Modular Tensor Categories from Six-Dimensional Field Theories

G. Moore, Rutgers University

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What am I doing here???

Simulation of Topological Field Theories by Quantum Computers

Michael H. Freedman[1], **Alexei Kitaev**[1,⋆], **Zhenghan Wang**[2]

[1] Microsoft Research, One Microsoft Way, Redmond, WA 98052-6399, USA
[2] Indiana University, Dept. of Math., Bloomington, IN 47405, USA

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Abstract: Quantum computers will work by evolving a high tensor power of a small (e.g. two) dimensional Hilbert space by local gates, which can be implemented by applying a local Hamiltonian H for a time t. In contrast to this quantum engineering, the most abstract reaches of theoretical physics has spawned "topological models" having a finite dimensional internal state space with no natural tensor product structure and in which the evolution of the state is discrete, $H \equiv 0$. These are called topological quantum field theories (TQFTs). These exotic physical systems are proved to be efficiently simulated on a quantum computer. The conclusion is two-fold:

News from the most abstract and exotic reaches of theoretical physics

Today I will report on some general recent developments in those abstract reaches of theoretical physics which have spawned a good deal of activity and excitement in the past few years.

This does not come with a warranty that it will be useful for quantum computation or condensed matter – but I've chosen topics that share some points of contact with previous successful interactions.

I am surveying results by many different people, and my viewpoints have been influenced by many discussions with

Dan Freed,

Davide Gaiotto,

Andy Neitzke,

and Edward Witten

Sources of the developments

- Defects and boundary conditions in QFT, and their relation to (higher) categories

- Distinguished *six-dimensional interacting conformal field theories*

- Knot homology and BPS states

- Constructions of BPS states and 4d field theories related to duality groupoids and MTC's

Defects in Local QFT

Pseudo-definition: Defects are local disturbances supported on positive codimension submanifolds

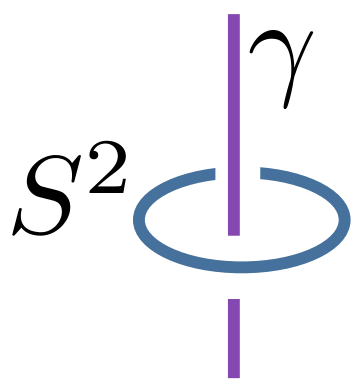
Example 1: d=0: Local Operators

Example 2: d=1: ``Line operators''

Wilson operators in gauge theory:

$$W(\gamma) = P \exp\left(\int_\gamma A\right)$$

't Hooft loops in 4d gauge theory



$$F \sim m \sin\theta d\theta d\phi + \cdots$$

Defects – II

- dim =2: Surface defects: Couple a 2-dimensional field theory to an ambient theory.

- etc

- codim = 2 Monodromy defects

- codim =1: Domain walls

N.B. A boundary condition (in space) in a theory T can be viewed as a domain wall between T and the empty theory. So the theory of defects subsumes the theory of boundary conditions (e.g. D-branes).

Multiplying Defects

Under some good conditions, defects can be multiplied – leading to generalized notions of the operator product expansion.

e.g. Supersymmetric line defects in GL-twisted N=4 SYM (Kapustin-Witten):

$$\mathrm{Tr}_{R_i} W(\gamma_1) \times \mathrm{Tr}_{R_j} W(\gamma_2) = \sum_k N_{ij}^k \mathrm{Tr}_{R_k} W(\gamma_2)$$

Sometimes the coefficients of OPE's of line defects can be graded vector spaces, whose graded traces satisfy Skein relations. (More on this below.)

Extended Field Theories

A key idea of the Atiyah-Segal definition of TFT is to encode the most basic aspects of locality in QFT. Many people have been extending this notion to include manifolds with corners and the division of submanifolds of higher codimension into pieces.

