A general formulation for the Casimir energy and its practical applications

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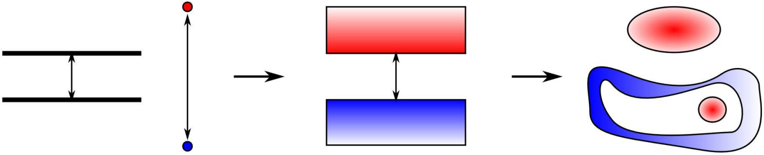
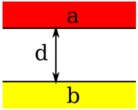
Related work

- 77 - Balian and Duplantier
- 91 - Jaekel and Reynaud
- 01, 05 - Bulgac, Magierski, Wirzba nucl-th/0102018, hep-th/0511056
- 06, 07 - Emig, Graham, Jaffe, Kardar cond-mat/0601055, 0707.1862
- 06, 07 - Kenneth and Klich quant-ph/0601011, 0707.4017

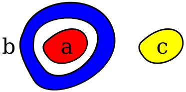


Result



Lifshitz formula

$$E = \frac{\hbar}{2\pi} \int_0^\infty d\kappa \int_0^\infty \frac{L^2}{2\pi} q_\| dq_\| \sum_{i=E,M} \ln\left(1 - r_a^i r_b^i e^{-2d\sqrt{\kappa^2 + {q_\|}^2}}\right)$$



General formula

$$E = \frac{\hbar}{2\pi} \int_0^\infty d\kappa \, \mathrm{Tr} \ln \begin{pmatrix} \mathbb{T}_a^{-1} & \mathbb{X}_{ab} & \mathbb{X}_{ac} \\ \mathbb{X}_{ba} & \mathbb{T}_b^{-1} & \mathbb{X}_{bc} \\ \mathbb{X}_{ca} & \mathbb{X}_{cb} & \mathbb{T}_c^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{T}_a & 0 & 0 \\ 0 & \mathbb{T}_b & 0 \\ 0 & 0 & \mathbb{T}_c \end{pmatrix}$$

Path integrals

$$\langle x_f | e^{-iHT} | x_i \rangle = \int \mathcal{D}x(t)\big|_{x(0),x(T)}\, e^{i\int_0^T \frac{1}{2}\dot{x}^2 - V(x)\,dt}$$

$$\mathrm{Tr}\ e^{-\beta H} = \int \mathcal{D}x|\, e^{-\int_0^\beta \mathcal{H}\,d\tau} =: Z$$

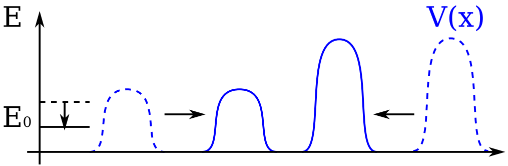
$$x(t) \to A^\mu(t,x,y,z) \quad \int dt \to \int d^4x \quad \mathcal{H} \to \mathbf{ED} + \mathbf{BH}$$

$$E_0 = -\lim_{T\to 0} kT \log Z$$

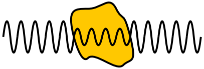
Casimir energy

$$E_0 = -\frac{\hbar}{2\pi}\int_0^\infty d\kappa \log \int \mathcal{D}A\; e^{-\frac{1}{\kappa^2}\int d^3x\, \mathbf{E}\left(\nabla\times\nabla\times+\kappa^2\right)\mathbf{E}^*+\mathbf{E}V(i\kappa)\mathbf{E}^*}$$

$$V(i\kappa) = \kappa^2\left(\epsilon(i\kappa, x) - 1\right)$$



$$\int \mathcal{D}A e^{-\int d^3x\, \mathbf{E}(\cdots)\mathbf{E}^* + \frac{1}{\kappa^2}\mathbf{E}V(i\kappa)\mathbf{E}^*}$$



$$\sim \int \mathcal{D}J\big|_{\text{obj}}\; e^{\int d^3x\, \mathbf{J}\left(\mathcal{G}_0 + V^{-1}(x,x')\right)\mathbf{J}^*}$$

$$\sim \det{}^{-1}\left(\begin{array}{c|c|c} \langle x_a|T_a^{-1}|x'_a\rangle & \langle x_a|\mathcal{G}_0|x'_b\rangle & \cdots \\ \hline \langle x_b|\mathcal{G}_0|x'_a\rangle & \langle x_b|T_b^{-1}|x'_b\rangle & \cdots \\ \hline \cdots & \cdots & \cdots \end{array}\right)$$

$$\mathcal{G}_{0,ij} = |\mathbf{E}_i\rangle \mathbb{X}_{ij} \langle \mathbf{E}_j| \qquad \mathbb{T}_i = \langle \mathbf{E}_i|T_i|\mathbf{E}_i\rangle$$

Result:

$$E = \frac{\hbar}{2\pi}\int_0^\infty d\kappa \ln\det \begin{pmatrix} \mathbb{T}_a^{-1} & \mathbb{X}_{ab} & \cdots \\ \mathbb{X}_{ba} & \mathbb{T}_b^{-1} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} \mathbb{T}_a & 0 & \cdots \\ 0 & \mathbb{T}_b & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

Scattering

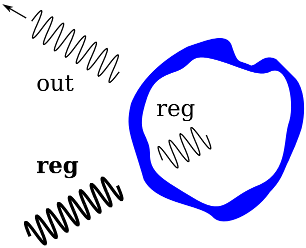
Wave equation

$$\left(\nabla \times \nabla \times + V\right)|\mathbf{E}^{\mathrm{tot}}\rangle = \omega^2 |\mathbf{E}^{\mathrm{tot}}\rangle$$

Lippmann-Schwinger:

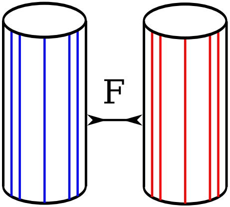
$$|\mathbf{E}^{\mathrm{tot}}\rangle = |\mathbf{E}^{\mathrm{hom}}\rangle - \mathcal{G}_0 \underbrace{\left(V \frac{1}{1 + \mathcal{G}_0 V}\right)}_{T} |\mathbf{E}^{\mathrm{hom}}\rangle$$

