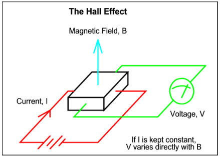
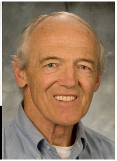
From topological order to long-range entanglement

Xiao-Gang Wen, Perimeter/MIT, Oct. 2012

In 1980's → there are phases beyond symmetry-breaking

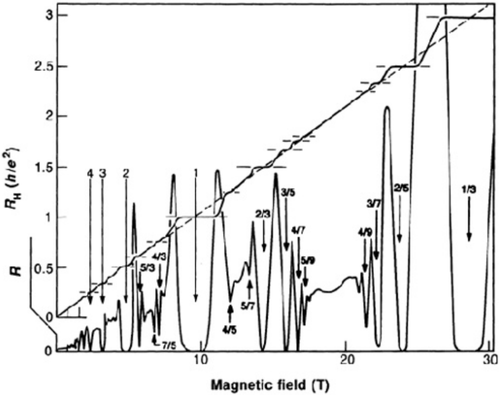




$E_y = R_H j_x, \quad R_H = \frac{p}{q}\frac{h}{e^2}$

- 2D electron gas in magnetic field has many **quantum Hall (QH) states**



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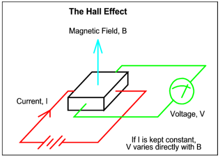
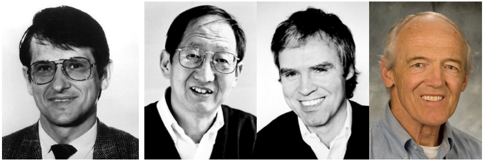
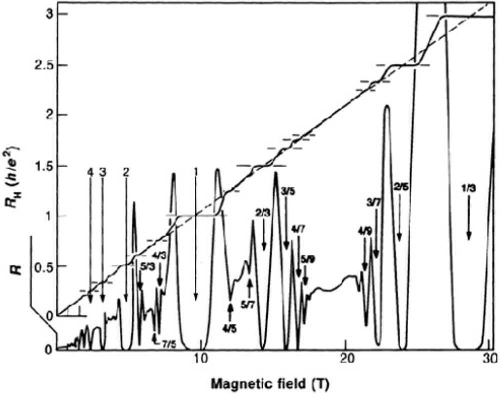


$E_y = R_H j_x, \; R_H = \frac{p}{q}\frac{h}{e^2}$

- 2D electron gas in magnetic field has many **quantum Hall (QH) states** that all have the **same symmetry**.
- Different QH states can-not be described by symmetry breaking theory.
- We call the new order **topological order** Wen 89



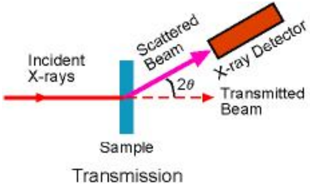


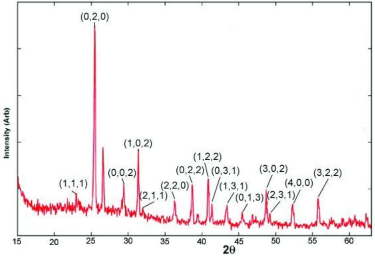
What is topological order? (What is spin liquid?)

To define a physical concept, such as symmetry-breaking order or topological order, is to design a probe to measure it

For example,

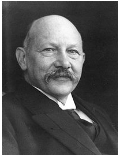
- crystal order is defined/probed by X-ray diffraction:

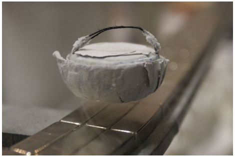




Symmetry-breaking orders through experiments

Order	Experiment
Crystal order	X-ray diffraction
Ferromagnetic order	Magnetization
Anti-ferromagnetic order	Neutron scattering
Superconducting order	Zero-resistance & Meissner effect
Topological order	???



