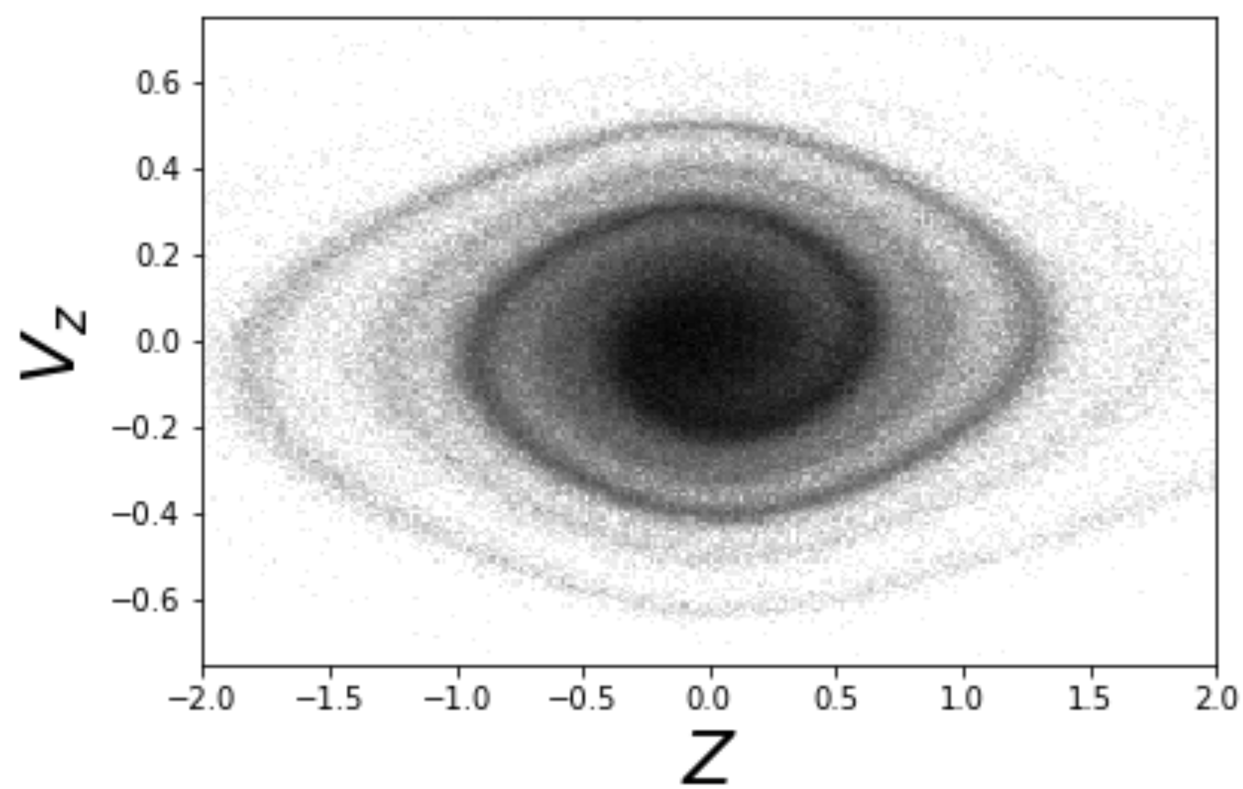
where $\rho_0 = \sigma_z^2/2\pi G z_0^2$.

