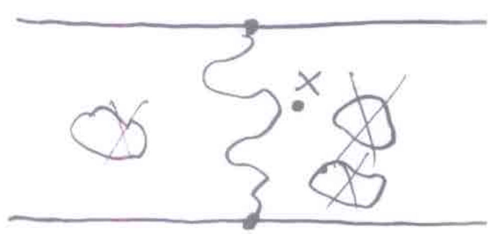
Kay Wiese
KITP 12 Sep 2008

Fluctuation forces exerted by a 2D self-avoiding polymer: the marvels of CFT and SLE

1) Introduction

In my talk, I will explain how to use recent developments in CFT and the so-called stochastic Löwner evolution (SLE) to calculate intersection and passing probabilities of a self-avoiding polymer in 2 dimensions, and the ensuing consequences for the enduced Casimir forces. This joint work with PLD is inspired by a talk we heard the first week here by Angie Maciolek on the effective forces induced by fluctuating off-critical interfaces in 2D Ising, i.e. interfaces with a tension. The question then arises what happens for critical interfaces. Though one can identify a critical interface in Ising on the honeycomb lattice, this can not be done locally, so this is not possible in an expeiment, due to the presence of bubbles



Get rid of bubbles $\Longleftrightarrow$ SA polymer.

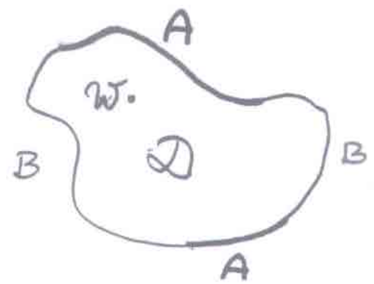
Q: What is the force exerted on point x?

I will use methods of CFT, so let me explain the concepts involved.

2.) Brownian Motion

Exit probability

Consider Brownian motion in a bounded domain $\mathcal{D}$



Q: What is proba $P(w)$ for a Brownian particle to start at w and leave the domain $\mathcal{D}$ through A, not B?

A: $P(w) = \phi(w)$, where $\phi(w)\big|_A = 1$, $\phi(w)\big|_B = 0$ and $\Delta \phi(w) = 0$ in $\mathcal{D}$.

P: W_t is a Brownian motion without drift
$$dW_t = dB_t \quad , \quad \langle dB_t \rangle = 0 \; , \; \langle dB_t^2 \rangle = dt$$
and B_t and $B_{t'}$ are independent. We use Itô calculus.

We now use a very powerful tool from stochastic processes, a martingale:

"Def:" A martingale is a stochastic process that does (on average) not change in time.

Claim: $\phi(W_t)$ is martingale.

Proof: $d\phi(W_t) = \phi'(W_t)\, dW_t + \frac{1}{2}\phi''(W_t)\, dW_t^2 + \ldots$

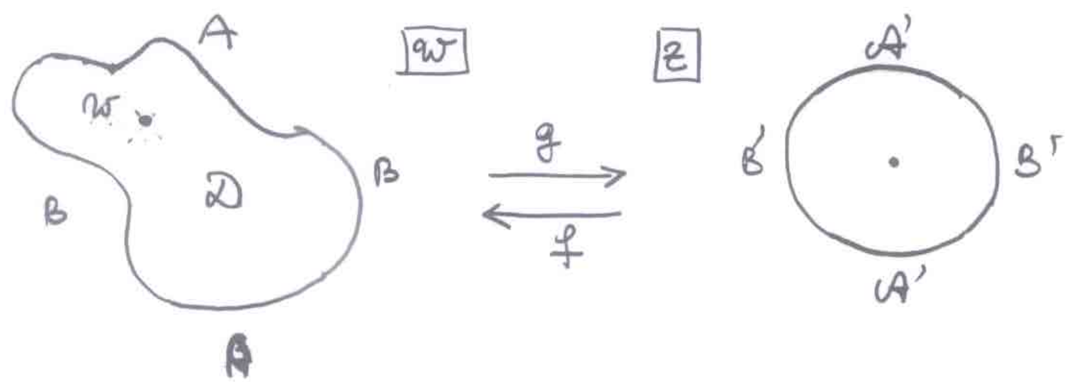
Thus on average:
$$d\langle \phi(W_t)\rangle = \underbrace{\langle \phi'(W_t)\, dW_t\rangle}_{0} + \frac{1}{2}\underbrace{\langle \phi''(W_t)\, dx_t^2\rangle}_{\Delta \phi(W_t)\, dt}$$

$$\Rightarrow \quad \frac{d}{dt}\langle \phi(W_t)\rangle = \Delta \phi(W_t)$$
$$= 0 \quad \text{iff} \quad \Delta \phi(w) = 0$$

(strong Markov property)

thus $\phi(x_{t_f})$ at the stopping time t_f has the same expectation as $\phi(x_0)$, but since we know the latter, we also know the former.