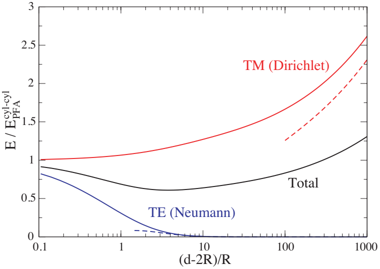
$$|\mathbf{E}^{\mathrm{tot}}\rangle = |\mathbf{E}^{\mathrm{reg}}_{lm}\rangle - \sum_{l'm'} |\mathbf{E}^{\mathrm{out}}_{l'm'}\rangle \underbrace{\langle \mathbf{E}^{\mathrm{reg}}_{l'm'} | T | \mathbf{E}^{\mathrm{reg}}_{lm}\rangle}_{\mathbb{T}_{l'm'lm}}$$



Example

$$\mathbf{E}^{\mathrm{tot}}(x)=\nabla\times\left(I_n(x_\parallel k)e^{iq_\perp x_\perp+in\theta}\hat{z}\right)+\mathbb{T}^{MM}_{nq_\perp nq_\perp}\nabla\times\left(K_n(x_\parallel k)e^{iq_\perp x_\perp+in\theta}\hat{z}\right)$$





Match boundaries

$$\mathbf{E}^{\parallel}(R)=0 \qquad \mathbf{B}^{\perp}(R)=0$$

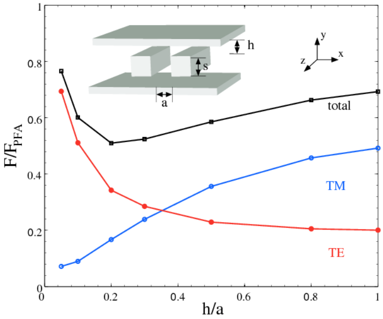
$$\mathbb{T}^{MM}_{nq_\perp n'q'_\perp}=-\frac{I'_n(kR)}{K'_n(kR)}\delta_{nn'}\delta(q_\perp-q'_\perp)$$

$$\mathbb{X}_{nq_\perp n'q'_\perp}=K_{n-n'}(kd)\delta(q_\perp-q'_\perp)$$

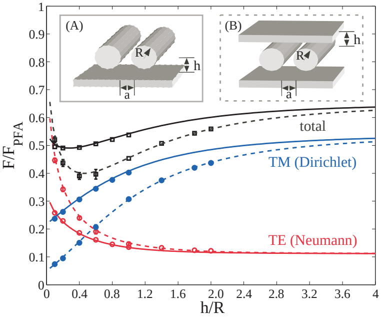
Result

$$E=-\frac{\hbar cL}{8\pi d^2\log^2(d/R)}$$

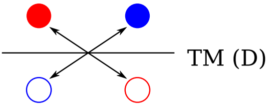
Non-monotonicity

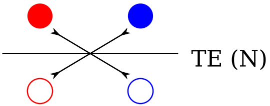


Rodriguez, AJ et al. (2007)



Rahi, SJ et al. (2008)



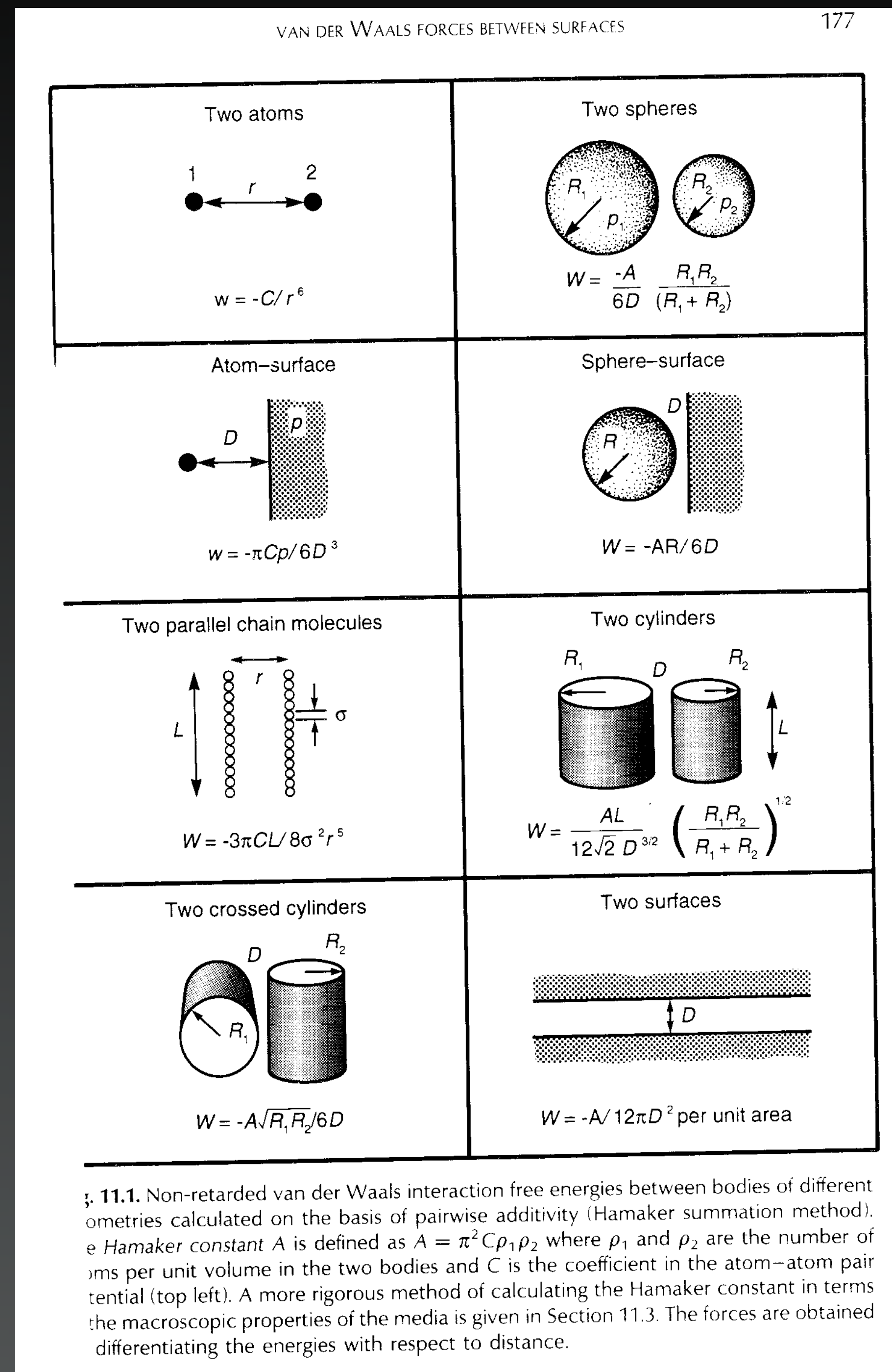


Outlook

- Material properties
- Objects *inside* one another
- Orientation dependence
- Attraction - repulsion

Applications...