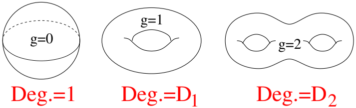


- All the above probes are linear responses. But topological order cannot be probed/defined through linear responses.

Topological orders through experiments (1990)

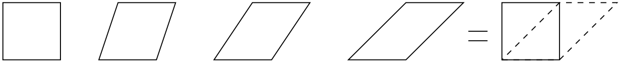
Topological order can be defined "experimentally" through two unusual topological probes (at least in 2D)

(1) **Topology-dependent ground state degeneracy** D_g Wen 89



(2) **Non-Abelian geometric's phases** of the degenerate ground state from deforming the torus: Wen 90

- Shear deformation T: $|\Psi_\alpha\rangle \to |\Psi'_\alpha\rangle = T_{\alpha\beta}|\Psi_\beta\rangle$



- 90° rotation S: $|\Psi_\alpha\rangle \to |\Psi''_\alpha\rangle = S_{\alpha\beta}|\Psi_\beta\rangle$

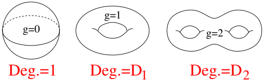
- T, S, define topological order "experimentally".
- T, S is a *universal probe* for any 2D topological orders, just like X-ray is a universal probe for any crystal orders.

Symmetry-breaking/topological orders through experiments

Order	Experiment
Crystal order	X-ray diffraction
Ferromagnetic order	Magnetization
Anti-ferromagnetic order	Neutron scattering
Superconducting order	Zero-resistance & Meissner effect
Topological order (Global dancing pattern)	Topological degeneracy, non-Abelian geometric phase

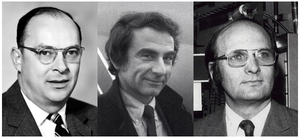
- The linear-response probe **Zero-resistance** and **Meissner effect** define **superconducting order**. Treating the EM fields as non-dynamical fields
- The topological probe **Topological degeneracy** and **non-Abelian geometric phases** T, S define a completely new class of order – **topologically order**.
- T, S determines the quasiparticle statistics. Keski-Vakkuri & Wen 93; Zhang-Grover-Turner-Oshikawa-Vishwanath 12; Cincio-Vidal 12

What is the microscopic picture of topological order?



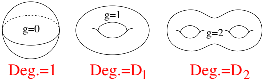
represent an experimental definition of topological order.

- But what is the microscopic understanding of topological order?
- Zero-resistance and Meissner effect $\to$ experimental definition of superconducting order.
- It took 40 years to gain a microscopic picture of superconducting order: **electron-pair condensation**



Bardeen-Cooper-Schrieffer 57

What is the microscopic picture of topological order?



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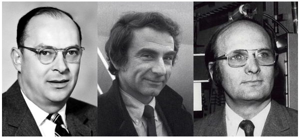

- It took 40 years to gain a microscopic picture of superconducting order: **electron-pair condensation**

Bardeen-Cooper-Schrieffer 57

- It took 20 years to gain a microscopic understanding of topological order: **long-range entanglements** Chen-Gu-Wen 10 (defined by local unitary trans. and motivated by topological entanglement entropy). Kitaev-Preskill 06,Levin-Wen 06

Pattern of long-range entanglements = topological order

For gapped systems with no symmetry:

- According to Landau theory, no symmetry to break
  $\rightarrow$ all systems belong to one trivial phase

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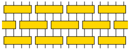
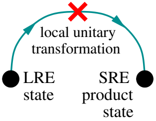
- Thinking about entanglement: Chen-Gu-Wen 2010
  - There are **long range entangled (LRE) states**
  - There are **short range entangled (SRE) states**

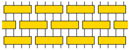
$|\text{LRE}\rangle \neq$  $|\text{product state}\rangle = |\text{SRE}\rangle$