D. Freed; R. Lawrence; J. Baez + J. Dolan ; G. Segal; D. Freed, M. Hopkins, J. Lurie, C. Teleman; A. Kapustin; K. Walker

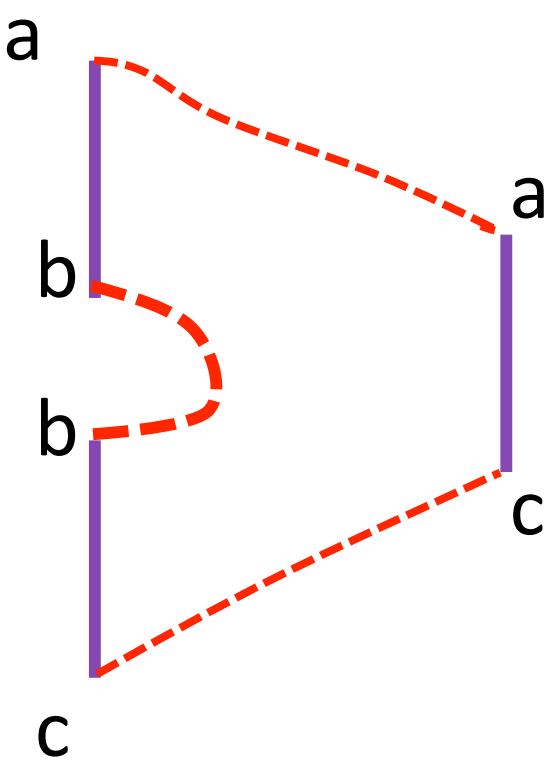
Example: 2-1-0 TFT:

$$F(M_2) \in \mathbb{C}$$ Partition Function

$$F(M_1) \in VECT$$ Hilbert Space

$$F(M_0) \in \mathrm{CAT}$$ Boundary conditions

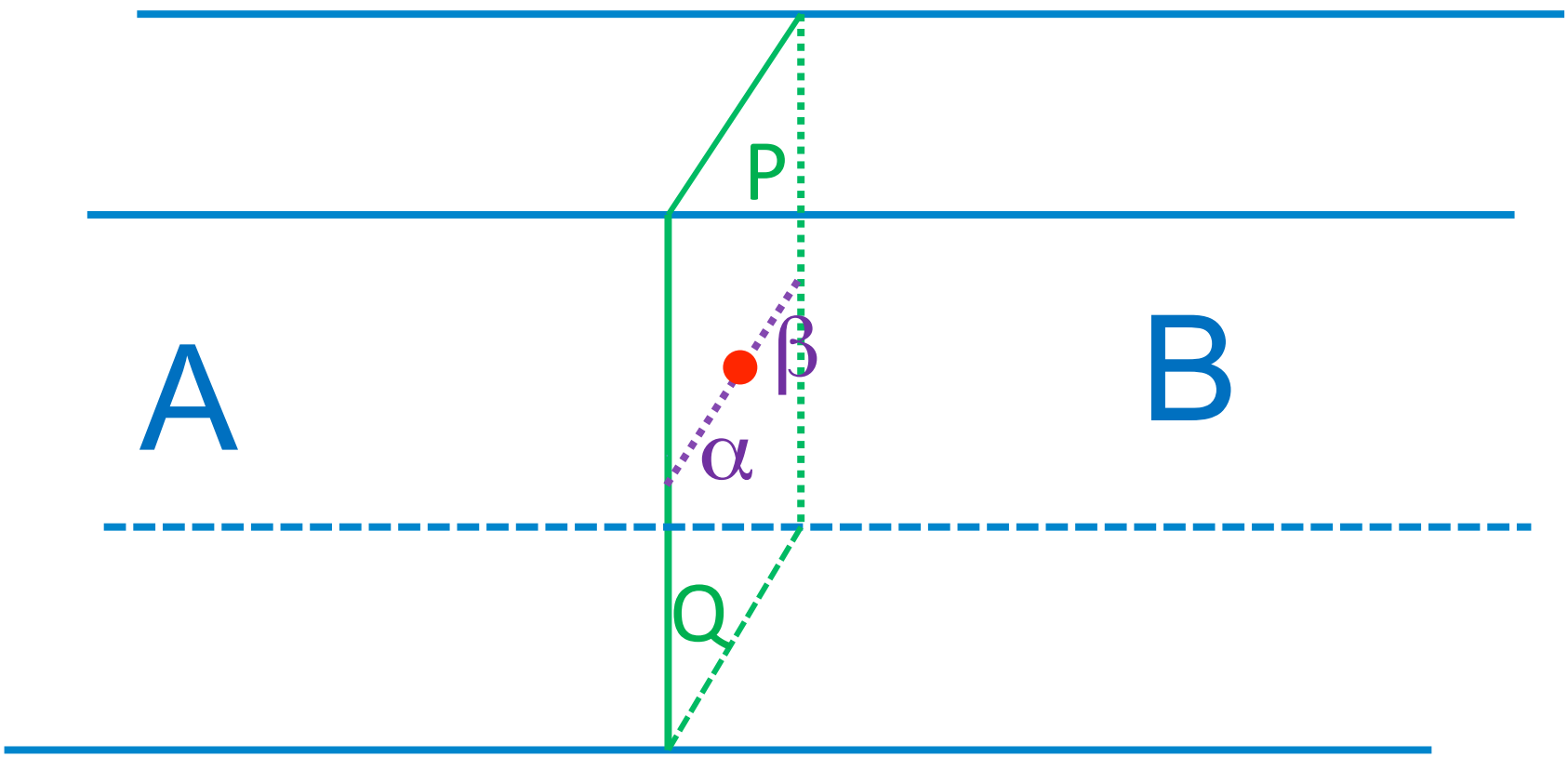
$$\mathcal{O}_{ab}$$



$$\mathcal{O}_{ab} \times \mathcal{O}_{bc} \to \mathcal{O}_{ac}$$

c.f. Moore & Segal, explained in this room, many times…

Defects Within Defects



Conclusion: Spatial boundary conditions in an n-dimensional ETFT are objects in an (n-1)-category:

k-morphism = (n-k-1)-dimensional defect in the boundary.

``KK reduction''

Given an n-dimensional ETFT F and a compact manifold C_d we can define an (n-d)-dimensional ETFT F_{KK}, called the ``Kaluza-Klein reduction along C_d'' by

$$F_{KK}(N_\ell) := F(N_\ell \times C_d) \qquad \ell = n-d, n-d-1, \ldots$$

Boundary conditions of F_{KK} form an (n-d-1)-category, so we have a hierarchy…

ETFT Hierarchy

$$F(M_n) \in \mathbb{C}$$ Partition Function

$$F(M_{n-1}) \in VECT$$ Hilbert Space

$$F(M_{n-2}) \in CAT$$ Boundary conditions

$$F(M_{n-k}) \in (k-1)CAT$$

One can show $F(M_{n-k})$ is the category of (k-1)-dimensional defects (c.f. Kapustin's ICM talk).

Field theories valued in field theories

DEFINITION: An m-dimensional field theory F valued in an n-dimensional field theory H, where n= m+d, is one such that

$$F(N_j) \in H(N_j) \qquad j= 0,1,\ldots , m$$

For example, if d=1 then the ``**partition function**'' of F on N_m is a ***vector in a vector space***, and not a complex number, and only if N_m is a boundary do we get a specific vector in that vector space.

An example of this is the chiral half of a RCFT, and we will meet another one below.

Knot Invariants & CSW

Recall that knot invariants such as the Jones polynomial are all associated with a 3-2-1 ETFT defined by Chern-Simons-Witten:

$$F_{csw}(M_3) \in \mathbb{C}$$ Reshetikhin-Turaev-Witten invariant

$$F_{csw}(M_2) \in VECT$$ Space of conformal blocks

$$F_{csw}(M_1) \in CAT$$ Modular tensor category (e.g. of integrable highest weight representations of level k current algebra)

and all this extends to include line defects.

Knot Categorification

Igor Frenkel asked: Is there an extended 4D field theory whose KK reduction is F_{csw} ?

e.g., compactification on S^1 → ``decategorification'' : Vector space → Euler character, Category → K-theory, …

In particular, for the Jones polynomial for a link L in R^3:

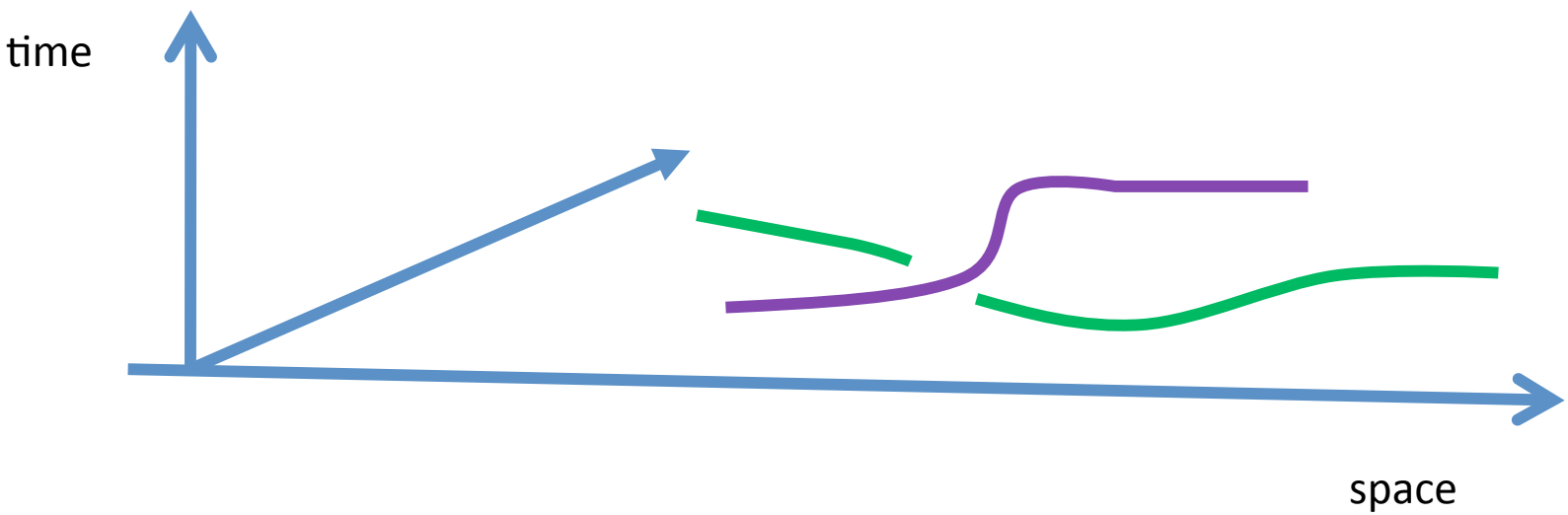
$$J(q;L) = \sum a_n q^n$$

The ``categorification'' would provide a (Z+Z graded) vector space

$$J(q;L) = \mathrm{Tr}_{\mathcal{H}(L)}(-1)^F q^P$$

Is it useful for quantum computation?

Difficulty 1: $\mathcal{H}$(L) is a space of states at fixed time:
Braiding is done along a spatial direction:



Difficulty 2: Unitarity?

Physical Answers

Following work in mathematics by M. Khovanov et. al. , physical interpretations of knot homology have been proposed by

- Gukov, Schwarz, and Vafa
- Gukov
- Witten

These proposals all center around an interesting set of six-dimensional superconformal field theories.

These are field theories valued in a higher dimensional field theory (with d=1) , and hence natural sources of categorification.

The six-dimensional theories

Claim, based on string theory constructions:

There is a family of stable field theories, T[N] with six-dimensional (2,0) superconformal symmetry, and certain characterizing properties. They are not free field theories for N>1. (Witten; Strominger; Seiberg).

These theories have not been constructed – even by physical standards - but the characterizing properties of these hypothetical theories can be deduced from their relation to string theory and M-theory.

These properties will be treated as axiomatic. Later they should be theorems.

Related free field theories with six-dimensional (2,0) superconformal symmetry can be rigorously constructed – although this has not been done in full generality, and itself presents some nontrivial issues in topology and physics.

T[N=1]: Abelian Theory

Fields: H, X^I, ψ^a $\quad I = 1, \ldots, 5,$ $a = 1, \ldots, 4$

$$\partial \cdot \partial X^I = 0 \qquad \gamma \cdot \partial \psi^a = 0$$

$$H = *H \quad dH = 0 \quad H \in \Omega^3(M_6)$$

Surface defects: $\mathbb{S}[\Sigma] = \exp[2\pi i \int_\Sigma B]$

These theories are defined on oriented spin 6-manifolds (with integral Wu-structure – Hopkins & Singer).

Six Characteristic Properties of T[N]

1. First, it is a field theory valued in a 7-dimensional TFT:
7-dimensional Chern-Simons field theory

$$S = N \int C dC$$

C is a 3-form gauge potential $\Omega^3(M_7)$

(More properly, we form a TFT using differential cohomology ….)

2. When T[N] is KK reduced on $R^{1,4} \times S^1$ where S^1 has radius R, with nonbounding, or Ramond spin structure, the long distance dynamics is governed by a (maximally supersymmetric) five-dimensional Yang-Mills theory with a gauge Lie algebra u(N) and coupling constant g^2_{YM} proportional to R.