Phase mixing of a 1D system in a fixed anharmonic potential

Equilibrium DF is perturbed by shifting in vz



Evolves via phase mixing

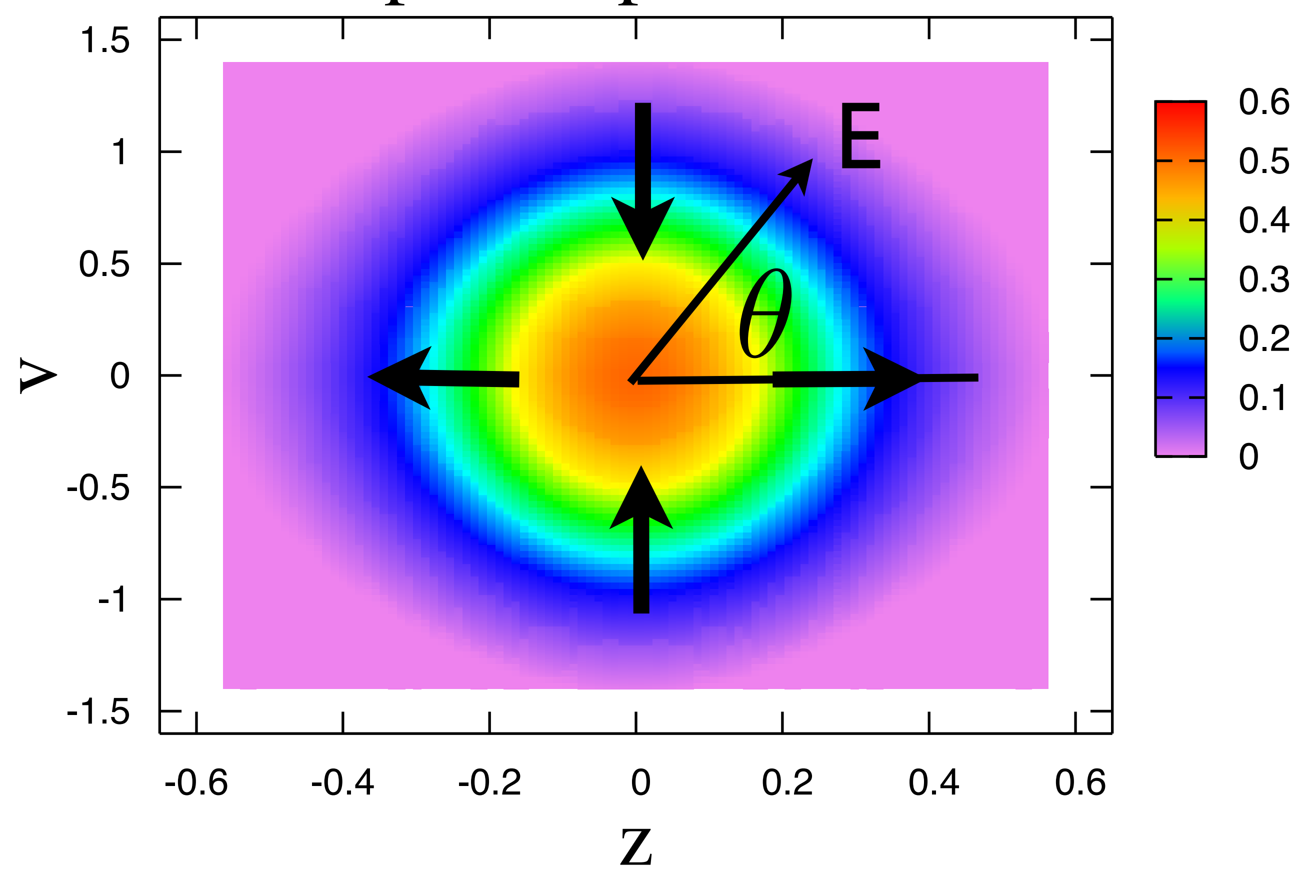


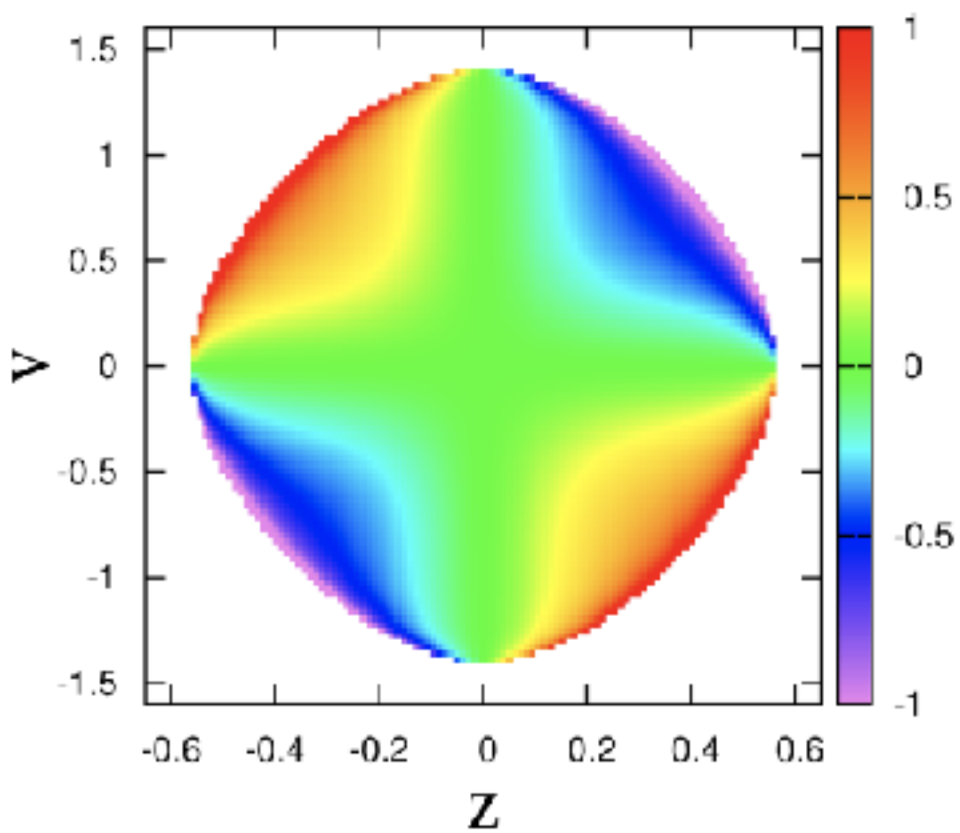
Self-gravitating 1D systems have discrete modes

