Microscopic theory of the Casimir effect at thermal equilibrium: large separation asymptotics

Joint work with Pascal Buenzli

Microscopic origin of universality in Casimir forces
J. Stat. Phys. 119, 273, 2005

Thermal quantum electrodynamics of non relativistic charged fluids
Phys.Rev. E, 75, 041125, 2007

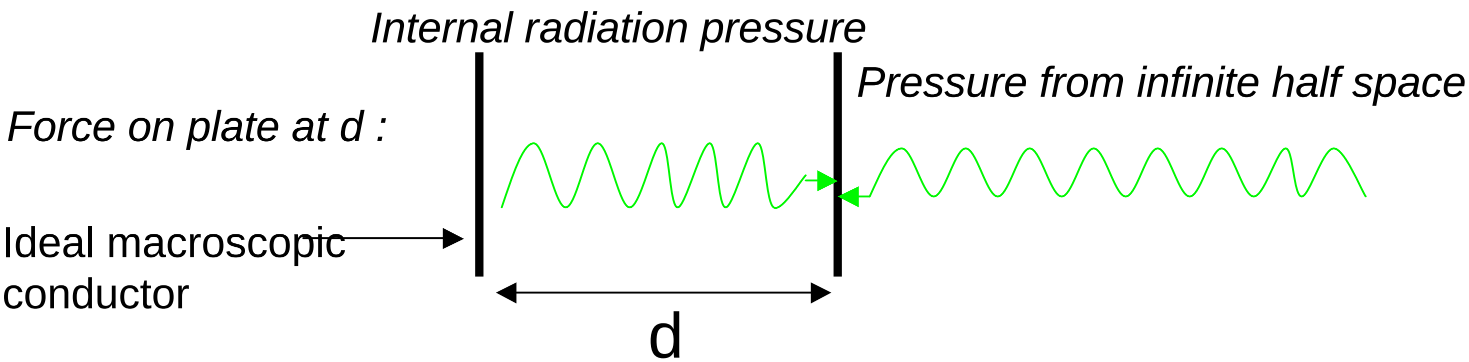
Microscopic theory of the Casimir effect at thermal equilibrium: large separation asymptotics
Phys. Rev. E, 77,011114, 2008

Main issue dicussed in this talk:

Role of thermal fluctuations in the electromagnetic Casimir effect

We briefly recall:

Standard calculation of the Casimir force at T=0



In standard calculation of the Casimir force, the plates are treated as **macroscopic** conductors (vanishing of the tangential electric field).

Boundary conditions leads to a d-dependence of the electromagnetic spectrum, which is the source of the Casimir force.

Field and charge fluctuations inside the conductors are ignored: ***dead conductors***

Casimir's result:

$$f^{\mathrm{vac}}(d) = -\frac{\pi^2 \hbar c}{240 d^4}$$

Part of the force due to thermal fluctuations

Fierz, 1960 ; Mehra, 1967

A new length: the thermal wave length of the photon : $\beta\hbar c$

Dimensionless parameter : $\alpha = \dfrac{\beta\hbar c}{d}$

α >> 1 ⟷ Low temperature or small distance

α <<1 ⟷ High temperature or large distance

The **free energy** of a photon :

Total free energy :

$$\Phi_{T,\Lambda} = -\sum_{\mathbf{k}\ \lambda}{}' \beta^{-1} \ln\left(\sum_{n=0}^{\infty} e^{-\beta\hbar\omega_{\mathbf{k}} n}\right)$$

Free energy per unit surface:

$$\varphi_T(d) = \lim_{L\to\infty} \frac{\Phi_{T,\Lambda}}{L^2}$$

Radiation pressure between the plates :

$$p_T^{\mathrm{rad}}(d) = -\frac{\partial}{\partial d}\varphi_T(d)$$

The total force is

$$f(d) = p_T^{\mathrm{rad}}(d) - p_T^{\mathrm{rad}}(\infty) + f^{\mathrm{vac}}(d)$$

