Classifying fractionalization: Symmetry classification of gapped Z_2 spin liquids in two dimensions

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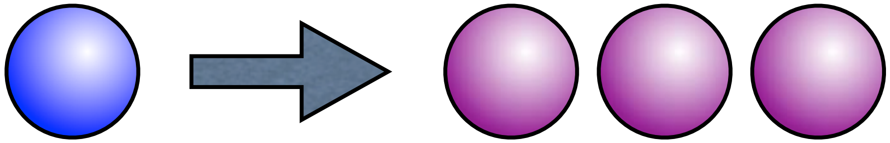
Fractionalization

- Gapped, topologically ordered phases (two dimensions)
  → Quasiparticle excitations with fractional quantum numbers

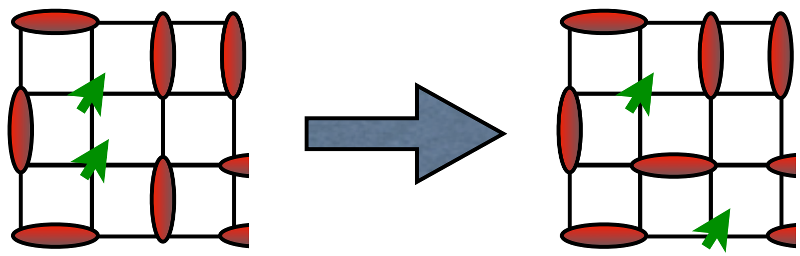


Charge e electron

Charge $e/3$ Laughlin quasiparticles

ν = 1/3 Fractional quantum Hall liquid

Gapped Z_2 spin liquid



S=1 “magnon”

S=1/2 spinons

- What are distinct types of fractionalization?
- How to describe/classify?
- How to detect in numerics/experiment?

Classification of spin liquids

- In presence of symmetry, there are many gapped Z_2 spin liquids (X.-G. Wen, ...)

With symmetry: Z_2A Z_2B Z_2C ....

Break all symmetry: Z_2

- Can we classify such distinct Z2 spin liquids?
- Simpler: symmetry classification

Z_2 spin liquids

Symmetry class 1

Distinct phases:
1a, 1b, 1c, ...

Symmetry class 2

Distinct phases:
2a, 2b, 2c, ...

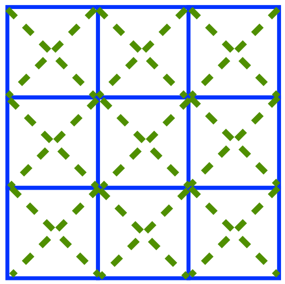
.....

Symmetry classes ⟺ Fractionalization classes (types of fractionalization)

Not in this talk: symmetry classes "beyond fractionalization."
(Next talk.)

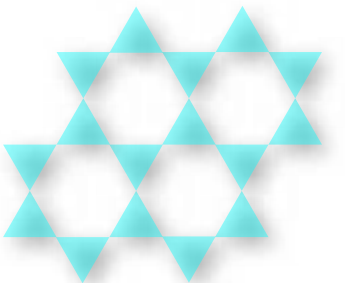
Why gapped Z_2 spin liquids?

- Simple setting to study interplay of symmetry & topological order
- Recent evidence they exist in simple, fairly realistic models:



J1-J2 square lattice

(H. C. Jiang, H. Yao & Balents;
L. Wang, Z.-C. Gu, Verstraete, X.-G. Wen)



Kagome lattice

(H. C. Jiang, Z. Y. Weng, D. N. Sheng;
S. Yan, Huse & White;
Depenbrock, McCulloch & Schollwöck;
H. C. Jiang, Z. Wang & Balents)

- Can we find direct evidence for fractionalization in these models?
- Can we tell *which* Z_2 spin liquids occur?

Outline

1. Prior work: parton constructions and projective symmetry group (PSG)

2. Review: topological order of gapped Z_2 spin liquids

3. Fractionalization of SO(3) spin

4. Fractionalization of crystal momentum

5. General symmetry classification

Signature in neutron scattering

With square lattice space group + time reversal + spin rotation symmetry:

$2{,}098{,}176 = (2^{22} - 2^{11})/2 + 2^{11} \approx 2^{21}$ symmetry classes

(Actually even more than this.)