Summation of 2-body interactions



ş. **11.1.** Non-retarded van der Waals interaction free energies between bodies of different ometries calculated on the basis of pairwise additivity (Hamaker summation method). e *Hamaker constant* A is defined as $A = \pi^2 C\rho_1\rho_2$ where ρ_1 and ρ_2 are the number of ıms per unit volume in the two bodies and C is the coefficient in the atom–atom pair tential (top left). A more rigorous method of calculating the Hamaker constant in terms the macroscopic properties of the media is given in Section 11.3. The forces are obtained differentiating the energies with respect to distance.

From: J.N. Israelachvili, *Intermolecular and Surface Forces*

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I. Atoms: Crossover vdW - Casimir

- Two-state approximation for dipole polarizability

$$\alpha = \frac{e^2}{m}\frac{f_{10}}{\omega_{10}^2 - \omega^2} \to \frac{(L/L_{10})^2}{(L/L_{10})^2 + u^2}\,\alpha_0 \quad \text{with} \quad \omega \to iuc/L$$

with static polarizability $\alpha_0 = f_{10} r_0 L_{10}^2$ and crossover length $L_{10} = c/\omega_{10},\ r_0 =$ Compton radius

- T-matrix: $T^{\mathrm{EE}}_{1m1m} = \frac{2}{3}\alpha(\kappa)\kappa^3$

- Interaction energy from T- and U-matrices:

$$\mathcal{E} = \frac{\hbar c}{2\pi}\frac{1}{L}\int_0^\infty du \log\left[\left(1 - 4(1+u)^2 e^{-2u}\frac{\alpha^2(u/L)}{L^6}\right)\left(1 - (1+u+u^2)^2 e^{-2u}\frac{\alpha^2(u/L)}{L^6}\right)^2\right]$$

- Casimir-Polder (retarded) limit, $L \gg L_{10}$

replace $\alpha(u/L)$ by α_0 expand in α_0/L^3 : $\boxed{\mathcal{E} = -\frac{23}{4\pi}\hbar c\,\frac{\alpha_0^2}{L^7}}$

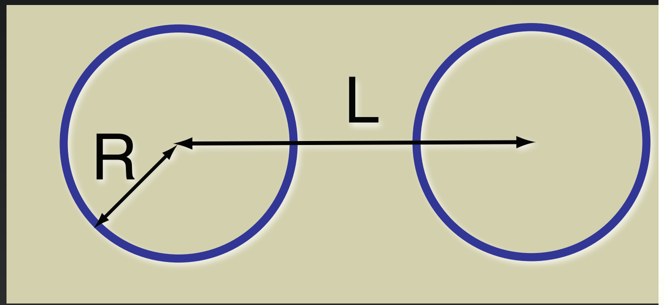
- London/vdW (non-retarded) limit, $L \ll L_{10}$

Frequency dependence of α important, but factors from U-matrices can be ignored.

Expansion in α_0 yields: $\mathcal{E} = -\frac{3}{4}\hbar c\,\frac{\alpha_0^2}{L_{10}L^6} = \boxed{-\frac{3}{4}\hbar\omega_{10}\,\frac{\alpha_0^2}{L^6}}$

II. Two spheres

- Dielectric spheres with $\epsilon(\omega),\ \mu(\omega)$
- T - matrix (Debye, dissertation, 1909) diagonal in polarization TE/TM and in l and independent of m



- Matrix elements for imaginary frequency $k = i\kappa$:

$$T^{11}_{lmlm} = (-1)^l \frac{\pi}{2} \frac{\eta I_{l+\frac{1}{2}}(z)\left[I_{l+\frac{1}{2}}(nz) + 2nzI'_{l+\frac{1}{2}}(nz)\right] - nI_{l+\frac{1}{2}}(nz)\left[I_{l+\frac{1}{2}}(z) + 2zI'_{l+\frac{1}{2}}(z)\right]}{\eta K_{l+\frac{1}{2}}(z)\left[I_{l+\frac{1}{2}}(nz) + 2nzI'_{l+\frac{1}{2}}(nz)\right] - nI_{l+\frac{1}{2}}(nz)\left[K_{l+\frac{1}{2}}(z) + 2zK'_{l+\frac{1}{2}}(z)\right]}$$

for TE channels (magnetic multipoles), $z = \kappa R\,, n = \sqrt{\epsilon(i\kappa)\mu(i\kappa)}\,, \eta = \sqrt{\epsilon(i\kappa)/\mu(i\kappa)}$
for TM channels (electric multipoles) same expression with

$$\begin{array}{ccc} \epsilon(i\kappa) & \rightarrow & \mu(i\kappa) \\ \mu(i\kappa) & \rightarrow & \epsilon(i\kappa) \end{array}$$

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Large separations

- Expansion in 1/L: we need low-frequency form of T - matrix
- Determined by **static** electric and magnetic multipole polarizability $\alpha_l^{\mathrm{E}} = [(\epsilon-1)/(\epsilon+(l+1)/l)]R^{2l+1}$ $\alpha_l^{\mathrm{M}} = [(\mu-1)/(\mu+(l+1)/l)]R^{2l+1}$

$$T_{lmlm}^{\mathrm{MM}} = \kappa^{2l}\left[\frac{(-1)^{l-1}(l+1)\alpha_l^{\mathrm{M}}}{l(2l+1)!!(2l-1)!!}\kappa + \gamma_{l3}^{\mathrm{M}}\kappa^3 + \gamma_{l4}^{\mathrm{M}}\kappa^4 + \ldots\right]$$

with finite-κ corrections

$$\gamma_{13}^{\mathrm{M}} = -[4+\mu(\epsilon\mu+\mu-6)]/[5(\mu+2)^2]R^5, \quad \gamma_{14}^{\mathrm{M}} = (4/9)[(\mu-1)/(\mu+2)]^2R^6$$