Short distance or low temperature limit $\alpha \longrightarrow \infty$

$$p_T^{\mathrm{rad}}(d) = \frac{\pi}{d^3\beta}\left[e^{-\alpha} + \mathcal{O}\left(e^{-2\alpha}\right)\right]$$

the pressure of a very thin black body is exponentially small

Low temperature-short distance

$$f(d) = -\underbrace{\frac{\pi^2\hbar c}{240 d^4}}_{\text{Casimir force}} - \frac{\pi^2}{45}\underbrace{\frac{1}{\beta^4\hbar^3c^3}}_{\text{black body pressure}} + \frac{\pi}{d^3\beta}\mathcal{O}\left(e^{-\alpha}\right), \quad \alpha \to \infty$$

Long distance or high temperature limit $\alpha \longrightarrow 0$

$$p_T^{\mathrm{rad}}(d) = \frac{\pi^2}{45}\frac{1}{\beta^4\hbar^3c^3} - \frac{\zeta(3)}{4\pi\beta d^3} + \frac{\pi^2\hbar c}{240 d^4} + \mathcal{O}\left(\exp\left(-\frac{b}{\alpha}\right)\right)$$

Black body radiation pressure

Classical term independent of Plank's constant

Casimir term with opposite sign

Exponentially small corrections

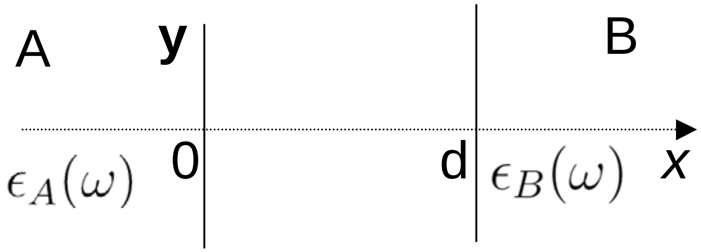
High temperature-long distance

$$f(d) = -\frac{\zeta(3)}{\boxed{4}\pi\beta d^3} + \mathcal{O}\left(\exp\left(-\frac{b}{\alpha}\right)\right), \qquad \alpha \longrightarrow 0$$

The part due to vacuum fluctuation is cancelled. Thermal fluctuations dominate. The asymptotic force is classical (no dependance of h and c)

Lifshitz theory of the force between dielectric bodies (1956) :

characterizes the physical properties of the dielectrics by their frequency dependent dielectric functions



- The actual fields are realisations of a ***stochastic process*** generated by a ***random polarization***

$$\mathbf{P}(\mathbf{x},\omega) = \overline{\mathbf{P}}(\mathbf{x},\omega) + \frac{\mathbf{K}(\mathbf{x},\omega)}{4\pi} \longleftarrow$$

Random polarization due to quantum and thermal fluctuations of matter and fields

and obey stochastic Maxwell equations.

- The ***random polarization*** obeys the ***fluctuation-dissipation*** theorem.

In the high temperature limit $T \to \infty$ and the perfect conductor limit $\epsilon \to \infty$ Lifshitz finds

$$f(d) \sim -\frac{\zeta(3)}{\boxed{8}\pi\beta d^3}$$

Schwinger, De Raad and Milton, Ann. Phys., 1978

« Lifshitz found a temperature dependence which disagrees with that found in other calculations. We show that the error arises only in the limit taken to recover the conductor case »

To make the Lifshitz formula effective one needs a model of the frequency dependent dielectric function $\epsilon(\omega)$

Drude model: $\epsilon(\omega) \sim \frac{4\pi i\sigma}{\omega}$ **Plasma model** : $\epsilon(\omega) \sim 1 - \frac{\omega_p^2}{\omega^2}$

α >> 1

$$f(d) \sim -\frac{\pi^2 \hbar c}{240 d^4} + O(T^4), \qquad r^{\mathrm{TE}}(0, \boldsymbol{k}) = 1 \qquad \text{plasma}$$

$$f(d) \sim -\frac{\pi^2 \hbar c}{240 d^4} + \frac{\zeta(3) k_B T}{8\pi d^3} + O(T^4), \qquad r^{\mathrm{TE}}(0, \boldsymbol{k}) = 0 \qquad \text{Drude}$$