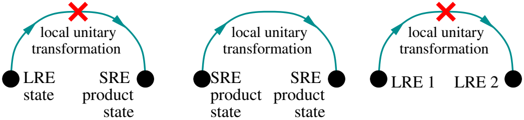


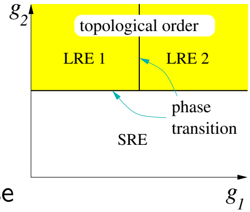
Pattern of long-range entanglements = topological order

For gapped systems with no symmetry:

- According to Landau theory, no symmetry to break
  $\to$ all systems belong to one trivial phase
- Thinking about entanglement: Chen-Gu-Wen 2010
  - There are **long range entangled (LRE) states** $\to$ many phases
  - There are **short range entangled (SRE) states** $\to$ one phase

$|\text{LRE}\rangle \neq$  $|\text{product state}\rangle = |\text{SRE}\rangle$





- All SRE states belong to the same trivial phase
- LRE states can belong to many different phases
  = different **patterns of long-range entanglements** defined by the LU trans.
  = different **topological orders**
  $\to$ A classification by **tensor category theory** Levin-Wen 05, Chen-Gu-Wen 2010

How to SEE topological order?

- Topological order is a property or a *pattern* in the ground state wave function

$$\Phi(x_1, x_2, ..., x_N), \qquad N \sim 10^{10} - 10^{23}$$

But how to see a pattern in a wave function that we cannot even write down?

- Symmetry breaking order is also a *pattern* in the ground state wave function, where we examine if the wave function is invariant under symmetry operation U or not:

$$U[\Phi(x_1, x_2, ..., x_N)] = ?= \Phi(x_1, x_2, ..., x_N)$$

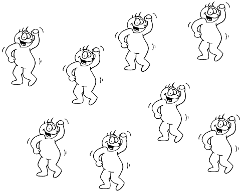
$\to$ pattern of symmetry breaking.

- *Use dancing picture to understand the pattern of topological order and pattern of symmetry breaking order.*

See symmetry breaking orders through pictures

Ferromagnet

Anti-ferromagnet



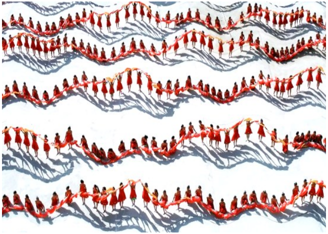
Superfluid of bosons

Superconductor of fermions

- every spin/particle is doing its own dancing,
  every spin/particle is doing the same dancing $\rightarrow$ **Long-range order**

Topological orders through pictures

FQH state



String liquid (spin liquid)

- **Global dance**:

  All spins/particles dance following a local dancing "rules"

  → The spins/particles dance collectively

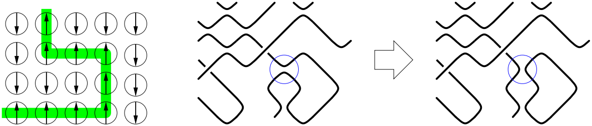
  → a global dancing pattern.

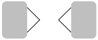
Local dancing rule $\to$ global dancing pattern

- Local dancing rules of a FQH liquid:
  (1) every electron dances around clock-wise
  (Φ_{FQH} only depends on $z = x + \mathrm{i}\,y$)
  (2) takes exactly three steps to go around any others
  (Φ_{FQH}'s phase change 6π)
  $\to$ Global dancing pattern $\Phi_{\text{FQH}}(\{z_1, ..., z_N\}) = \prod(z_i - z_j)^3$

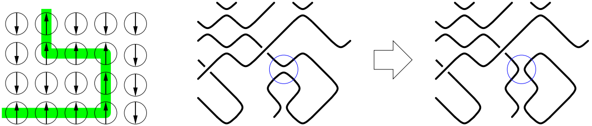
- Local dancing rules are enforce by the Hamiltonian
  to lower the ground state energy.