Prior work: projective symmetry group (PSG) classification

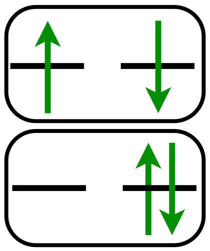
(X. G. Wen)

- Consider *e.g.* S=1/2 spin model, represent with S=1/2 fermionic partons

Hilbert space

$$\vec{S} = \frac{1}{2} f^\dagger_\alpha \vec{\sigma}_{\alpha\beta} f_\beta$$

$$f^\dagger_\alpha f_\alpha = 1$$



S=1/2 doublet (physical states) S=1/2, G=0

Unphysical doublet S=0, G=1/2

- Mean-field Hamiltonian: $H_{MFT} = \sum_{(r,r')} \left[t f^\dagger_{r\alpha} f_{r'\alpha} + \Delta (f^\dagger_{r\uparrow} f^\dagger_{r'\downarrow} + f^\dagger_{r'\uparrow} f^\dagger_{r\downarrow}) + \cdots \right]$

Action of symmetry:

1. Non-trivial gauge transformations: $T_x : f_{r\alpha} \rightarrow e^{i\lambda_r} f_{r+\boldsymbol{x},\alpha}$
2. Acts projectively: $T_x T_y = e^{i\phi} T_y T_x$

- Classify distinct ways symmetry can act, up to unitary (gauge) equivalence.
- Each such class is called a “PSG.” Really, PSGs comprise a particular class of projective *representations* of the symmetry group
- PSG provides a *mean-field* symmetry classification

PSG and symmetry classification

- Mean-field + fluctuating gauge field → low-energy effective theory.
  Can be gapped Z_2 spin liquid.
- Issue 1: Parton description is not an essential property of a Z_2 spin liquid.
- Issue 2: PSG is a mean-field classification

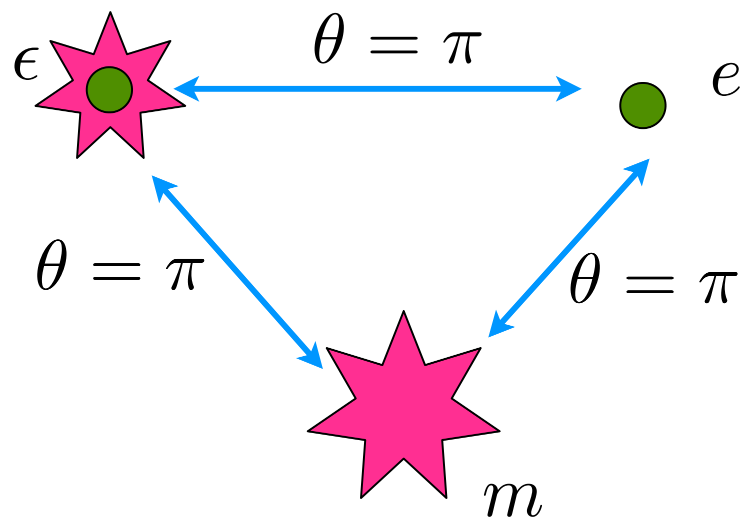
<u>Other prior work:</u>

- Ying Ran & Xiao-Gang Wen, 2002, unpublished
- Alexei Kitaev, Ann. Phys. 2006, Appendix F