Mathur 1990, Weinberg 1991, LW & Bonner 2015

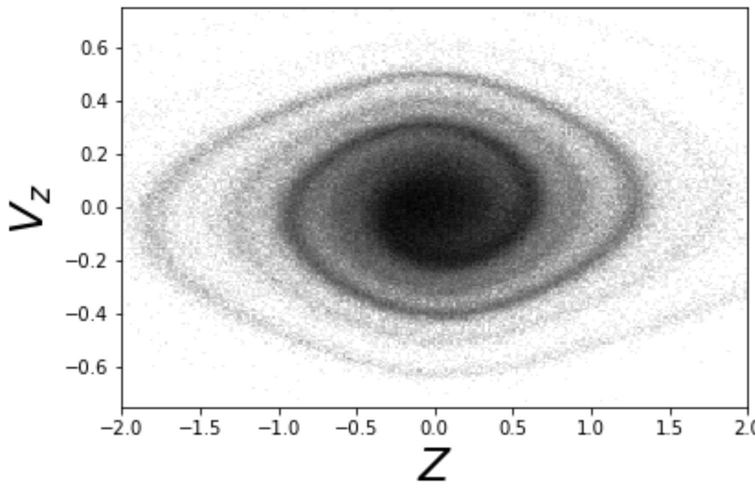
breathing mode

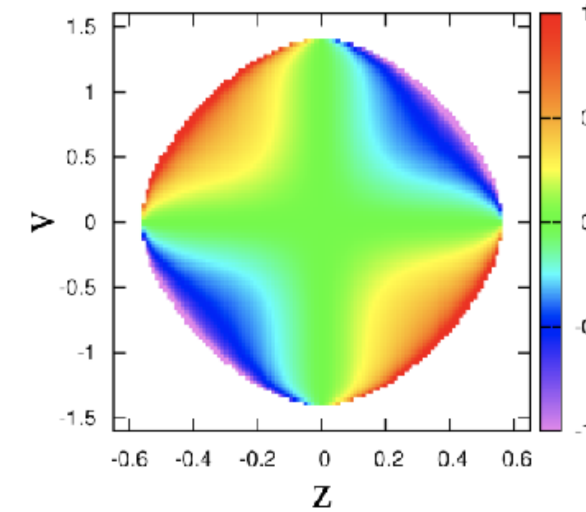
phase space DF





In general, we expect both phase mixing and modal oscillations to be relevant for dynamics