3. The theory in Minkowski space has a moduli space of vacua given by $(\mathbb{R}^5 \otimes \mathbb{R}^N)/S_N$

Low energy dynamics is described by N free tensormultiplets and we view the space of vacua as parametrizing:

$$\langle X^{I,i} \rangle \quad I = 1, \ldots, 5 \quad i = 1, \ldots, N$$

4. There are dynamical string-like excitations around generic vacua which are simultaneously electric and magnetic sources for the free tensormultiplets H^i:

$$dH^i = \lambda^i \delta(W_2 \subset \mathbb{R}^6)$$

5. There are surface defects $\mathbb{S}[\mathcal{R},\mathbb{S}]$ associated to representations $\mathcal{R}$ of u(N). Far out on the moduli space they are well approximated by

$$\mathbb{S}[\mathcal{R}, \Sigma] \sim \sum_w \exp[2\pi i \int_\Sigma w \cdot B + \cdots]$$

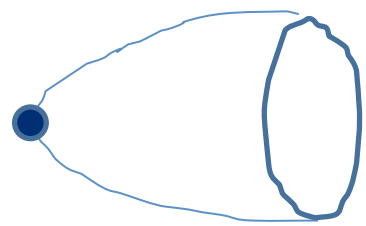
6. There are (mysterious) codimension two supersymmetric defects D. They have global symmetries G[D].

Witten Construction – I

Study T[N] on six-manifold: $M_6 = \mathbb{R} \times W \times D$

TIME

D: p



W: 3-manifold containing a surface defect L x {p}

More generally, the surface defect is supported on a link cobordism $L_1 \rightarrow L_2$:

Witten Construction – II

Now, KK reduce by U(1) isometry of the cigar D with fixed point p to obtain 5D SYM on $\mathbb{R} \times W \times \mathbb{R}_+$

TIME

Hilbert space of states depends on W and L.

Space of supersymmetric (BPS) states $\mathcal{H}_{BPS}(W, L)$

This space is constructed from a chain complex using infinite-dimensional Morse theory.

Reminder: Supersymmetry & Morse Theory

Susy particle moves on a manifold X with potential energy determined from a Morse function h: X → R

Configuration space: $\Omega^*(X)$

Supersymmetric operators: $Q = e^h d e^{-h}$

Get one semiclassical groundstate for each critical point $|\varphi_a\rangle$

Instantons are gradient flows $|\varphi_a\rangle \to |\varphi_b\rangle$

defining a differential on the Morse-Smale-Witten complex.

Witten Construction III

Semiclassical BPS states are solutions (A,f) to

$$F - \phi^2 + *d_A\phi = 0$$

$$d_A * \phi = 0$$

Here A is a u(N) gauge field and f is a 1-form valued in NxN Hermitian matrices, both are defined on the 4-manifold: V = W x R_+

The boundary conditions at y=0 are determined by the link L.

Claim: With suitable boundary conditions there are isolated solutions.

Then similar 5D equations describe the instantons connecting these solutions. In this way we get a $Z + Z$ -graded complex.

One of the Z-gradings is provided by instanton number.

Witten Construction IV: Connecting to the Jones Polynomial

However, these equations are ALSO the equations for BPS configurations of $\mathcal{N}$=4 u(N) SYM on V = W x R_+. BC's encode an 't Hooft line defect inserted in W at y=0. So the Euler character of the space of BPS states is also the $\mathcal{N}$=4 partition function with an 't Hooft loop.

Now, invoking S-duality, that $\mathcal{N}$=4 SYM partition function is also the partition function of the u(N) theory with Wilson loop operators along L at y=0.

Finally, by a previous Witten paper on analytic continuation of Chern-Simons:

Path integral of $\mathcal{N}$=4 SYM on V = W x R_+.	=	Path integral of Chern-Simons on W (with ``level'' determined by Yang-Mills coupling and twisting)

Witten Construction: Conclusion

Therefore, the path integral of the theory T[N] on $S^1 \times R^3 \times D$ with a surface defect at $S^1 \times L \times p$ is equal to the Jones polynomial.

But, by construction, it is also a graded trace on a Hilbert space of ``***BPS states***,'' moreover: that Hilbert space is the cohomology of a (Morse-Smale-Witten) chain complex, defined by 4 and 5-dimensional PDEs.

So, Witten's answer to Frenkel's question involves the ***BPS states*** in the 6-dimensional theory T[N].