Conformal invariance



Claim: If $z = g(w)$ is a conformal transformation, then $P(x)$ is mapped conformally, or

$$\phi(w,\bar w) = \phi(\overbrace{f(z)}^{w}, \overbrace{\bar f(\bar z)}^{\bar w})$$

$$\Rightarrow \Delta\phi(z,\bar z) = 4\,\partial_z\partial_{\bar z}\,\phi(f(z),\bar f(\bar z))$$

$$= 4\, f'(z)\,\bar f'(\bar z)\,\underbrace{\partial_w\bar\partial_{\bar w}\phi(w,\bar w)}_{=0}$$

$\begin{cases} z = x+iy \\ \bar z = x-iy \end{cases}$

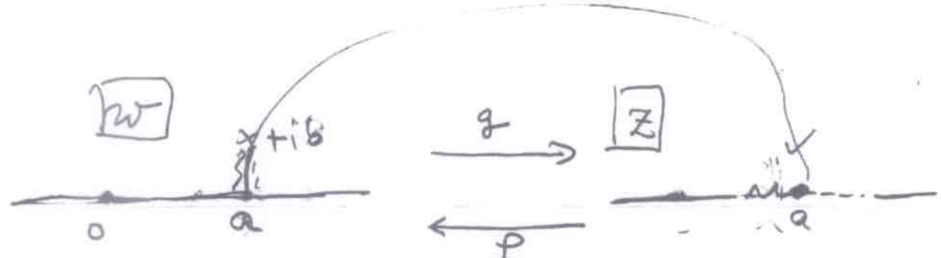
$\Delta\, z\bar z = \Delta(x^2+y^2) = 4$

$\partial_z\partial_{\bar z}\, z\bar z = 1 \Rightarrow \Delta = 4\partial\bar\partial$

Thus, if one suceeds in mapping $\mathcal{D} \xrightarrow{g}$ disc and $w \to 0$, then $\phi(w) = \dfrac{|A'|}{|A' \cup B'|}$.

All games played are intimately connected to find a conformal map from a complicated to a simple geometry

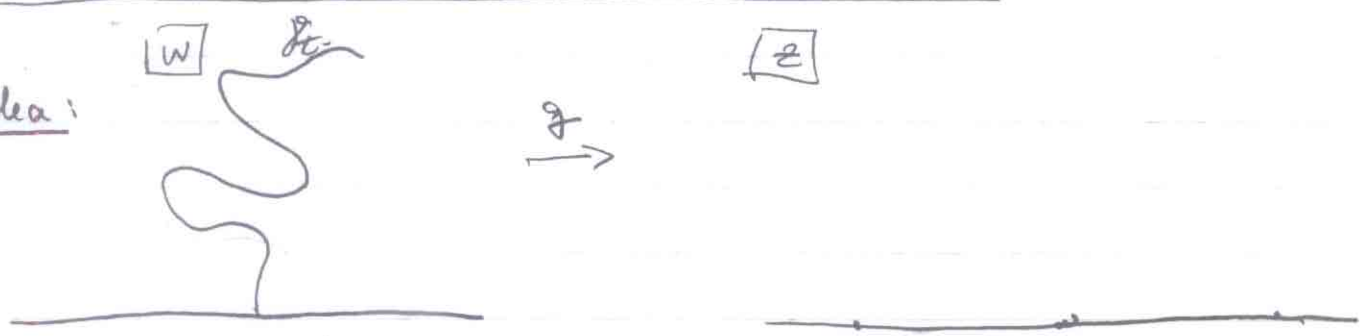
Example:



$$g(w) = \sqrt{(w-a)^2 + b^2} + a \qquad (*)$$

3. Stochastic Löwner evolution

Idea:



K_t = critical curve. Construct remover map (uniformising)

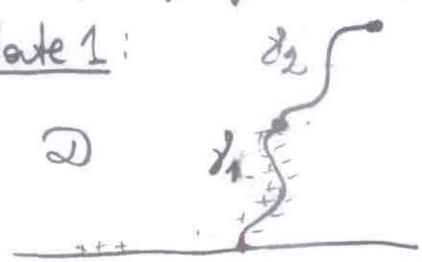
Take $(*)$ and set $b = 2\sqrt{t}$

$$\Rightarrow \boxed{\partial_t g_t(w) = \frac{2}{\sqrt{(w-a)^2+4t}} = \frac{2}{g_t(w) - a_t}}$$

Löwner's equation $\circledast$

Now use this for physics. Imagine γ = interface (Ising, Potts, ...)