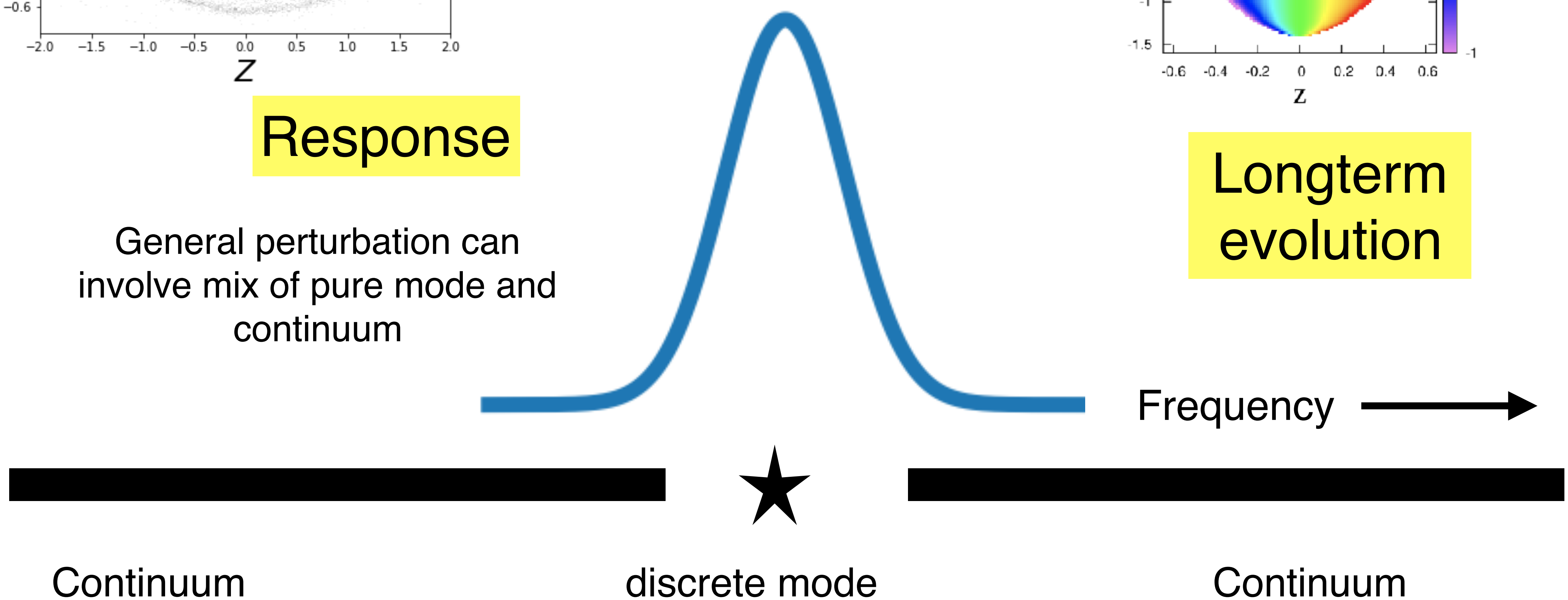




Response

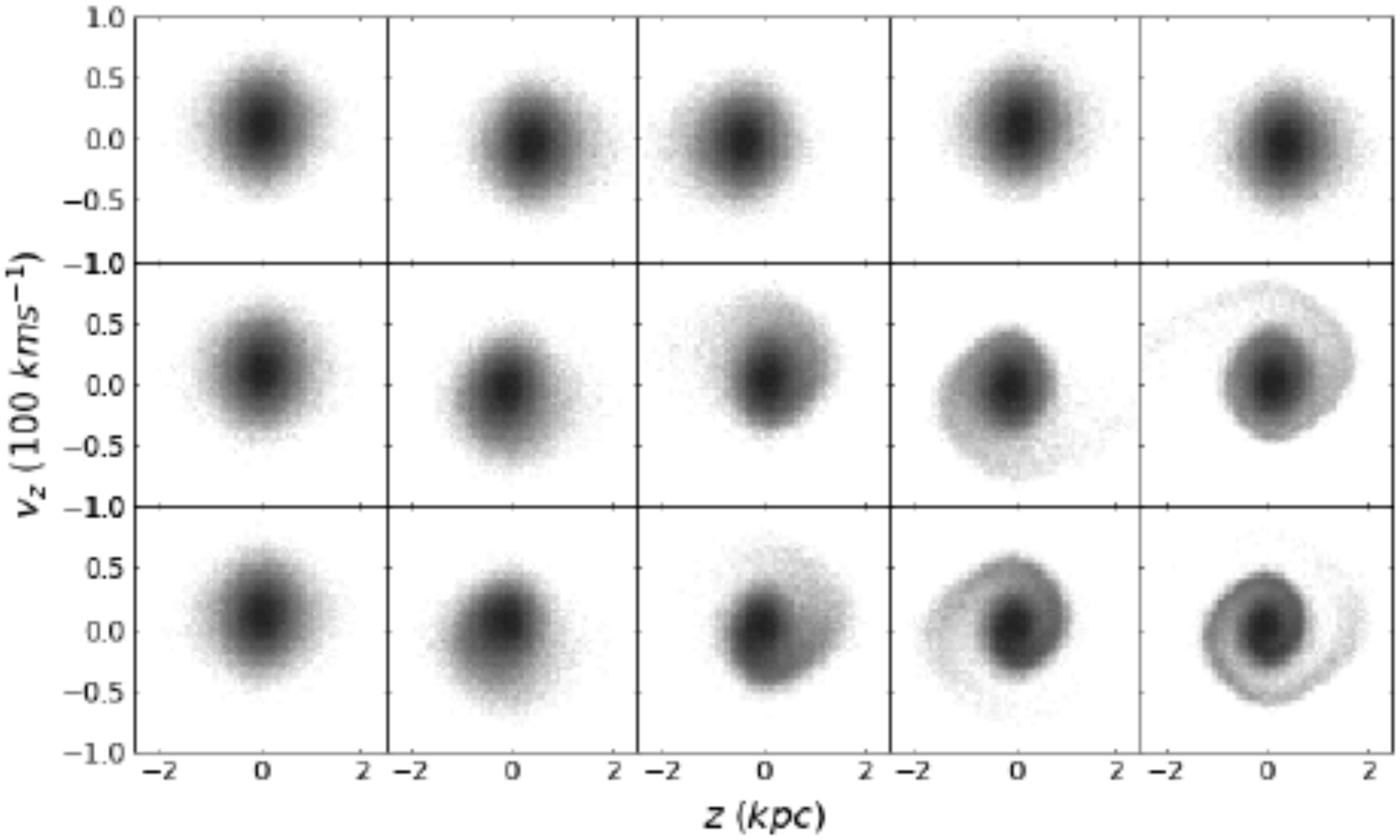
Longterm evolution

General perturbation can involve mix of pure mode and continuum



Explains why it is difficult to excite a pure mode in any realistic simulation (Weinberg 1991)

To explore the competition between phase mixing and self-gravity we consider a simple 1D (slab) model for stellar disk where we can tune the amount of self-gravity vs external potential