- Casimir energy:

$$\mathcal{E} = -\frac{\hbar c}{\pi}\left\{\left[\frac{23}{4}\left((\alpha_1^{\mathrm{E}})^2 + (\alpha_1^{\mathrm{M}})^2\right) - \frac{7}{2}\alpha_1^{\mathrm{E}}\alpha_1^{\mathrm{M}}\right]\frac{1}{L^7} + \frac{9}{16}\left[\alpha_1^{\mathrm{E}}(59\alpha_2^{\mathrm{E}} - 11\alpha_2^{\mathrm{M}} + 86\gamma_{13}^{\mathrm{E}} - 54\gamma_{13}^{\mathrm{M}}) + \mathrm{E}\leftrightarrow\mathrm{M}\right]\frac{1}{L^9} + \frac{315}{16}\left[\alpha_1^{\mathrm{E}}(7\gamma_{14}^{\mathrm{E}} - 5\gamma_{14}^{\mathrm{M}}) + \mathrm{E}\leftrightarrow\mathrm{M}\right]\frac{1}{L^{10}} + \ldots\right\}$$

Casimir-Polder

Feinberg-Sucher

New

No L^{-8} term!

Ideally conducting spheres

- Limit of perfect metal spheres follows from taking $\epsilon \to \infty, \quad \mu \to 0$

$$\mathcal{E} = -\frac{\hbar c}{\pi}\frac{R^6}{L^7}\sum_{n=0}^{\infty} c_n \left(\frac{R}{L}\right)^n$$

$$c_0 = \frac{143}{16},\ c_1 = 0,\ c_2 = \frac{7947}{160},\ c_3 = \frac{2065}{32},\ c_4 = \frac{27705347}{100800},\ c_5 = -\frac{55251}{64},$$
$$c_6 = \frac{1373212550401}{144506880},\ c_7 = -\frac{7583389}{320},\ c_8 = -\frac{2516749144274023}{44508119040},\ c_9 = \frac{274953589659739}{275251200}$$

(4 scatterings and l=5 partial waves)

- This is asymptotic expansion (not convergent) $\longrightarrow$ not applicable to shorter separations

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Exact results at all separations

- Numerical evaluation of the determinant and integral of

$$\mathcal{E} = \frac{\hbar c}{2\pi} \int_0^\infty d\kappa \, \ln \det(1 - \mathbb{T}^1 \mathbb{U}^{12} \mathbb{T}^2 \mathbb{U}^{21})$$

  by truncating matrices at finite multipole order l
  $\rightarrow 2l(2+l) \times 2l(2+l)$ - matrix
  no scattering / frequency expansion!

- This yields series $\mathcal{E}_l$ of energies

- Exponential convergence to exact result for $l \rightarrow \infty$:

$$|\mathcal{E}_l - \mathcal{E}| \sim e^{-\delta\,(L/R-2)\,l}, \quad \delta \sim \mathcal{O}(1)$$

- Slowest convergence for smallest separations, $L \rightarrow 2R$

- Program can be applied to all shapes, materials, fields

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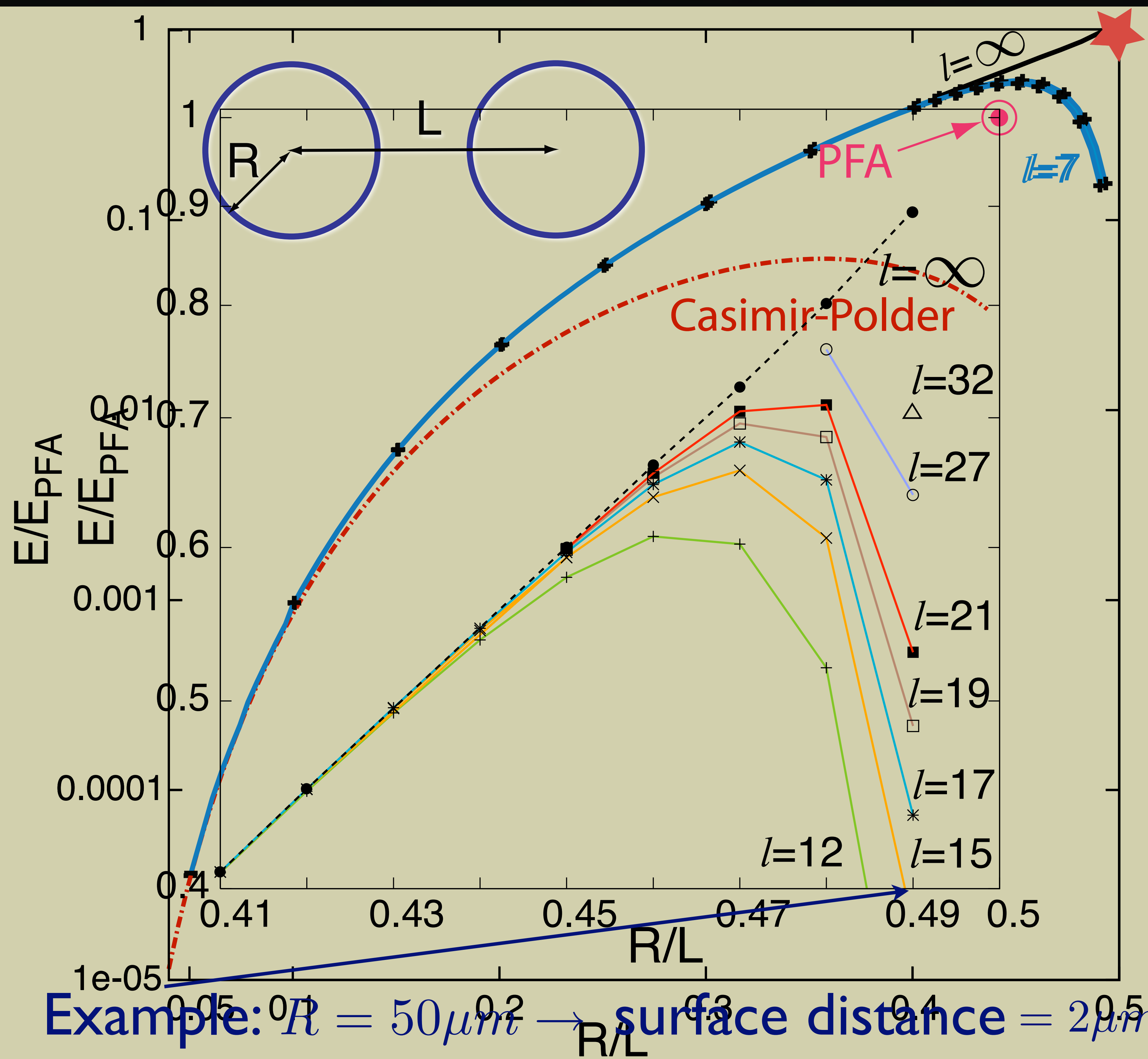
Two conducting spheres