Postulate 1:



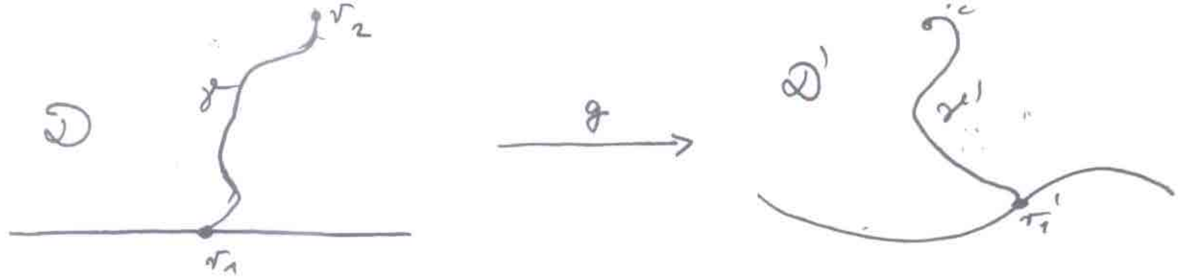
the conditional probability measure

$$\frac{\mu(\gamma_2 | \gamma_1 ; \mathcal{D})}{\mu(\gamma_2 ; \mathcal{D} \backslash \gamma_1)}$$

Postulate 2: Conformal invariance.

$\circledast$ Uniqueness: $g_t(z) = z + \frac{2t}{z} + O(\frac{1}{z^2})$

5



$$(g * \mu)(\gamma; \mathcal{D}, v_1, v_2) = \mu(g(\gamma); \mathcal{D}', v_1', v_2')$$

<u>Theorem (Oded Schramm)</u>

$$\boxed{a_t = \sqrt{\kappa}\; B_t, \quad \text{where } B_t = \text{standard Brownian motion}}$$

$$\langle \partial_t B_t \, \partial_{t'} B_{t'} \rangle = \delta(t - t')$$

<u>Why?</u> property 3 demands that the increments in a_t are independent for all time intervals δt. The only such process is a Brownian with drift. The latter is absent due to reflection symmetry.

<u>Remark:</u> $g(w) \xrightarrow{w \to \infty} w + \mathcal{O}\left(\frac{1}{w}\right)$ (uniqueness requirement)

<u>Idea:</u> Different values of κ will give measure on critical curves in different critical systems.

<u>Examples:</u>

$\kappa = 2$: LERW

$\kappa = \frac{8}{3}$: SAW

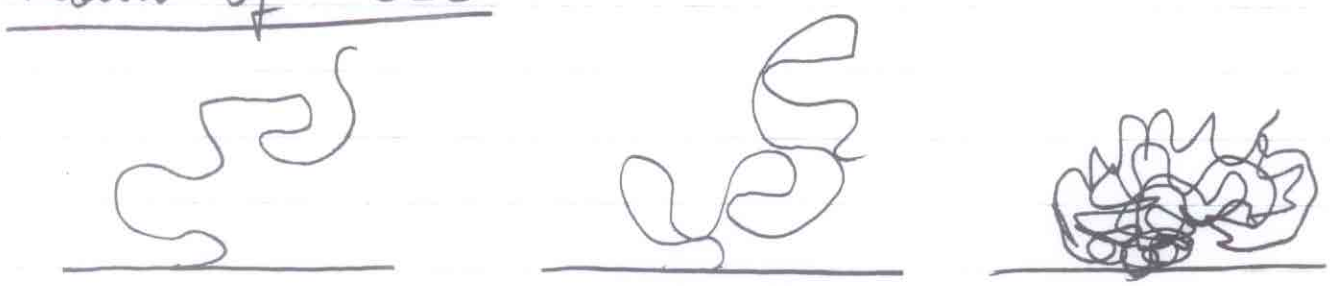
$\kappa = 3$: cluster boundaries in Ising