Self-gravitating

Test particles

Darling & LW 2019

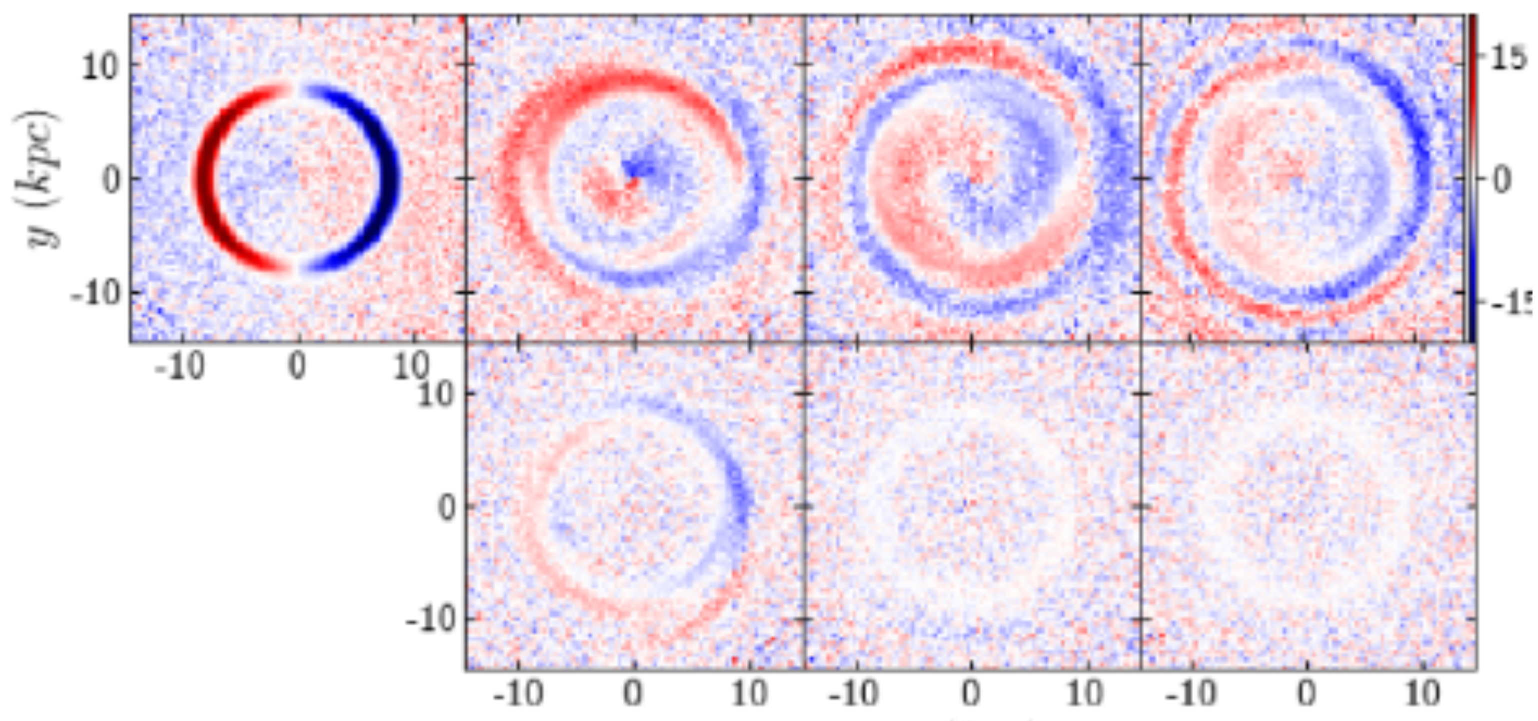
Importance of self-gravity

Perturb a disk by tilting a ring at these Solar circle

Use high-res, low mass particles in the ring to boost resolution where we want to probe phase space.

(This is somewhat dubious since particles mix in radius. However, the resolution in Gaia is 1 M_{sun}, which is well below what is possible in simulations.)