α <<1

$$f(d) \sim -\frac{\zeta(3) k_B T}{4\pi d^3}, \qquad r^{\mathrm{TE}}(0, \boldsymbol{k}) = 1 \qquad \text{plasma}$$

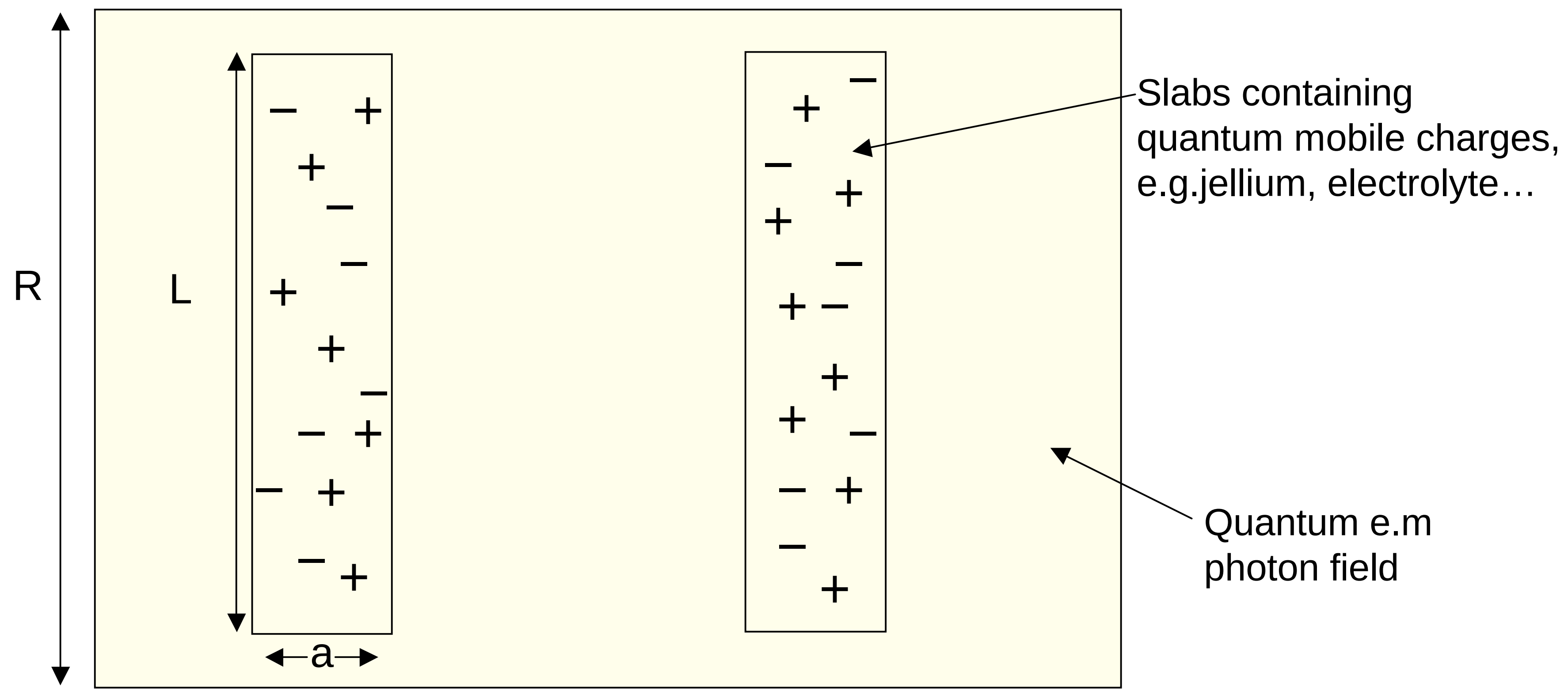
$$f(d) \sim -\frac{\zeta(3) k_B T}{8\pi d^3}, \qquad r^{\mathrm{TE}}(0, \boldsymbol{k}) = 0. \qquad \text{Drude}$$

Decide about the 1/2 factor from first principle without
Using the Lifshitz theory $\longrightarrow$ fully microscopic theory