$\kappa = 4$: level lines of Gaußian free field

$\kappa = 6$; cluster boundaries in percolation

$\kappa = 8$: dense phase of selfavoiding walks
spanning trees

Phases of SLE



$0 \leq \kappa \leq 4$ $\qquad$ $\kappa < 4 < 8$ $\qquad$ $\kappa > 8$

SLE - duality $\qquad \kappa \longrightarrow \dfrac{16}{\kappa}$

example : SAW $\longleftrightarrow$ percolation

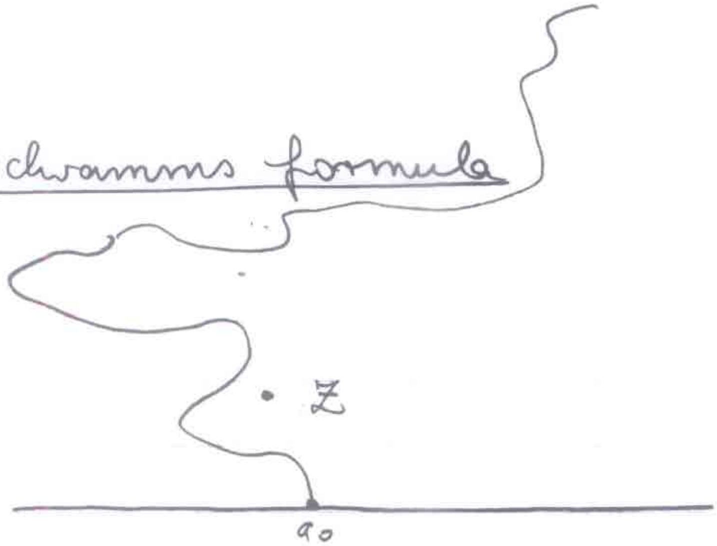
central charge: $\quad c = \dfrac{(3\kappa - 8)(6 - \kappa)}{2\kappa}$

fractal dimension of curve

$$d_f = 1 + \frac{\kappa}{8}$$

(7)

Schramms formula



What is the probability that a curve γ passes to the left of a point z?

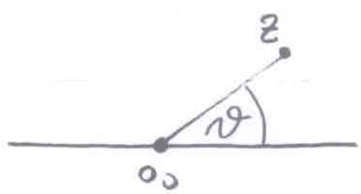
Due to our postulates, and using SLE,

$$P(z,\bar z, a_0) = P\left(z+\frac{2\,dt}{z-a_0},\ \bar z+\frac{2\,dt}{\bar z-a_0},\ a_0+\sqrt{\kappa}\,dB_t\right)$$

$$\Rightarrow \boxed{\left[\frac{2}{z-a_0}\frac{\partial}{\partial z}+\frac{2}{\bar z-a_0}\frac{\partial}{\partial \bar z}+\frac{\kappa}{2}\left(\frac{\partial}{\partial a_0}\right)^2\right]P(z,\bar z,a_0)=0}$$

$$P(z,\bar z,a_0) = P(z-a_0,\ \bar z-a_0) = P\Big(\underbrace{\frac{z-a_0}{\bar z-a_0}}_{e^{2i\vartheta}}\Big)$$

$(z-a_0)= r\,e^{i\vartheta}$



$$\frac{e^{2i\vartheta}}{r^2}\left\{2\left[e^{-2i\vartheta}-e^{+2i\vartheta}\right]P'+\frac{\kappa}{2}\left[2\cdot(e^{2i\vartheta}-1)P'+(e^{2i\vartheta}-1)^2P''\right]\right.$$

NR: $\displaystyle\frac{\partial}{\partial a_0}\frac{z-a_0}{\bar z-a_0} = -\frac{1}{\bar z-a_0}+\frac{z}{(\bar z-a_0)^2} = \frac{e^{i\vartheta}}{r}\left(e^{2i\vartheta}-1\right)$

$$\frac{\partial^2}{\partial a_0^2}\frac{z-a_0}{\bar z-a_0} = -\frac{2}{(\bar z-a_0)^2}+\frac{2(z-a_0)}{(\bar z-a_0)^3} = \frac{2e^{i2\vartheta}}{r^2}\left(e^{2i\vartheta}-1\right)$$

$$\mathcal{P}(e^{2i\vartheta}) := \tilde{\mathcal{P}}(\vartheta)$$

$$\Rightarrow \quad \tilde{\mathcal{P}}'(\vartheta) = \mathcal{P}'(e^{2i\vartheta})\, e^{2i\vartheta}\, 2i$$

$$\tilde{\mathcal{P}}''(\vartheta) = -4\left[\mathcal{P}'(e^{2i\vartheta})\, e^{2i\vartheta} + \mathcal{P}''(e^{2i\vartheta})\, e^{4i\vartheta}\right]$$

Solve for $\tilde{\mathcal{P}}(\vartheta)$

$$\Rightarrow \quad 2(\kappa-4)\cos(\vartheta)\,\tilde{\mathcal{P}}'(\vartheta) + \kappa \sin\vartheta\, \tilde{\mathcal{P}}''(\vartheta) = 0$$

$$\Rightarrow \quad \frac{d}{d\vartheta} \ln \tilde{\mathcal{P}}'(\vartheta) = \frac{\tilde{\mathcal{P}}''}{\tilde{\mathcal{P}}'} = \frac{2(4-\kappa)}{\kappa} \cot\vartheta$$

$$\Rightarrow \quad \ln \tilde{\mathcal{P}}'(\vartheta) = \frac{2}{\kappa}(4-\kappa) \ln(\sin\vartheta) + \text{const}$$

$$\Rightarrow \quad \boxed{\tilde{\mathcal{P}}'(\vartheta) \sim (\sin\vartheta)^{\frac{8}{\kappa}-2}}$$