mid plane displacement across the disk

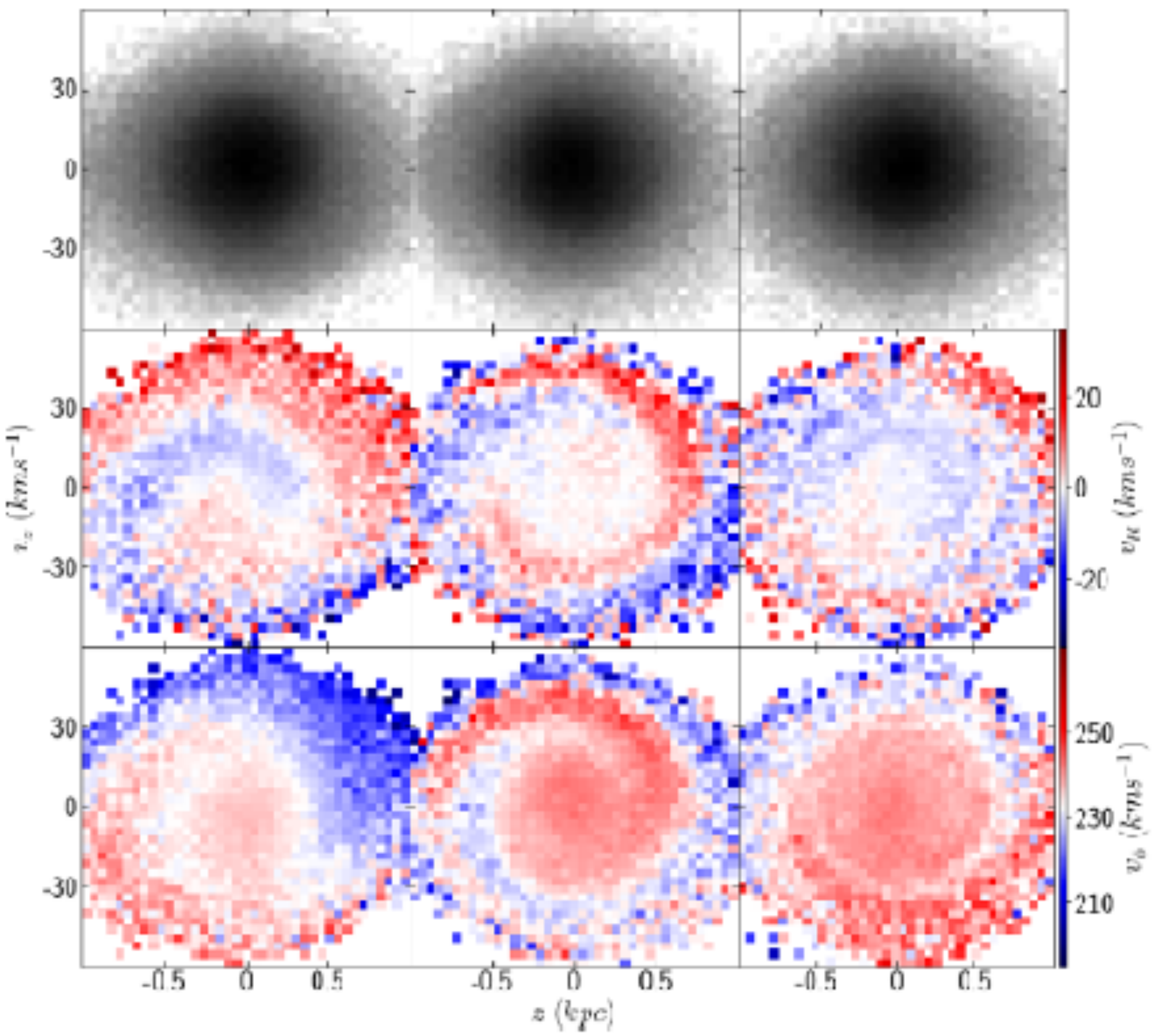


Fully live disk and halo
Includes self-gravity for perturbation

Test particles in static potential

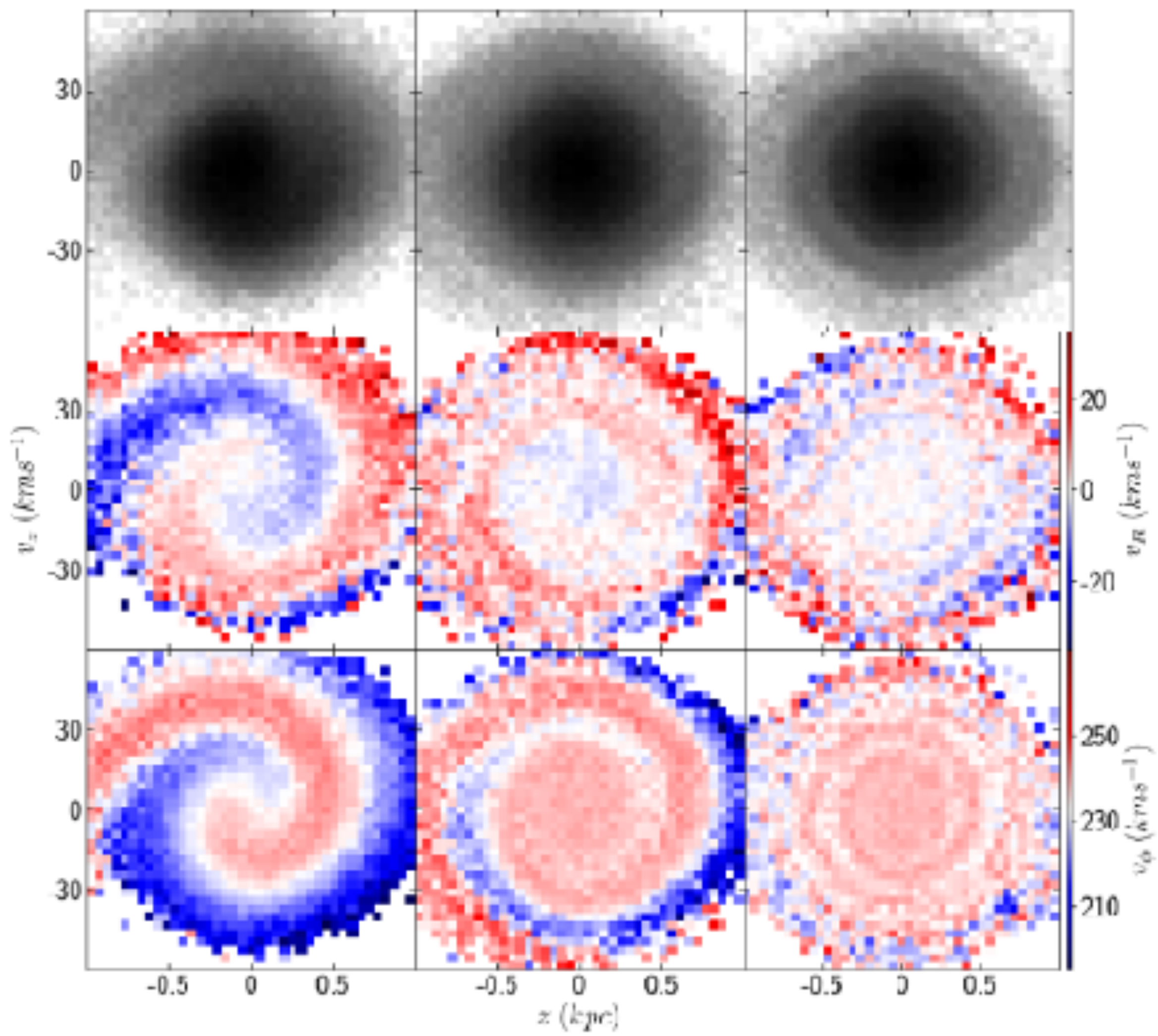
Darling & LW 2019

Live disk and halo



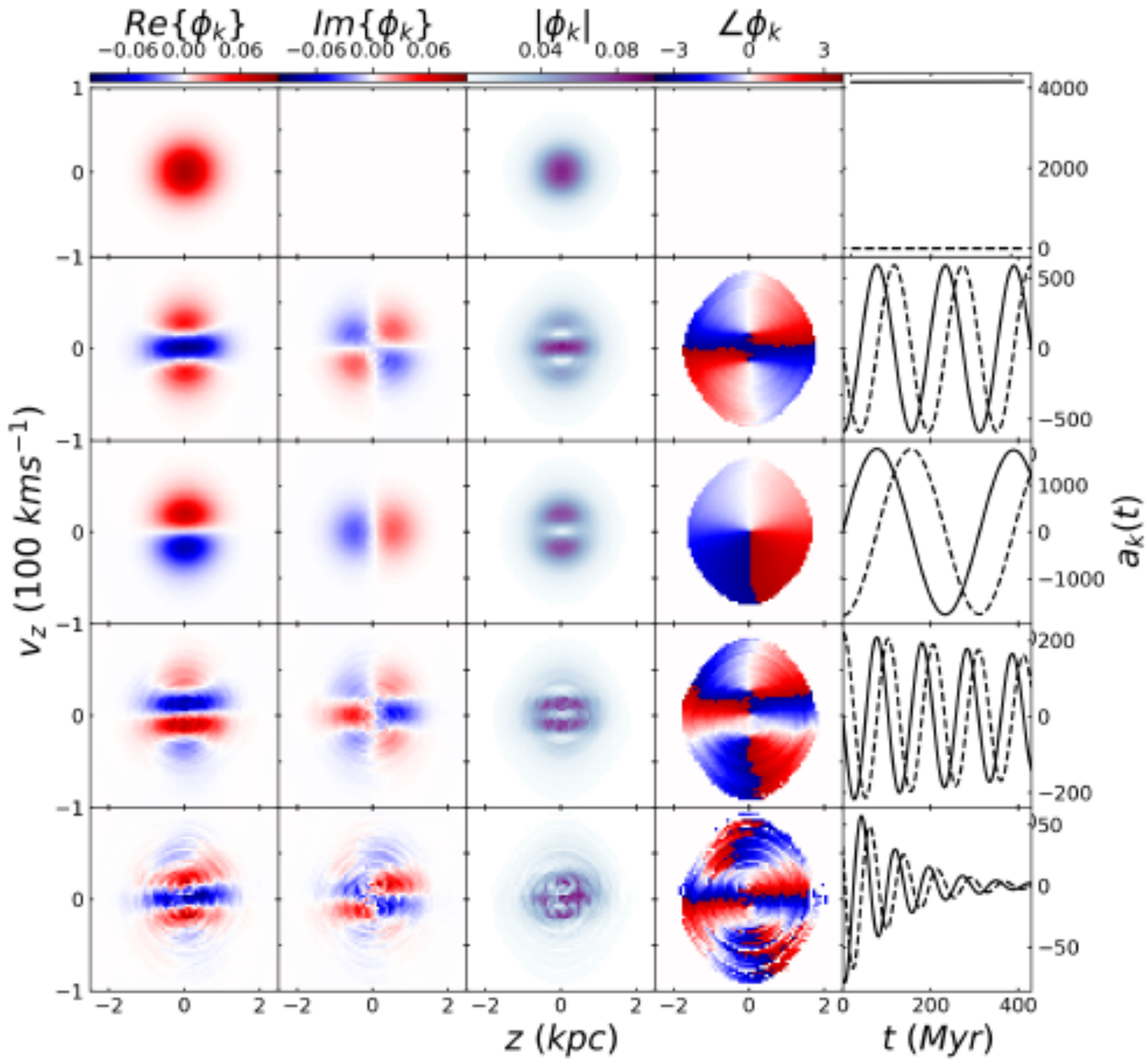
250 Myr 500Myr 1Gyr

Test particle simulation