Local dancing rule → global dancing pattern



- Local dancing rules of a string liquid:
  (1) Dance while holding hands (no open ends)
  (2) $\Phi_{\text{str}}\Big($  $\Big) = \Phi_{\text{str}}\Big($  $\Big)$, $\Phi_{\text{str}}\Big($  $\Big) = \Phi_{\text{str}}\Big($  $\Big)$
  → Global dancing pattern $\Phi_{\text{str}}\Big($  $\Big) = 1$

Local dancing rule → global dancing pattern



- Local dancing rules of a string liquid:
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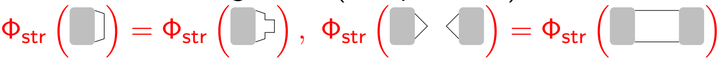
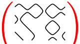
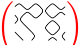
  (2) 
  → Global dancing pattern Φ_{str} () $= 1$

- Local dancing rules of another string liquid:
  (1) Dance while holding hands (no open ends)
  (2) 
  → Global dancing pattern Φ_{str} () $= (-)^{\#\text{ of loops}}$

- Two string-net condensations → two topological orders Levin-Wen 05

What is the significance of topological order?

Global dancing pattern is a nice picture for topological order.
But does it mean anything?
Does topological order have any experimental significance?
Does topological order have any new experimental properties, that is different from any symmetry breaking order?

How to measure/study topological order in experiments?

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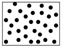
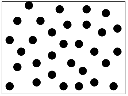
How to measure/study topological order in experiments?

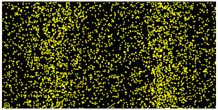
- Topological orders produce **new kind of waves** (collective excitations above the topo. ordered ground states).
  → *change our view of universe*
- The defects of topological order carry **fractional statistics** (including non-Abelian statistics) and **fractional charges** (if there is symmetry).
  → *a medium for topological quantum memory and computations.*
- Some topological orders have topologically protected **gapless boundary excitations**
  → *perfect conducting surfaces despite the insulating bulk.*

Topological order (closed oriented strings) $\to$ emergence of electromagnetic waves (photons)

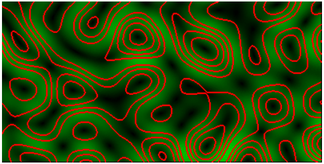
- Wave in superfluid state $|\Phi_{\text{SF}}\rangle = \sum_{\text{all position conf.}}$ |  ⟩:

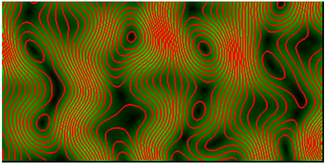


density fluctuations:
Euler eq.: $\partial_t^2 \rho - \partial_x^2 \rho = 0$
$\to$ Longitudinal wave



- Wave in closed-string liquid $|\Phi_{\text{string}}\rangle = \sum_{\text{closed strings}}$ |  ⟩:

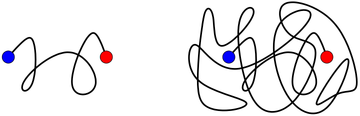




String density $\mathbf{E}(\mathbf{x})$ fluctuations $\to$ waves in string condensed state.
Strings have no ends $\to$ $\partial \cdot \mathbf{E} = 0$ $\to$ **only two transverse modes**.
$\to$ Maxwell eq.: $\dot{\mathbf{E}} - \partial \times \mathbf{B} = \dot{\mathbf{B}} + \partial \times \mathbf{E} = \partial \cdot \mathbf{B} = \partial \cdot \mathbf{E} = 0.$

Foerster-Nielsen-Ninomiya 80; Wen 02; Senthil-Motrunich 02; Hermele-Fisher-Balents 04;...