Integration and fixing the normalizations yields:

$$\tilde{\mathcal{P}}(\vartheta) = \frac{1}{2} + \frac{\Gamma(\frac{4}{\kappa})}{\Gamma(\frac{4}{\kappa}-\frac{1}{2})\sqrt{\pi}} \cos\vartheta\; {}_2F_1\left(\frac{1}{2}, \frac{3}{2}-\frac{4}{\kappa}, \frac{3}{2}, \cos^2\vartheta\right)$$

$$\kappa = 8/3 \quad \Rightarrow \quad \tilde{\mathcal{P}}'(\vartheta) \sim \sin\vartheta$$

$$\Rightarrow \quad \boxed{\tilde{\mathcal{P}}(\vartheta) = \frac{1}{2}(1 + \cos\vartheta) = \cos^2\left(\frac{\vartheta}{2}\right)}$$

normalizations which were fixed

Other simple values of κ: $\kappa = 4$: $\mathcal{P}$ = uniform

in general: $\kappa = \frac{8}{m+2}$, $m \in \mathbb{N}_0$; $\kappa = 2$

4. Casimir forces

free energy $\mathcal{F} = -kT \ln \mathcal{P}(z,\bar{z}) = 2kT \ln \cos\left(\frac{\vartheta}{2}\right)$

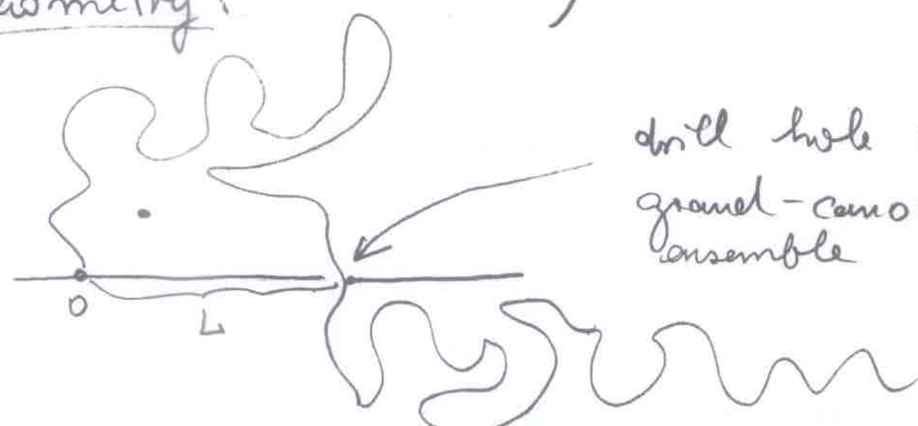
force $\vec{f} = -\vec{\nabla}\mathcal{F} = -kT \frac{\vec{n}_\theta}{r} \tan\left(\frac{\theta}{2}\right)$

$\boxed{w}$

$g(z) = \frac{z+L}{L-z}$

Another geometry:

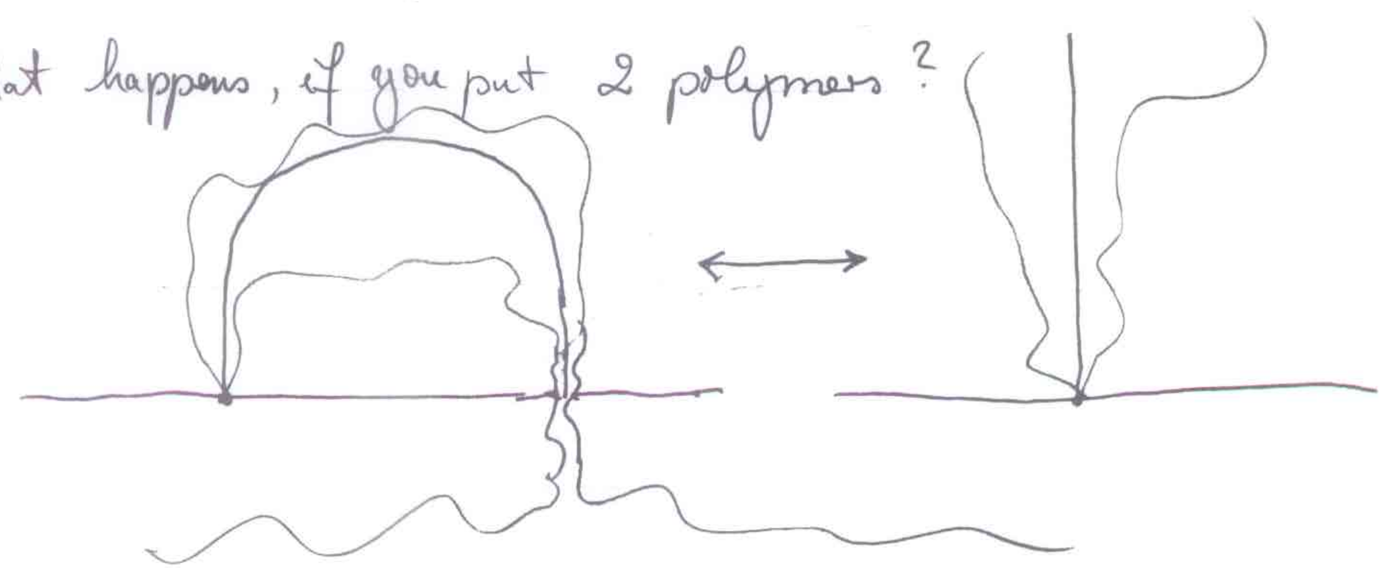
$\boxed{z}$



drill hole to have grand-canonical ensemble

There is a confining force.

What happens, if you put 2 polymers?



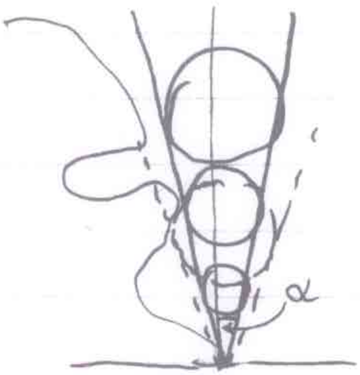
b)

2 SA polymers,

$$\boxed{\mathcal{P}^{(2)}_{m}(z) = \mathcal{P}^{(1)}_{\ell}(z)\ \mathcal{P}^{(1)}_{r}(z)\ \frac{16}{5}}$$

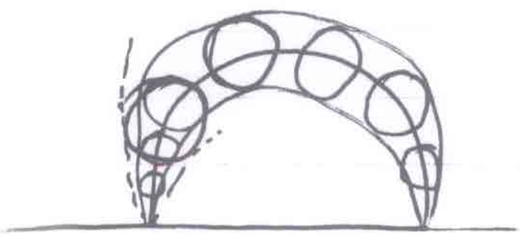
$\Rightarrow$ Force is the same as if the 2 polymers would not see each other. (Why?)

From the half-plane geometry we know that there is no force along the red line.
$\Rightarrow$ equilibrium line
How can we get an equilibrium position?



From scale invariance, the probability for the SAW to pass left (or right) of a disc is a function of cone opening α, not its size.

map to strip by inversion (Möbius)