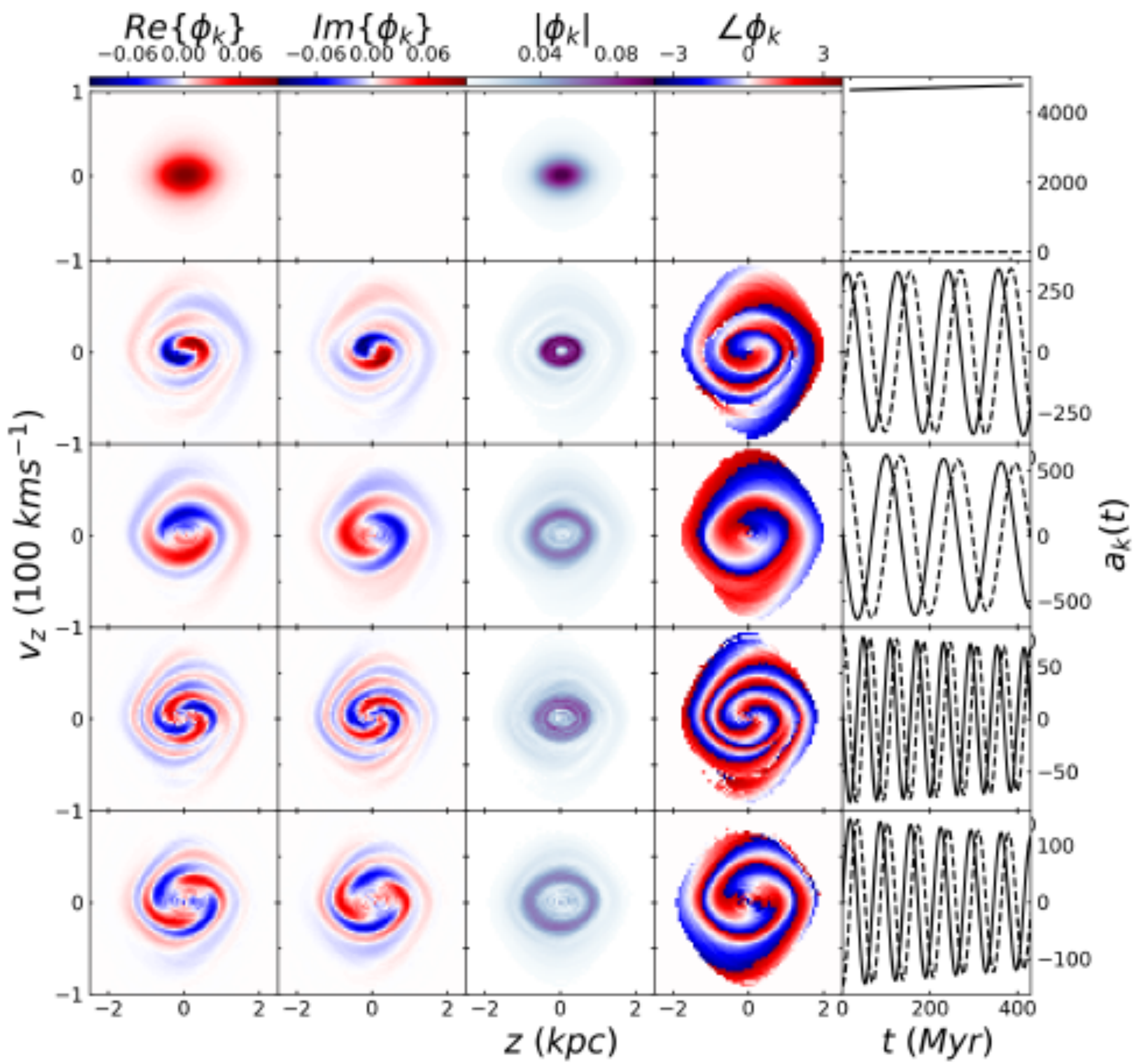


250 Myr 500Myr 1Gyr

First 5 DMD modes for a nearly self-gravitating disk



Dominant modes for nearly test-particle case



The obvious next step is to apply the method to 3D simulations of galactic dynamics.

The challenge: Choosing an appropriate set of observables that captures structures in the full 6D phase space.

An opportunity: Identify “modes” that connect in-plane and vertical motions and that may include different azimuthal *m*’s…”modes” that would not necessarily show up in a spectral (linear) decomposition.

The dream: Match observations (Gaia snapshot) to some template from simulations using DMD to understand past and future evolution and constrain the potential and DM distribution.