Topological order →
Emergence of electrons (fermions, and even anyons)



- In string condensed states, the ends of string be have like point particles
  - with quantized (gauge) charges
  - with Fermi statistics

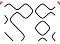
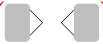
Levin-Wen 2003

- **String-net/topological-order provides a way to unify gauge interactions and Fermi statistics in 3D**

Emergence of fractional spin/statistics (from the local dancing rules)

- Why end of string carry spin-1/2 and Fermi statistics?

  Levin-Wen 05;Fidkowski-Freedman-Nayak-Walker-Wang 06

- $\Phi_{str}\Big($  $\Big) = 1$ string liquid $\Phi_{str}\Big($  $\Big) = \Phi_{str}\Big($  $\Big)$

  360° rotation:  $\rightarrow$  and  $=$  $\rightarrow$ 

   $+$  has a spin 0 mod 1.  $-$  has a spin $1/2$ mod 1.

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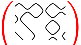
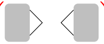
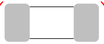
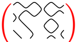
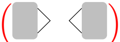
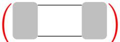
- Φ_{str} () $= 1$ string liquid Φ_{str} () $= \Phi_{\text{str}}$ ()

  360° rotation: $\rightarrow$ and $=$ $\rightarrow$

  $+$ has a spin 0 mod 1. $-$ has a spin $1/2$ mod 1.

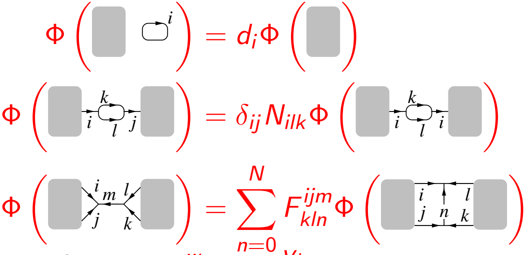
- Φ_{str} () $= (-)^{\#\text{ of loops}}$ string liquid Φ_{str} () $= -\Phi_{\text{str}}$ ()

  360° rotation: $\rightarrow$ and $= -$ $\rightarrow -$

  $+ \, i$ has a spin $-1/4$ mod 1. $- \, i$ has a spin $1/4$ mod 1.

More general dancing rules → Tensor category theory

The local dancing rules can be described by data $d_i, N_{ijk}, F^{ijm}_{kln}$:



Levin-Wen 05

which must satisfy $F^{ijk}_{j^* i^* 0} = \frac{v_k}{v_i v_j} N_{ijk}, \quad v_i^2 = d_i$

$$F^{ijm}_{kln} = F^{lkm^*}_{jin} = F^{jim}_{lkn^*} = F^{imj}_{k^* nl} \frac{v_m v_n}{v_j v_l}$$

$$\sum_{n=0}^{N} F^{mlq}_{kpn} F^{jip}_{mns} F^{jsn}_{lkr} = F^{jip}_{qkr} F^{riq}_{mls}$$

The theory about the solutions = tensor category theory

→ classify 2D gapped phases with no symmetry (topological order)

Gapped phases w/ symmetry → SET and SPT phases

- there are LRE symmetric states → Symm. Enriched Topo. phases
  - 100s symm. spin liquid through the PSG of topo. excit. Wen 02
  - 8 trans. symm. enriched Z_2 topo. order in 2D, 256 in 3D Kou-Wen 09
  - $1000, 000$s symm. Z_2 spin liquid through $[\mathcal{H}^2(SG, Z_2)]^2 \times$ Hermele 12
  - Classify SET phases through $\mathcal{H}^3[SG \times GG, U(1)]$ Ran 12
- there are SRE symmetric states → one phase

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- there are LRE symmetric states → Symm. Enriched Topo. phases
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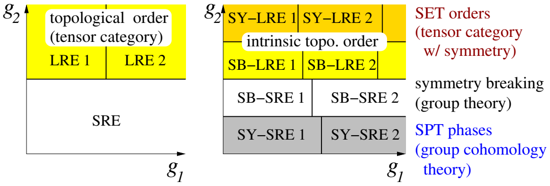
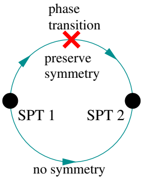
  - $1000,000$s symm. Z_2 spin liquid through $[\mathcal{H}^2(SG, Z_2)]^2 \times$ Hermele 12
  - Classify SET phases through $\mathcal{H}^3[SG \times GG, U(1)]$ Ran 12
- there are SRE symmetric states → many different phases