Z_2 topological order: particle types

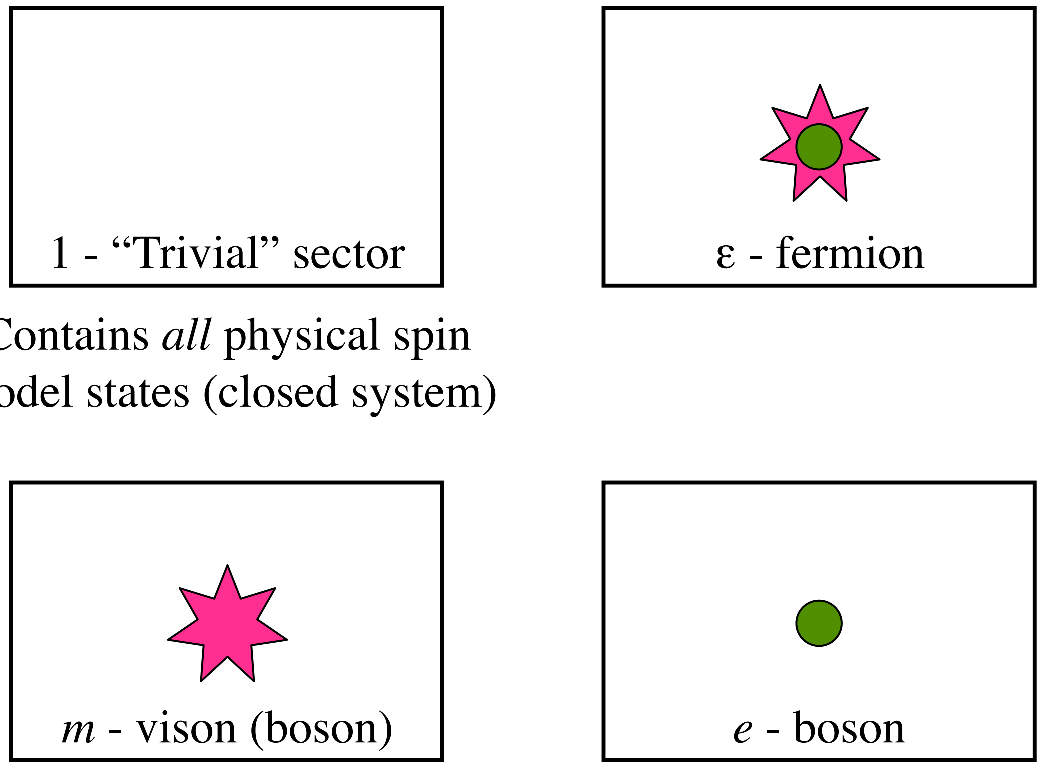
- Two bosons (e and m). One fermion (ε). Also one "trivial" boson (1).
- Often: e = spinon, m = vison, ε = spinon+vison bound state
- Or: ε = spinon, m = vison, e = spinon+vison bound state

  - Fusion rules: $\epsilon \times \epsilon = m \times m = e \times e = 1$
    $\epsilon \times m = e \; , \; \epsilon \times e = m \; , \; e \times m = \epsilon$
  - Mutual statistics:



Superselection sectors

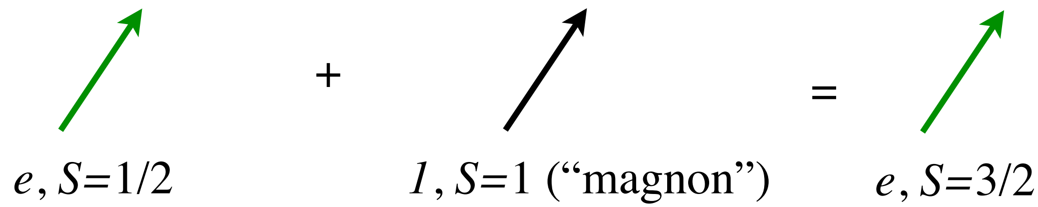
- Cannot locally create single isolated *e*, *m* or ε. Create in pairs and separate.
- Topological superselection sectors



- Sectors are closed under action of local operators

Fractionalizing spin

- e-particle could have S=0, 1/2, 1, 3/2, ...



- Only integer vs. half-odd-integer spin matters → two fractionalization classes

e	m	ε
1/2	0	1/2
0	1/2	1/2
1/2	1/2	0
0	0	0

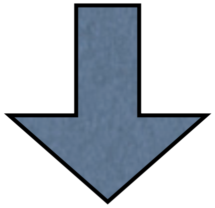
S mod 1

Same, under *relabeling* e↔m

- Three symmetry classes if *only* SO(3) spin rotation symmetry present

Fractionalizing spin

- S mod 1 = 0, 1/2 → Two fractionalization classes for SO(3)