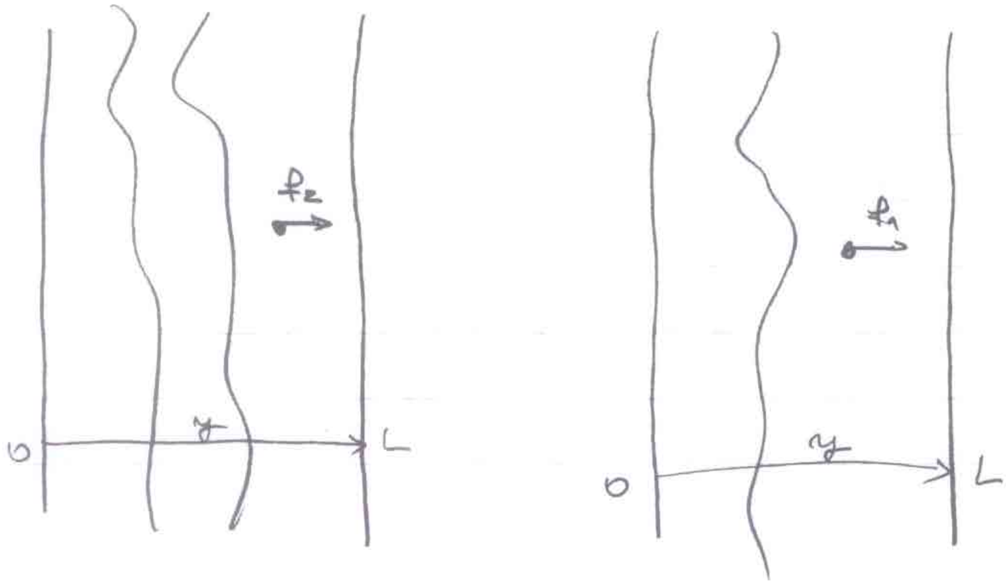


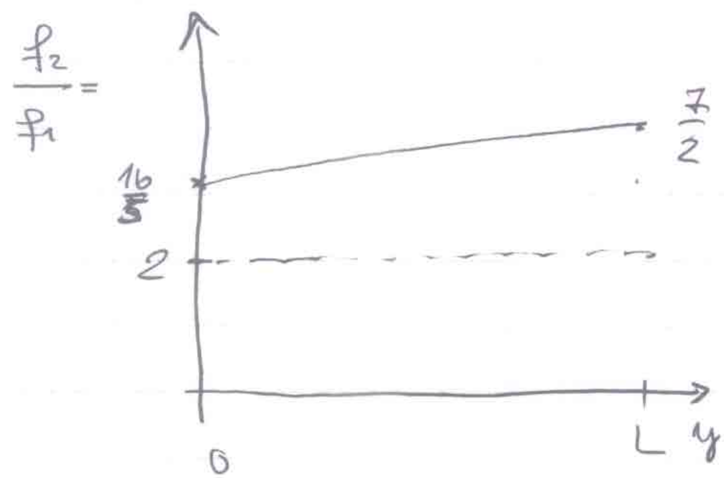
The free energy certainly increases with cone size.
$\Rightarrow$ force to ∞ in the left, and to the center in the right geometry.

What happens to a diffusing particle? Its pdf. is

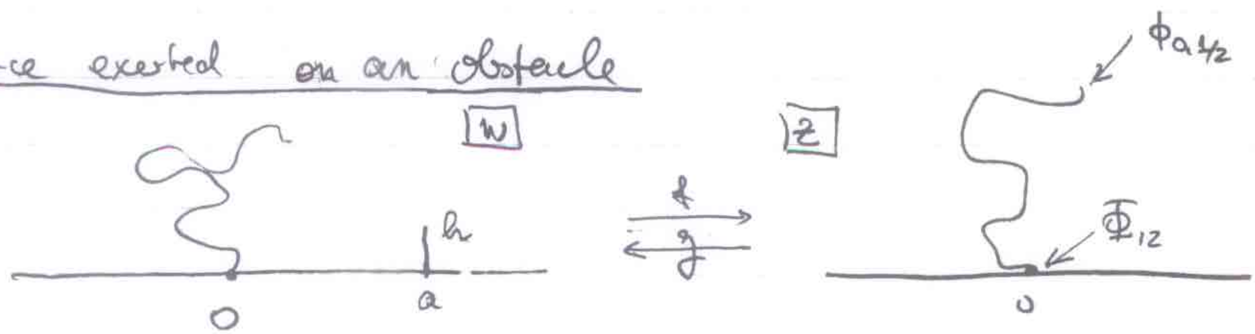
$$p_{\text{particle}}(z) = \frac{e^{-F/kT}}{Z} = \frac{e^{-(kT \ln \mathcal{P})/kT}}{Z} \sim \mathcal{P}(z)$$

11





Force exerted on an obstacle



$$f(w) = \sqrt{(w-a)^2 + h^2} + \sqrt{a^2+h^2} \qquad (a > 0)$$

$$f(\infty) = \infty \;; \quad f'(\infty) = 1 \;, \quad f(0) = 0$$

Then $\quad P_{\text{no hit}} \underset{\uparrow}{=} f'(0)^{5/8} = \left(\frac{a^2}{a^2+h^2}\right)^{\frac{5}{16}} \qquad (*)$

(Lawler, Schramm, Werner)

Can be understood also from Coulomb gas, noting that Φ_{12} = curve creation on a boundary, $h_{..}^{x\geq 3/8} = 5/8$

r.t. (*) is the transformation of a primary operator.

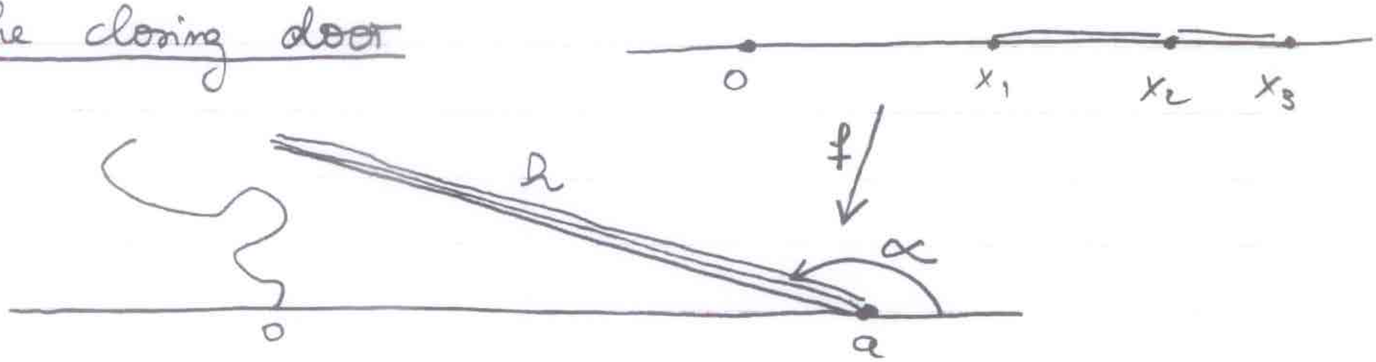
Remark: $\Phi_{0,\frac{1}{2}}$ creates curve in bulk,