We may call them **symmetry protected trivial (SPT)** phase

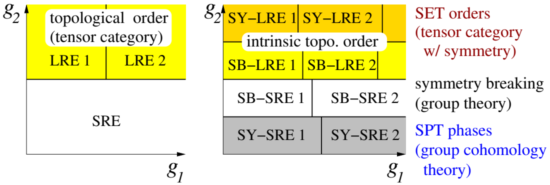


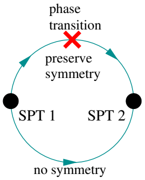


Gapped phases w/ symmetry → SET and SPT phases

- there are LRE symmetric states → Symm. Enriched Topo. phases
  - 100s symm. spin liquid through the PSG of topo. excit. Wen 02
  - 8 trans. symm. enriched Z_2 topo. order in 2D, 256 in 3D Kou-Wen 09
  - 1000,000s symm. Z_2 spin liquid through $[\mathcal{H}^2(SG, Z_2)]^2 \times$ Hermele 12
  - Classify SET phases through $\mathcal{H}^3[SG \times GG, U(1)]$ Ran 12
- there are SRE symmetric states → many different phases

We may call them **symmetry protected trivial (SPT)** phase





- Haldane phase of 1D spin-1 chain w/ $SO(3)$ symm. Haldane 83

Gapped phases w/ symmetry → SET and SPT phases

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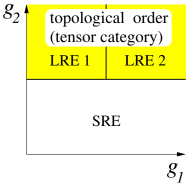
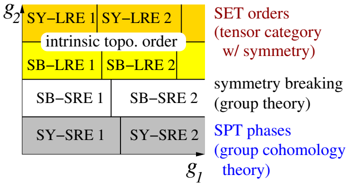
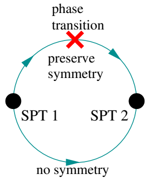
- there are SRE symmetric states → many different phases

We may call them **symmetry protected trivial (SPT)** phase
or **symmetry protected topological (SPT)** phase

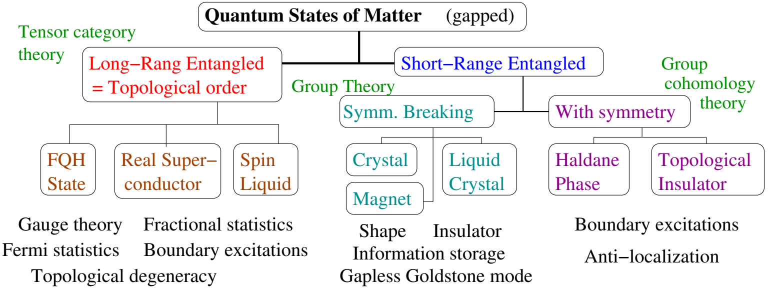






- Haldane phase of 1D spin-1 chain w/ $SO(3)$ symm. Haldane 83
- 1 topo. ins. w/ $U(1) \times T$ symm. in 2D, Kane-Mele 05; Bernevig-Zhang 06
  15 in 3D Moore-Balents 07; Fu-Kane-Mele 07

Highly entangled quantum matter: A new chapter of condensed matter physics



- Group theory → Symmetry breaking order → *shape, superfluid, phonon, magnets, magnon, liquid crystals, ...*
- Tensor category theory → Topological order → *FQH effect, anyons, fermions, fractional charge/spin, spin liquid, photon, perfect conducting edges, ...*
- Group cohomology theory → SPT order → *symmetry protected boundary excitations, ...*