Theories of class S

In some roughly orthogonal developments, GMN studied T[N] theories on $M_6 = M_4 \times C$

Here C is a punctured Riemann surface with special codimension two defects inserted at the punctures z_a.

Kaluza-Klein reduction on C produces an interesting ***four-dimensional*** theory $T[N,C,D_a]$:

1. The theory only depends on the conformal structure of C.

2. The theory has global symmetry group $\Pi_a G_a$

3. The BPS states, and supersymmetric line and surface defects all have elegant descriptions in terms of the geometry of the Riemann surface C and certain coverings of C.

Dictionary for u(2)

1. Space of UV gauge couplings = Moduli space of complex structures on C

2. Moduli space of IR quantum vacua is the family of holomorphic quadratic differentials:

$$\phi \sim \left(\frac{m_a^2}{(z-z_a)^2} + \cdots \right) (dz)^2$$

3. Seiberg-Witten curve is a 2:1 covering of C defined by the subspace of T^*C :

$$\lambda^2 = \phi$$

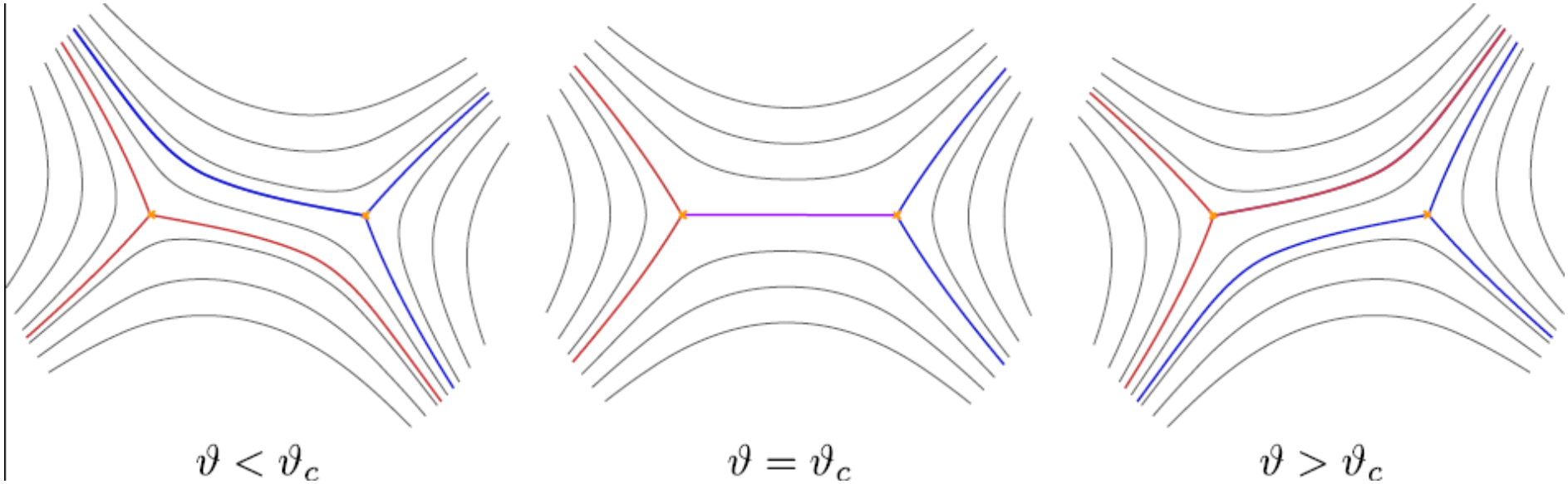
λ = canonical differential pdq in T^*C

BPS States

There is a very geometric description of BPS states defined by foliations of C from integral curves of l:

$$\langle \partial_t, \lambda \rangle = e^{i\vartheta}$$

BPS states are closed or ``finite'' curves connecting branch points of the double cover. They only occur for critical values of the phase ϑ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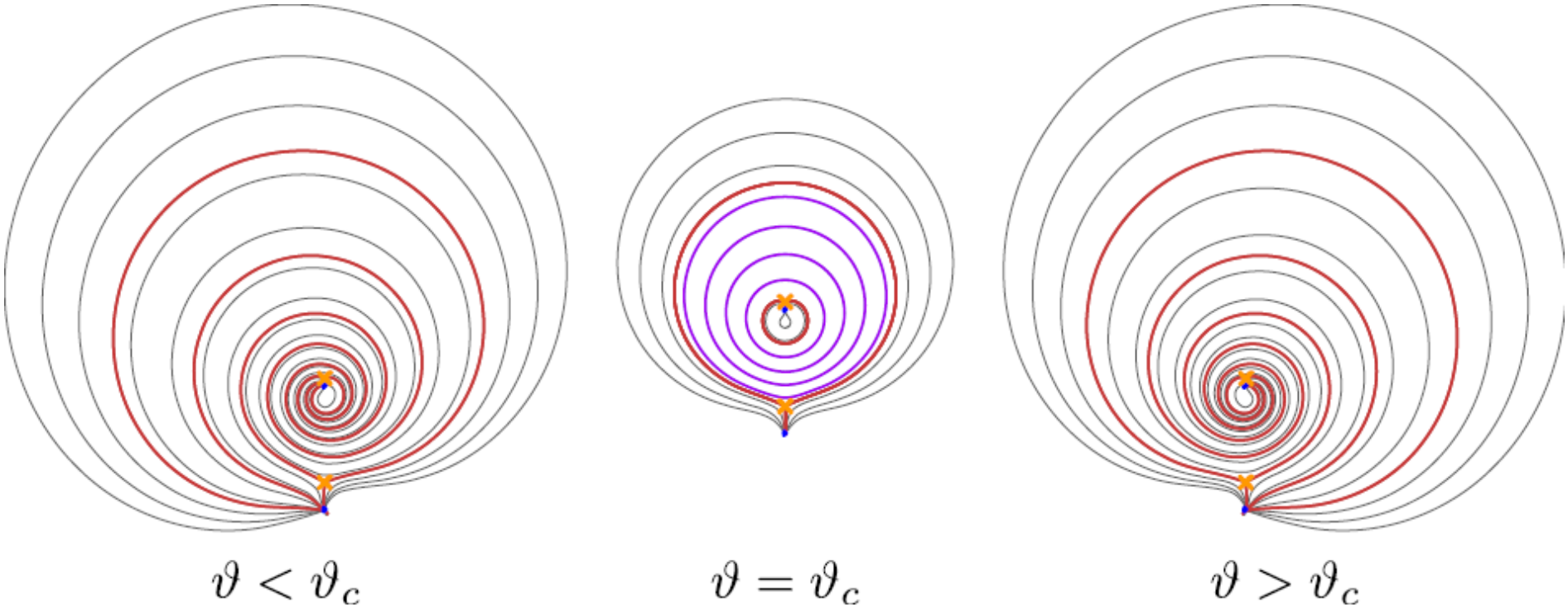
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BPS States with line defects

Looking at analogous BPS states associated with line defects leads to a ``categorification'' of the algebra of Wilson loop operators acting on statespaces H(C) in a MTC:

To a homotopy class $\wp \subset C$ and a representation $\mathcal{R}$

$$\Rightarrow \quad \mathcal{O}(\wp, \mathcal{R}) \in End(H(C))$$

$$\mathcal{O}(\wp_i)\mathcal{O}(\wp_j) = \sum_k c_{ij}^k(q)\mathcal{O}(\wp_k)$$ ``Skein relations''