$h_{0,\frac{1}{2}} = \frac{5}{96}$

$\Rightarrow \quad \Phi_{0,\frac{1}{2}} = h_{0,\frac{1}{2}} + \bar{h}_{0,\frac{1}{2}} = \frac{5}{48}$

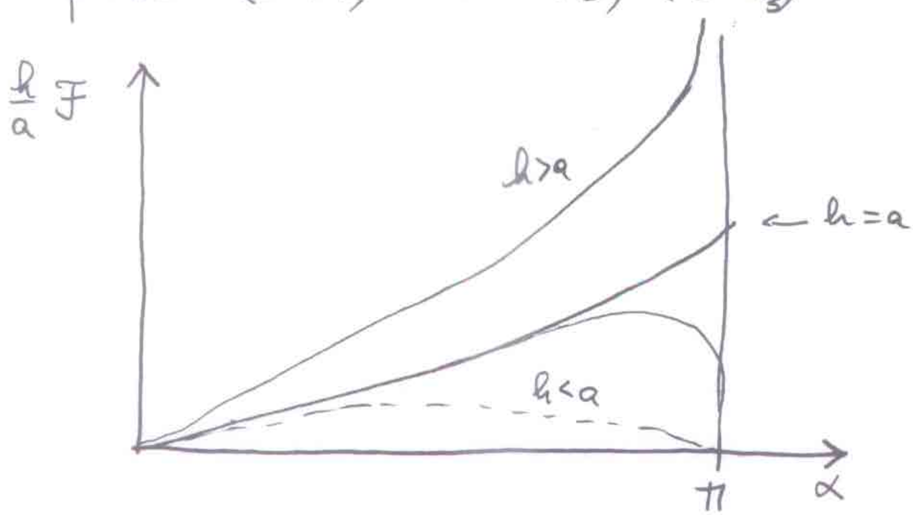
$\Rightarrow \quad f'(\text{endpoint})^{\frac{5}{48}}$.

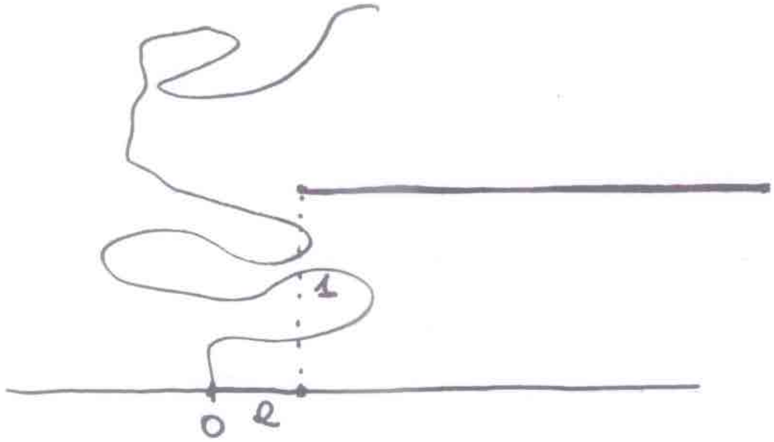
The closing door



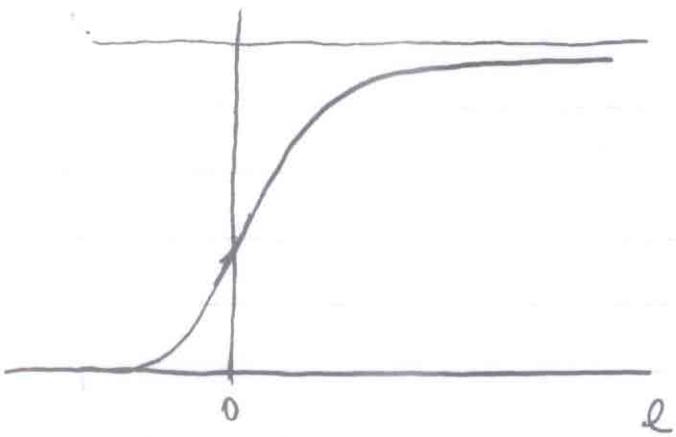
Schwarz - Christophel

$$f'(z) = (z-x_1)^{-\frac{\alpha}{\pi}} \; (z-x_2) \; (z-x_3)^{\frac{\alpha}{\pi}-1}$$





$P_{\text{no hit}} =$



$$P_{\text{no hit}} = \left(1 + \frac{1}{W(\exp(\ell\pi - 1))} \right)^{-\frac{5}{8}}$$

where $z = W(z)\, e^{W(z)}$

This can be interpreted as extremal statistics :
P(rightmost point on line $i + \mathbb{R}$ is smaller than ℓ)