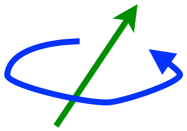


Specify fractionalization
classes for all particle types

- Three symmetry classes

<u>Mathematics: projective representations</u>

- S mod 1 = 0 → $R_s(2\pi\hat{n}) = 1$
- S mod 1 = 1/2 → $R_s(2\pi\hat{n}) = -1$



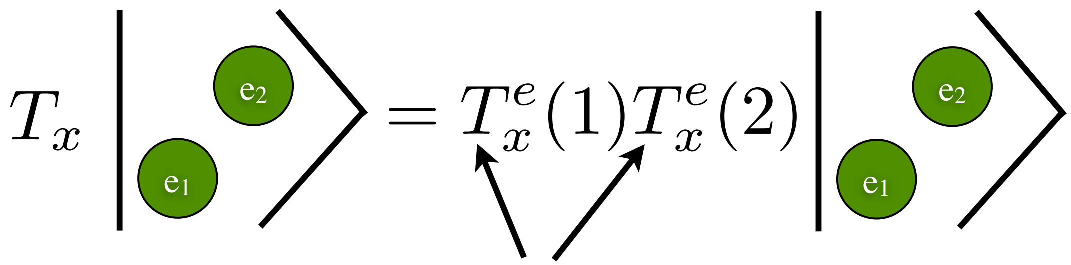
- Important: $R_s(2\pi\hat{n})$ must be a constant on each sector. Otherwise one gets a topologically trivial *S*=1/2 particle

- These are (the only) two different "Z_2 central extensions" of SO(3)
- Can summarize in terms of group cohomology: $H^2(\mathrm{SO}(3), Z_2) = Z_2$

Fractionalizing crystal momentum

- Translation symmetry: $T_x T_y = T_y T_x$, $T_x T_y T_x^{-1} T_y^{-1} = 1$ → Holds for physical states (1-sector)

- Acting on state with two e-particles:

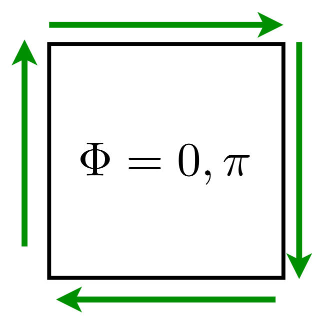


These operators localized near corresponding e-particles

$$T_x T_y T_x^{-1} T_y^{-1} = 1$$

↓

$$T_x^e T_y^e (T_x^e)^{-1} (T_y^e)^{-1} = \pm 1$$



Interpretation: e-particle feels 0 or π flux per plaquette

- Note: we assume e and m particles not exchanged under translation. This is “beyond fractionalization.” See next talk!

Fractionalizing crystal momentum

	e	m	ε
$T_x T_y T_x^{-1} T_y^{-1}$	-1	1	-1
	-1	-1	1
	1	1	1

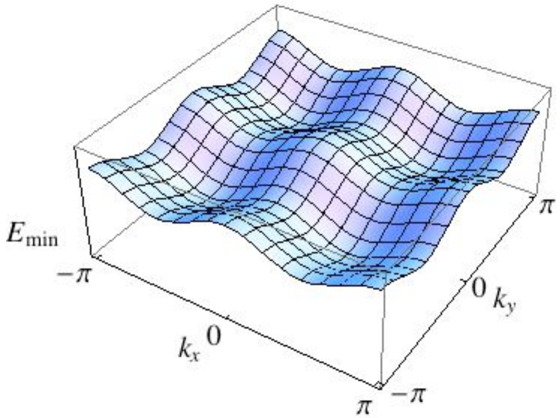
- Translation symmetry: 2 fractionalization classes & 3 fractionalization classes
- These classes all realized in Kitaev toric code model (vary signs of vertex & plaquette terms)

- Again, there are two "Z_2 central extensions" of G = Z × Z translation symmetry
- Group cohomology: $H^2(G, Z_2) = Z_2$

Fractionalizing crystal momentum: possible neutron signature

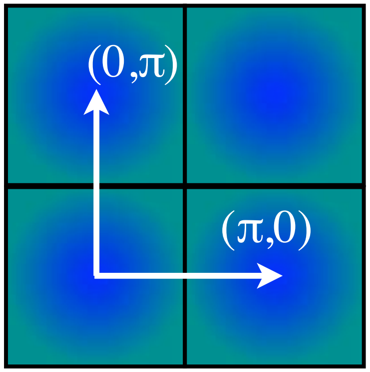
- Suppose *e*-particle (spinon) has *S*=1/2 *and* $T_x T_y T_x^{-1} T_y^{-1} = -1$

Bottom of *S*=1 two-spinon continuum:



Extra periodicity in ***k***:

$$E_{\rm min}[\vec{k}] = E_{\rm min}[\vec{k} + (\pi, 0)] = E_{\rm min}[\vec{k} + (0, \pi)]$$



- Most general irreducible representation in this fractionalization class is two-dimensional, given by:

$$T_x = e^{ik_x} \sigma^z \qquad T_y = e^{ik_y} \sigma^x$$

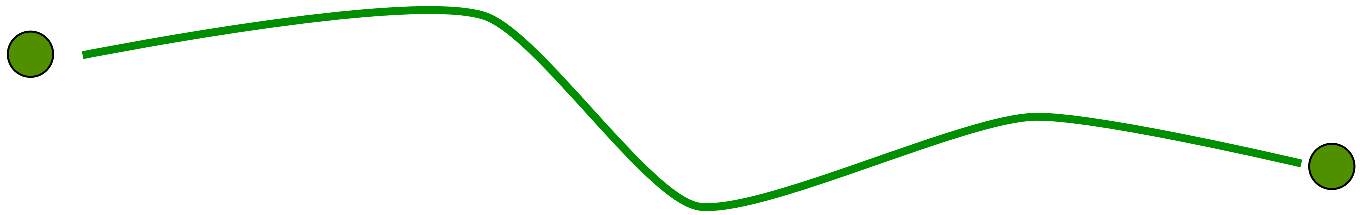
- Implies four degenerate scattering states with crystal momenta:

$$\vec{q}, \vec{q} + (\pi, 0), \vec{q} + (0, \pi), \vec{q} + (\pi, \pi)$$

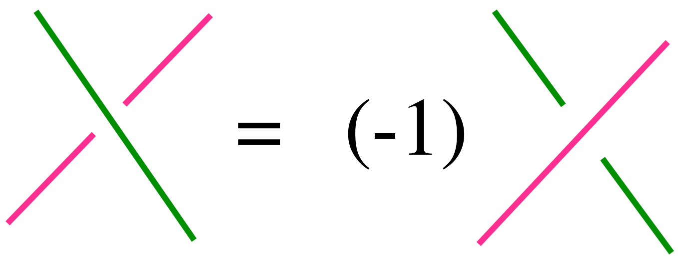
$$\vec{q} = \vec{k}_1 + \vec{k}_2$$

String operators

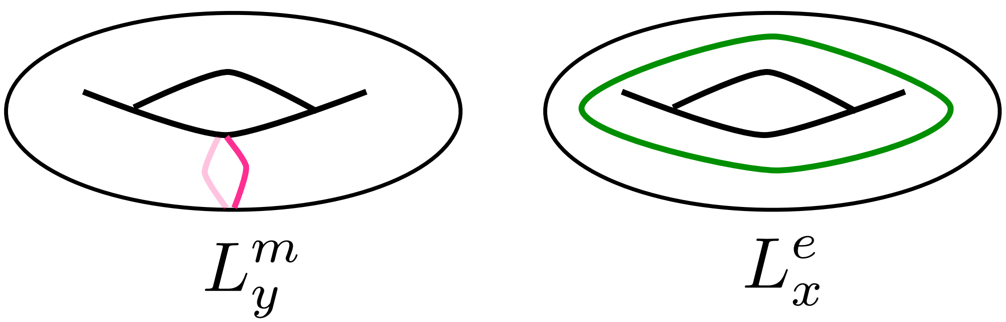
- To move an *e*-particle, or to create two isolated *e*'s, act with string operator:



- *e*- and *m*-strings anti-commute at crossing points:



- Loop operators/algebra:

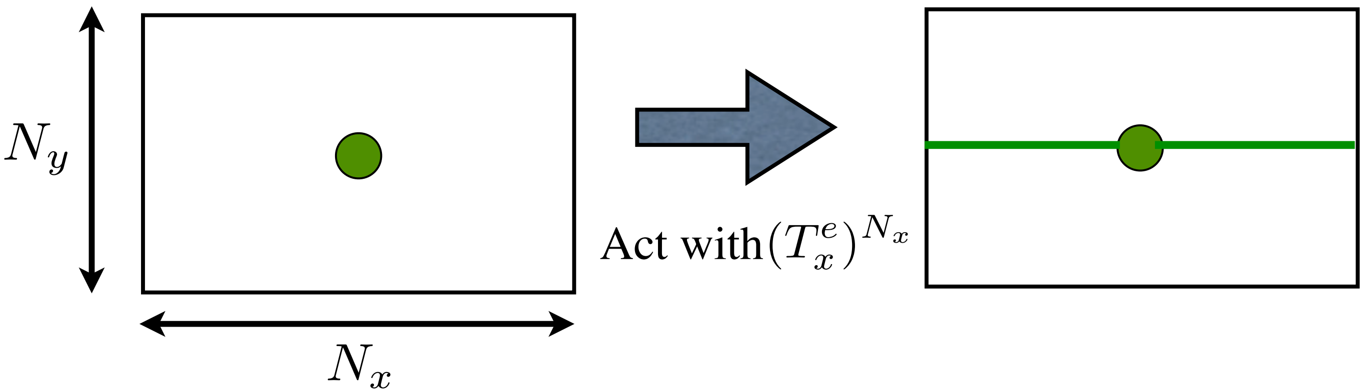


$$\{L_x^e, L_y^m\} = 0$$
$$\{L_y^e, L_x^m\} = 0$$
$$(L_{x,y}^e)^2 = (L_{x,y}^m)^2 = 1$$

⇒ *D*=4 irrep (4-fold ground state degeneracy)

Relation to ground state quantum numbers

- Degenerate ground states can have nontrivial quantum numbers



- Suggests associations: $L^e_x \simeq (T^e_x)^{N_x}$, $L^m_x \simeq (T^m_x)^{N_x}$, $\ldots$

- Action of symmetry on loop operators: $T_y L^e_x T_y^{-1} \to T^e_y (T^e_x)^{N_x} (T^e_y)^{-1}$

- From this can work out *relative* momenta among four ground states.

General symmetry group

- Some mathematics...
- Consider symmetry group G, elements $g \in G$, projective representation $\Gamma(g)$

$$\Gamma(g_1)\Gamma(g_2) = \omega(g_1, g_2)\Gamma(g_1 g_2), \; \omega(g_1, g_2) \in Z_2$$

"Factor set" — From fusion rules

- Associativity constraint: $\omega(g_1, g_2)\omega(g_1 g_2, g_3) = \omega(g_1, g_2 g_3)\omega(g_2, g_3)$
- Abelian group structure: $(\omega_A \omega_B)(g_1, g_2) = \omega_A(g_1, g_2)\omega_B(g_1, g_2)$
- "Gauge" transformation:

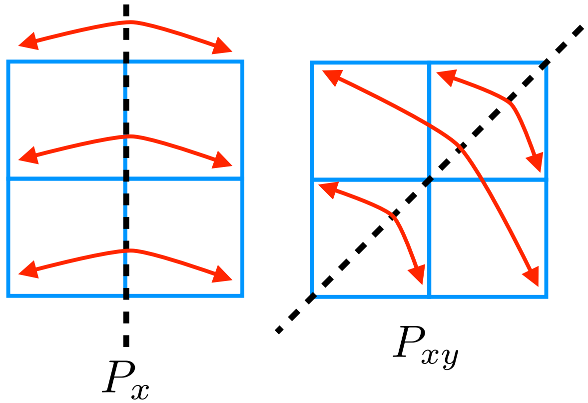
$$\Gamma(g) \to \lambda(g)\Gamma(g) \implies \omega(g_1, g_2) \to \lambda^{-1}(g_1)\lambda^{-1}(g_2)\lambda(g_1 g_2)\omega(g_1, g_2)$$

- Classify factor sets up to "gauge" equivalence.
- Factor set classes also form Abelian group: $H^2(G, Z_2)$ ← 2nd cohomology group, coefficients in Z_2

Fractionalization class (for one sector) ⟷ Element of $H^2(G, Z_2)$

Square lattice example

- G = Square lattice space group × time reversal × spin rotation.
- Square lattice space group generators: T_x, P_x, P_{xy}
- Note that: $T_y = P_{xy} T_x P_{xy}^{-1}$
- Time reversal $\mathcal{T}$
- Spin rotation (by θ about $\hat{n}$-axis): $R(\theta\hat{n})$