Example: $R = 50\mu m \rightarrow$ surface distance $= 2\mu m$

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III. Sphere - Plane

- Large separations: dielectric sphere - conducting plane:

$$\begin{aligned}\mathcal{E} &= -\frac{\hbar c}{\pi}\left\{\frac{3}{8}(\alpha_1^{\mathrm{E}}-\alpha_1^{\mathrm{M}})\frac{1}{L^4}+\frac{15}{32}(\alpha_2^{\mathrm{E}}-\alpha_2^{\mathrm{M}}+2\gamma_{13}^{\mathrm{E}}-2\gamma_{13}^{\mathrm{M}})\frac{1}{L^6}\right.\\ &+ \boxed{\frac{1}{1024}\left[23(\alpha_1^{\mathrm{M}})^2-14\alpha_1^{\mathrm{M}}\alpha_1^{\mathrm{E}}+23(\alpha_1^{\mathrm{E}})^2\right.}+2160(\gamma_{14}^{\mathrm{E}}-\gamma_{14}^{\mathrm{M}})\Big]\frac{1}{L^7}\\ &+ \left.\frac{7}{7200}\left[572(\alpha_3^{\mathrm{E}}-\alpha_3^{\mathrm{M}})+675\left(9(\gamma_{15}^{\mathrm{E}}-\gamma_{15}^{\mathrm{M}})-55(\gamma_{23}^{\mathrm{E}}-\gamma_{23}^{\mathrm{M}})\right)\right]\frac{1}{L^8}+\dots\right\}\end{aligned}$$

$(1/2)(1/2^7)\times$ Casimir-Polder for 2 atoms

- Perfect metal sphere:

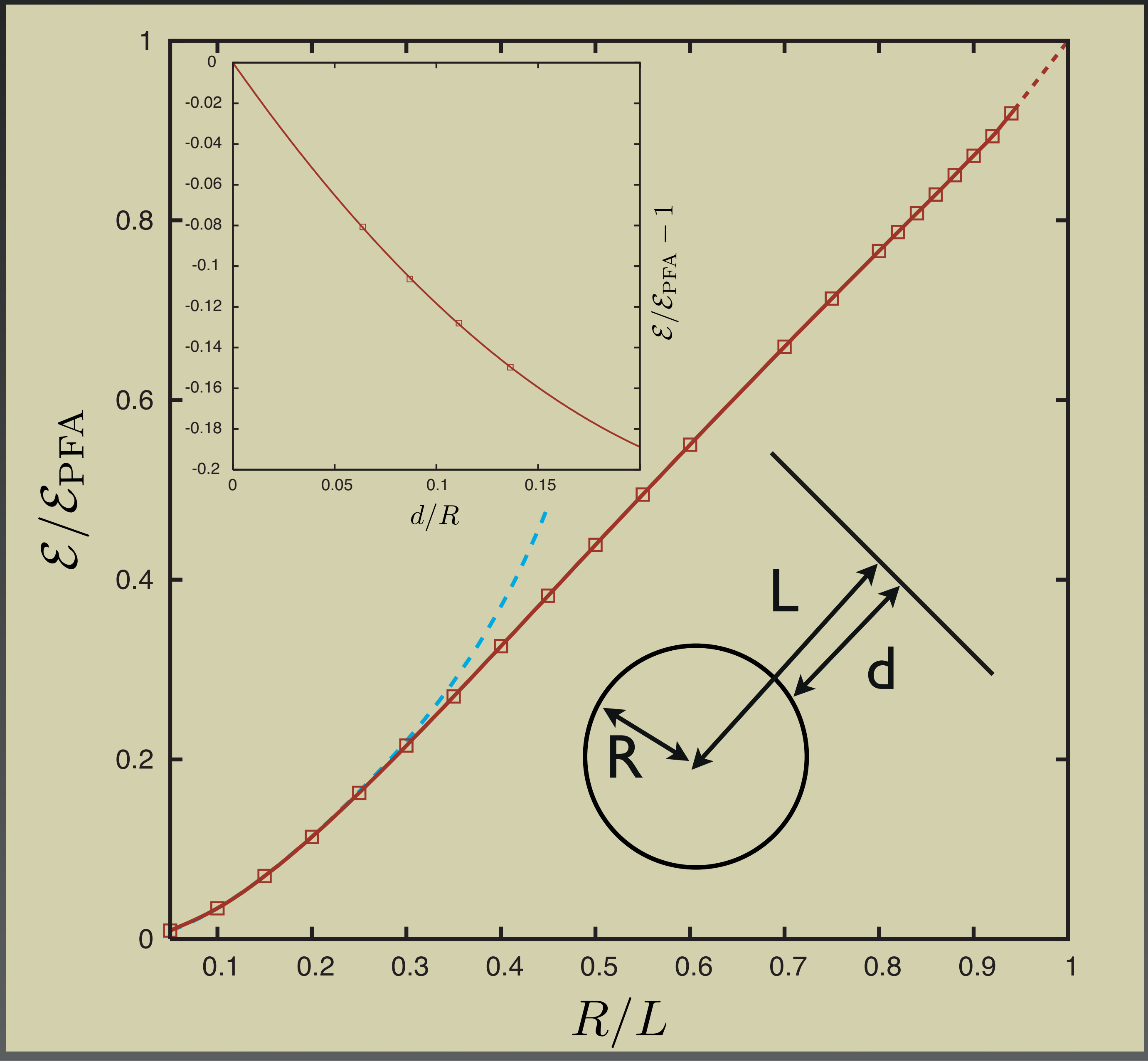
$$\mathcal{E}=\frac{\hbar c}{\pi}\frac{1}{L}\sum_{j=4}^{\infty}b_j\left(\frac{R}{L}\right)^{j-1}$$

$$\begin{aligned}b_4 &= -\frac{9}{16},\quad b_5=0,\quad b_6=-\frac{25}{32},\quad b_7=-\frac{3023}{4096}\\ b_8 &= -\frac{12551}{9600},\quad b_9=\frac{1282293}{163840},\\ b_{10} &= -\frac{32027856257}{722534400},\quad b_{11}=\frac{39492614653}{412876800}\end{aligned}$$