$$c_{ij}^k(q) = \mathrm{Tr}_{V_{ij}^k}(-1)^F q^P$$

BPS States with Surface Defects

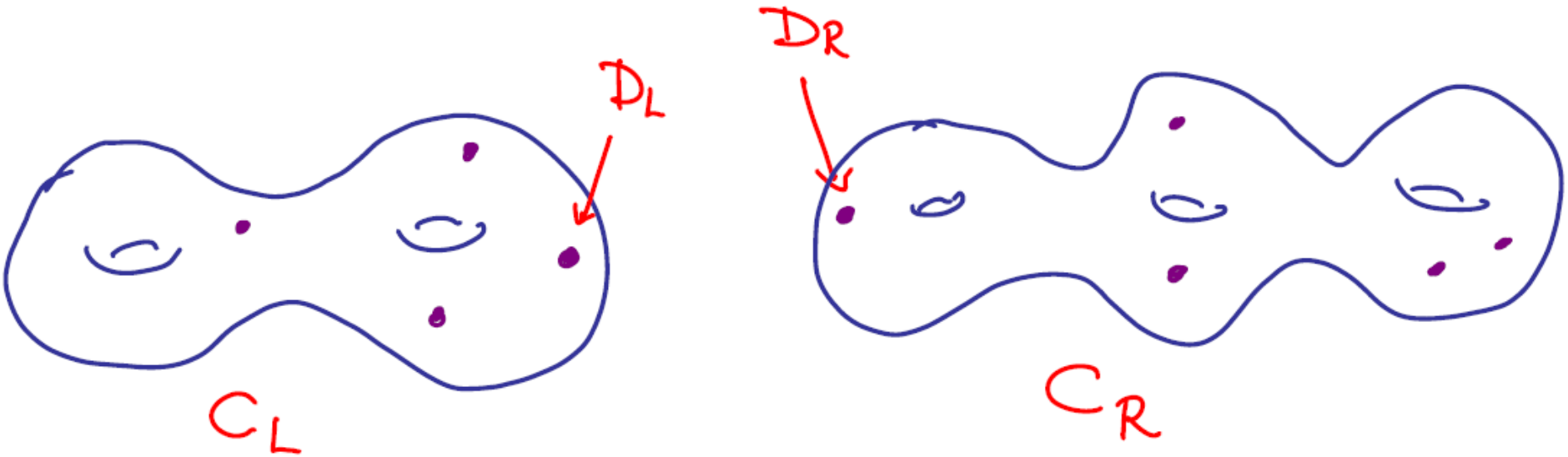
Studying analogous BPS states in the presence of surface defects, and line defect interfaces between these surface defects, appears to be a very promising way to find a categorification of knot invariants on $C \times R$

(Discussions in progress with Gaiotto, Neitzke, and Witten.)

Gaiotto Decompositions – I

A final very interesting connection to MTC's is suggested by a beautiful paper of D. Gaiotto.

Key idea: Gauging = Gluing

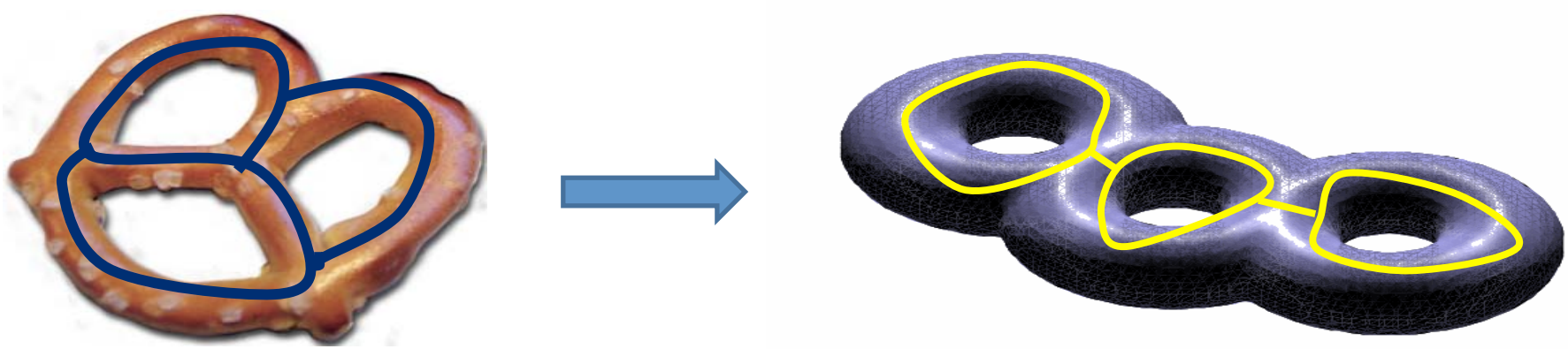


Gauging the diagonal $G \subset G_L \times G_R$ symmetry with $q = e^{2\pi i\tau}$ produces the theory with glued surface with $z_L z_R = q$

Gaiotto Decompositions II

Recall the duality groupoid of C :

We consider decompositions into 2g-2 + n trinions together with an embedded trivalent graph: These label asymptotic regions of $\mathrm{Teich}_{g,n}[C]$



Morphisms are given by basic B,F,S moves.

Gaiotto Decompositions – III

Pants decompositions → Presentation of T[N,C,m] in a weak-coupling region. (e.g. as a specific Lagrangian field theory)

Edges of the duality groupoid → ``S-dualities''

Speculation

Roughly speaking, there is a generalization of a MTC where

Objects → Special codimension 2 defects in T[N] with global symmetries

Spaces of conformal blocks → four-dimensional quantum field theories

Happy Birthday Mike!