- Generators + relations specify the symmetry class in one sector:

$$P_x^2 = \sigma_{px}$$
$$P_{xy}^2 = \sigma_{pxy}$$
$$(P_x P_{xy})^4 = \sigma_{pxpxy}$$
$$T_x T_y T_x^{-1} T_y^{-1} = \sigma_{txty}$$
$$T_x P_x T_x P_x^{-1} = \sigma_{txpx}$$
$$T_y P_x T_y^{-1} P_x^{-1} = \sigma_{typx}$$
$$\mathcal{T}^2 = \sigma_T$$

$$\mathcal{T} T_x \mathcal{T}^{-1} T_x^{-1} = \sigma_{Ttx}$$
$$\mathcal{T} P_x \mathcal{T}^{-1} P_x = \sigma_{Tpx}$$
$$\mathcal{T} P_{xy} \mathcal{T}^{-1} P_{xy} = \sigma_{Tpxy}$$
$$R(2\pi\hat{n}) = \sigma_R$$
$$R(\theta\hat{n})\mathcal{T} = \mathcal{T} R(\theta\hat{n})$$
$$R(\theta\hat{n}) P_x = P_x R(\theta\hat{n})$$
$$R(\theta\hat{n}) P_{xy} = P_{xy} R(\theta\hat{n})$$
$$R(\theta\hat{n}) T_x = T_x R(\theta\hat{n})$$

(+ Lie algebra of spin rotations)

- Here the σ's = ±1
- 11 independent Z_2 parameters → $H^2(G,Z_2) = (Z_2)^{11}$

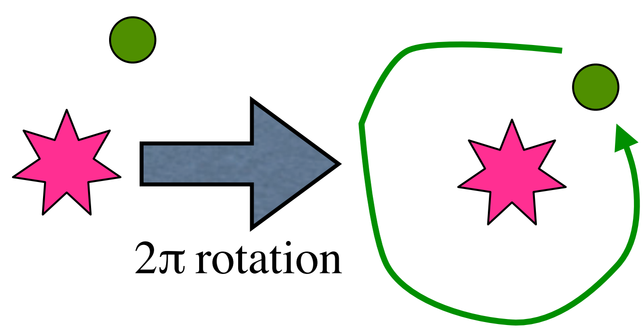
Square lattice example

- Fractionalization classes = elements of $H^2(G,Z_2) = (Z_2)^{11}$
- Specify fractionalization class for *e* and *m* independently
  $\rightarrow$ determines class for ε
- Accounting for relabeling *e*↔*m*: $(2^{22}-2^{11})/2 + 2^{11} \approx 2^{21}$ symmetry classes
- How to determine ε fractionalization class?
  - Take product, e.g: $(\mathcal{T}^\epsilon)^2 = (\mathcal{T}^e)^2(\mathcal{T}^m)^2$
  - *Except*: $(P_x^\epsilon P_{xy}^\epsilon)^4 = -(P_x^e P_{xy}^e)^4(P_x^m P_{xy}^m)^4$

Sign from mutual statistics

  - "Twisting" of H^2 group product:

$$\omega_\epsilon(g_1, g_2) = \omega_{\text{twist}}(g_1, g_2)\omega_e(g_1, g_2)\omega_m(g_1, g_2)$$



How are these results established?

- General arguments
- Explicit construction for Kitaev toric code model

PSG classification revisited

- (Focus on S=1/2 fermionic partons, for concreteness.)
- PSG classifies projective representations up to unitary equivalence. Compared to fractionalization class, this includes extra (presumably non-universal) information.
- For any PSG, can read off fractionalization class for ε.
- Given PSG + effective theory, can compute fractionalization class for m.
- On square lattice, Wen found 272 Z_2 PSGs (for a *single* sector). Should be compared with 2^{10} classes for same sector (all have $S=1/2$, fixes one parameter).
- Some classes not realized (for this particular parton theory)
- There are pairs of distinct PSGs belonging to same class. But in all cases I know, one PSG is gapless. Is there an example of two distinct Z_2-gapped PSGs, belonging to same fractionalization class?

Open issues

- Symmetry classes “beyond fractionalization” (see next talk)
- Full classification of gapped Z_2 spin liquids
- Generalize to other topological orders. (We have an answer for Abelian topological order with only translation + local symmetries.)
- Three dimensions?
- How can symmetry class be determined given ground state wavefunction, excited states? Application to numerics on kagome & J_1-J_2 Heisenberg models?
- Experimental signatures?
- Can we find a candidate gapped spin liquid material?