From atom to PFA

- Small separations: corrections to proximity force approx.

$$\mathcal{E} = \mathcal{E}_{\mathrm{PFA}}\left[1 + \theta_1 \frac{d}{R} + \theta_2 \left(\frac{d}{R}\right)^2 + \dots\right] \qquad \theta_1 = -1.42 \pm 0.02, \quad \theta_2 = 2.39 \pm 0.14$$



IV. Orientation dependence

- For objects of arbitrary shape energy in dipole approximation, exact to leading order in 1/d:

$$\begin{aligned}
\mathcal{E}_1^{12} &= -\frac{\hbar c}{d^7}\frac{1}{8\pi}\Big\{13\left(\alpha^1_{xx}\alpha^2_{xx}+\alpha^1_{yy}\alpha^2_{yy}+2\alpha^1_{xy}\alpha^2_{xy}\right)\\
&+ 20\,\alpha^1_{zz}\alpha^2_{zz}-30\left(\alpha^1_{xz}\alpha^2_{xz}+\alpha^1_{yz}\alpha^2_{yz}\right)+(\alpha\to\beta)\\
&- 7\left(\alpha^1_{xx}\beta^2_{yy}+\alpha^1_{yy}\beta^2_{xx}-2\alpha^1_{xy}\beta^2_{xy}\right)+(1\leftrightarrow 2)\Big\}
\end{aligned}$$

applies to all shapes and materials!

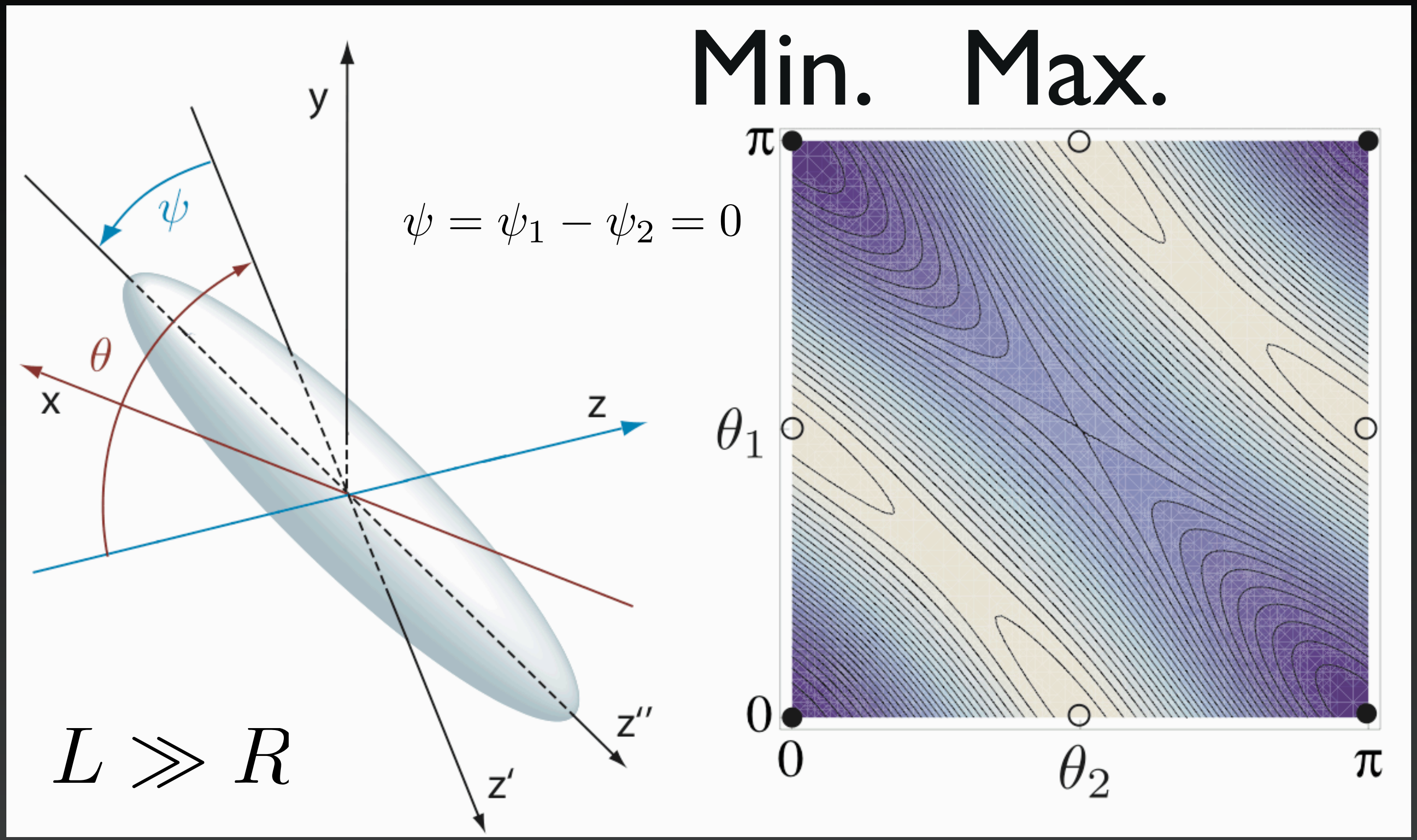
- Ellipsoids (ellipsoid with two equal axes $r_1 = r_2 = R, \quad r_3 = L/2$): electric, magnetic polarizability tensor

$$\alpha^0_{ii} = \frac{V}{4\pi}\frac{\epsilon-1}{1+\cancel{(\epsilon-1)n_i}}\,,\ \beta^0_{ii} = \frac{V}{4\pi}\frac{\mu-1}{1+\cancel{(\mu-1)n_i}}$$

for rarefied media
no orientation dependence!

$$n_i = \frac{r_1 r_2 r_3}{2}\int_0^\infty \frac{ds}{(s+r_i^2)\sqrt{(s+r_1^2)(s+r_2^2)(s+r_3^2)}}$$

Two needles



- Energy at large separation: minimal for parallel aligned needles

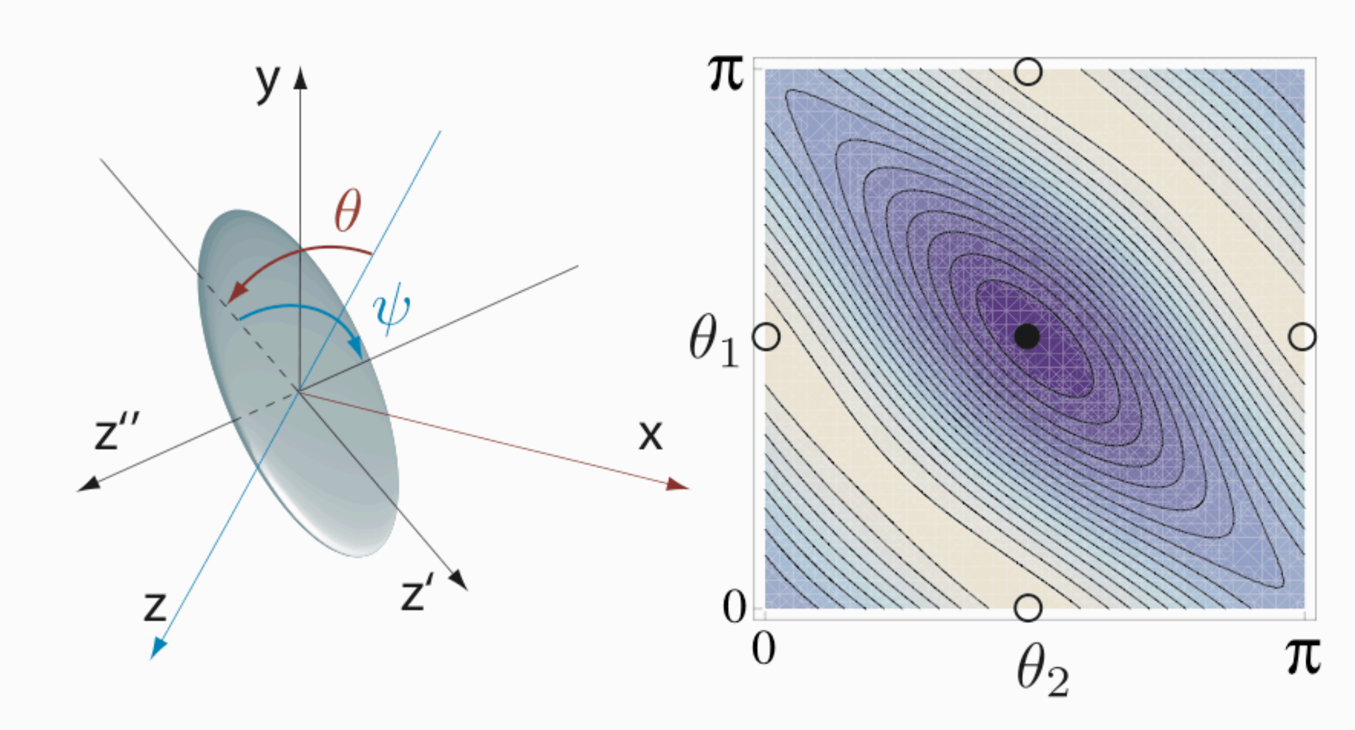
$$\mathcal{E}_1^{12}(\theta_1,\theta_2,\psi) = -\frac{\hbar c}{d^7}\left\{\frac{5L^6}{1152\pi\left(\ln\frac{L}{R}-1\right)^2}\left[\cos^2\theta_1\cos^2\theta_2 + \frac{13}{20}\cos^2\psi\sin^2\theta_1\sin^2\theta_2 - \frac{3}{8}\cos\psi\sin 2\theta_1\sin 2\theta_2\right] + \mathcal{O}\left(\frac{L^4R^2}{\ln\frac{L}{R}}\right)\right\}$$

- L^6 - scaling disappears for certain orientations, and energy is

$$\mathcal{E}_1^{12}\left(\frac{\pi}{2},\theta_2,\frac{\pi}{2}\right) = -\frac{\hbar c}{1152\pi\, d^7}\frac{L^4R^2}{\ln\frac{L}{R}-1}\left(73 + 7\cos 2\theta_2\right)$$

maximal energy for “crossed” needles

Two pancakes



- Energy: minimal for pancakes lying on same plane

$$\mathcal{E}_1^{12} = -\frac{\hbar c}{d^7}\left\{ \frac{R^6}{144\pi^3}\left[765 - 5(\cos 2\theta_1 + \cos 2\theta_2) + 237\cos 2\theta_1 \cos 2\theta_2 \right.\right.$$

$$\left.\left. +372\cos 2\psi \sin^2\theta_1 \sin^2\theta_2 - 300\cos\psi \sin 2\theta_1 \sin 2\theta_2 \right] + \mathcal{O}(R^5 L) \right\}$$

- R^6 - scaling does not disappear for any orientation, independent of thickness L

Plane - spheroid

- Perfectly conducting plane:

$$\mathcal{E}_1^{1m} = -\frac{\hbar c}{d^4}\frac{1}{8\pi}\mathrm{Tr}(\alpha - \beta) + \mathcal{O}(d^{-5})$$

  independent of orientation in dipole approximation!

- Preferred orientation? Consider deformed sphere with "radius" $R + \delta f(\vartheta, \varphi)$

- Energy for $f = Y_{20}(\vartheta, \varphi)$

$$\mathcal{E}_f = -\hbar c \frac{1607}{640\sqrt{5}\pi^{3/2}}\frac{\delta R^4}{d^6}\cos(2\theta)$$

  minimal if needle points to plane / pancake perpendicular to mirror

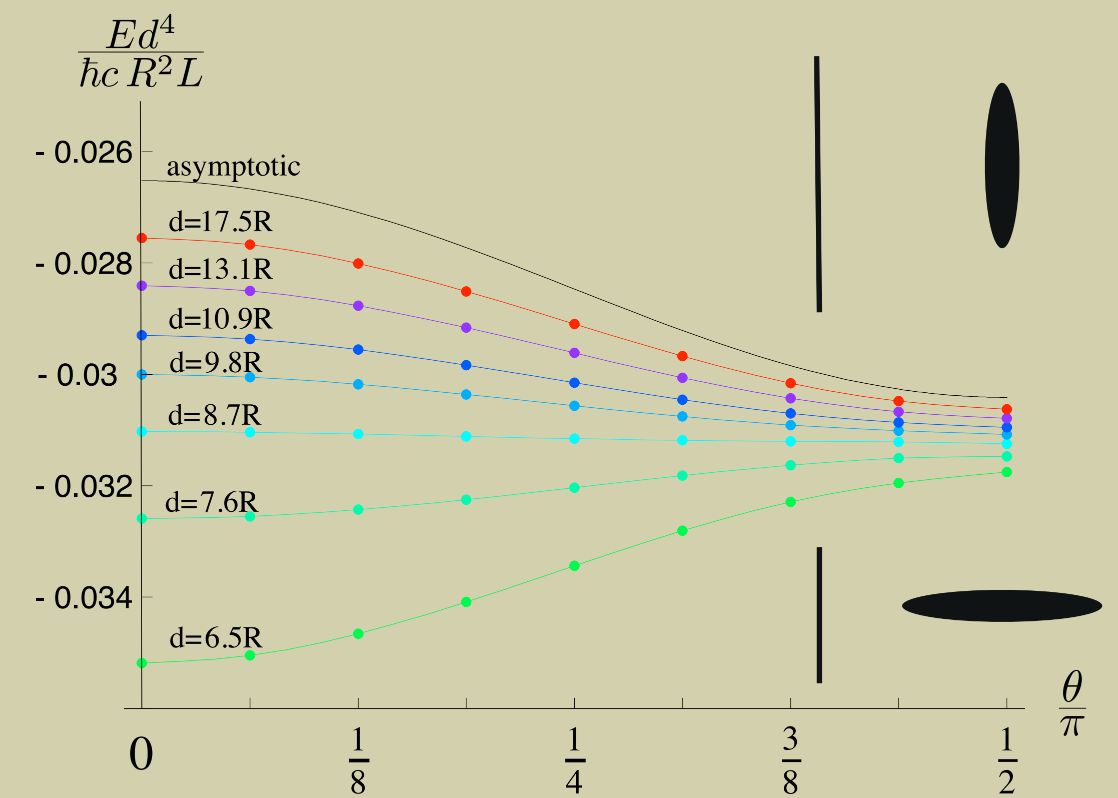
Plane - spheroid

- Neumann boundary condition on both surfaces:
  Energy for object of general shape (β = magn. polarizability of perf. cond.)

$$\mathcal{E} = -\frac{\hbar c}{d^4}\left[\frac{1}{64\pi^2}V - \frac{1}{16\pi}(\beta_{xx}+\beta_{yy}+3\beta_{zz})\right]$$

$$= -\frac{\hbar c}{d^4}\frac{R^2 L}{96\pi}\left[9-\cos(2\theta)+\mathcal{O}((R/L)^2)\right]$$

- Min. energy at large separation for needle parallel to plane.
  PFA suggest the opposite orientation at short separation
- Full T-matrix of spheroid (known in spheroidal basis): energy at all separations:



V. Nano tubes/wires

- Consider dielectric spheroids ("needles") and infinitely long cylinders (T-matrix known exactly)
- Correlations between shape and material properties
- Plasma response: $\epsilon(\omega) = 1 - \omega_p^2/\omega^2$
  2 wires of *finite length* L << separation d: non-retarded limit:

$$\mathcal{E} = -\frac{1}{2304}\frac{\hbar\omega_p}{d^6}\frac{RL^5}{\log^{3/2}(L/R)}$$

[cf. J. Dobson]

- *Infinite long* wires for $d \ll R\exp[I_0(R\omega_p/c)/(R\omega_p I_1(R\omega_p/c))] \to R\exp[\sqrt{2}c/R\omega_p] \quad \text{for} \quad R\omega_p/c \ll 1$

$$\begin{aligned}\frac{\mathcal{E}}{L} &= -\frac{\hbar c}{16\pi}\sqrt{\frac{R\omega_p/c\; I_1(R\omega_p/c)}{I_0(R\omega_p/c)}}\frac{1}{d^2\log^{3/2}(2d/R)} \\ &\to -\frac{\hbar}{16\sqrt{2}\pi}\frac{R\omega_p}{d^2\log^{3/2}(2d/R)} \quad \text{for} \quad R\omega_p \ll 1\end{aligned}$$

VAN DER WAALS ATTRACTION BETWEEN TWO CONDUCTING CHAINS

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The Van der Waals force between two conducting chains is shown from the zero point energies of the strongly spatially dispersive plasmon modes to vary at small separations L as L^{-3} instead of L^{-6} as for nonconducting chains.

Force / length:

$$F(L) \approx -\hbar\omega_p a/8\sqrt{2}\pi L^3 [\ln(2L/a)]^{3/2}$$

[mentioned by J. Dobson]

Nano tubes/wires

- Drude metal: $\epsilon(\omega) = 1 + 4\pi i \frac{\sigma}{\omega}$
- 2 wires of *finite length* L << separation d: (non-retarded)

$$\mathcal{E} = -\frac{1}{72}\frac{\hbar\sigma}{d^6}\frac{L^4 R^2}{\log^1(L/R)}$$

- *Infinitely long* wires for $d \gg R,\, \sigma R^2/c$

$$\frac{\mathcal{E}}{L} = -\frac{\pi}{32}\hbar\sigma\frac{R^2}{d^3 \log^1(2d/R)}$$

- Compare to universal (perfect metal) energy for $\sigma \to \infty$

$$\frac{\mathcal{E}}{L} = -\frac{\hbar c}{8\pi}\frac{1}{d^2 \log^2(2d/R)}$$