Two principles

- Quantum electrodynamics of non relativistic charged particles
- Equilibrium statistical mechanics

The model: **living conductors**



Hamiltonian of non relativistic charges coupled to the electromagnetic field through Maxwell equations in transverse gauge

$$H_{L,R} = \sum_{i=1}^{N} \frac{\left(\mathbf{p}_i - \frac{e_{\gamma_i}}{c}\mathbf{A}(\mathbf{r}_i)\right)^2}{2m_{\gamma_i}} + \sum_{i<j}^{N} \frac{e_{\gamma_i} e_{\gamma_j}}{|\mathbf{r}_i - \mathbf{r}_j|} + \sum_{i=1}^{N} V_{\mathrm{ext}}(\gamma_i, \mathbf{r}_i) + H_0^{\mathrm{rad}}$$

Coupling of particles to the radiation field

Coulomb interaction

External potential confining the particles in the slabs

Free field energy

Free field energy in terms of photon creation and annihilation operators:

$$H_0^{\mathrm{rad}} = \sum_{\mathbf{k}\lambda} \hbar\omega_{\mathbf{k}}\, a_{\mathbf{k}\lambda}^{\dagger} a_{\mathbf{k}\lambda}$$

Vector potential in transverse gauge *Div* **A**=0 :

$$\mathbf{A}(\mathbf{r}) = \left(\frac{4\pi\hbar c^2}{R^3}\right)^{1/2} \sum_{\mathbf{k}\lambda} g(k) \frac{\mathbf{e}_{\mathbf{k}\lambda}}{\sqrt{2\omega_{\mathbf{k}}}} (a_{\mathbf{k}\lambda}^{\dagger} e^{-i\mathbf{k}\cdot\mathbf{r}} + a_{\mathbf{k}\lambda} e^{i\mathbf{k}\cdot\mathbf{r}})$$

Ultraviolet cut-off

$$g(k) = 0 \qquad k > k_{\mathrm{cut}} = \bar{m}c/\hbar.$$

The total free energy is

Partition function of the free photon field

$$\Phi_{R,L,d} = -k_B T \ln \mathrm{Tr} e^{-\beta H} = -k_B T \ln \left(\frac{\mathrm{Tr} e^{-\beta H}}{Z_0^{\mathrm{rad}}} \right) - k_B T \ln Z_0^{\mathrm{rad}}$$

The force between the slabs per unit surface

$$f(d) = \lim_{L\to\infty} \lim_{R\to\infty} -\frac{1}{L^2} \frac{\partial}{\partial d} \Phi_{R,L,d} = k_B T \lim_{L\to\infty} \lim_{R\to\infty} \frac{1}{L^2} \frac{\partial}{\partial d} \ln \left(\frac{\mathrm{Tr} e^{-\beta H}}{Z_0^{\mathrm{rad}}} \right)$$

Problem: find the asymptotic behaviour of the force for large separation d

Hierarchy of lengths:

$$\lambda_{\mathrm{mat}} = \hbar\sqrt{\beta/m}$$

thermal wave length of the particles

thermal wave length of the photons

$$\lambda_{\mathrm{ph}} = \beta\hbar c$$

$$\lambda_{\mathrm{cut}} = \frac{\lambda_{\mathrm{mat}}}{\sqrt{\beta m c^2}} \ll \lambda_{\mathrm{mat}} \ll \lambda_{\mathrm{ph}} = \sqrt{\beta m c^2}\lambda_{\mathrm{mat}} \ll d,$$

$$\lambda_{\mathrm{screen}} \ll a, b \ll d.$$

Result

exact: involves no approximations or intermediate assumptions

$$f(d) = -\frac{\zeta(3)}{8\pi\beta d^3} + R(\beta,\hbar,d)\,, \qquad R(\beta,\hbar,d) = \mathcal{O}(d^{-4})$$

Universal classical Casimir amplitude

The asymptotic force is:

- independent of h and c
- the factor is 1/8 and not 1/4, supporting that TE modes do not contribute in this regime
- universal with respect to the microscopic constitution of the plates
- does not require regularization procedures

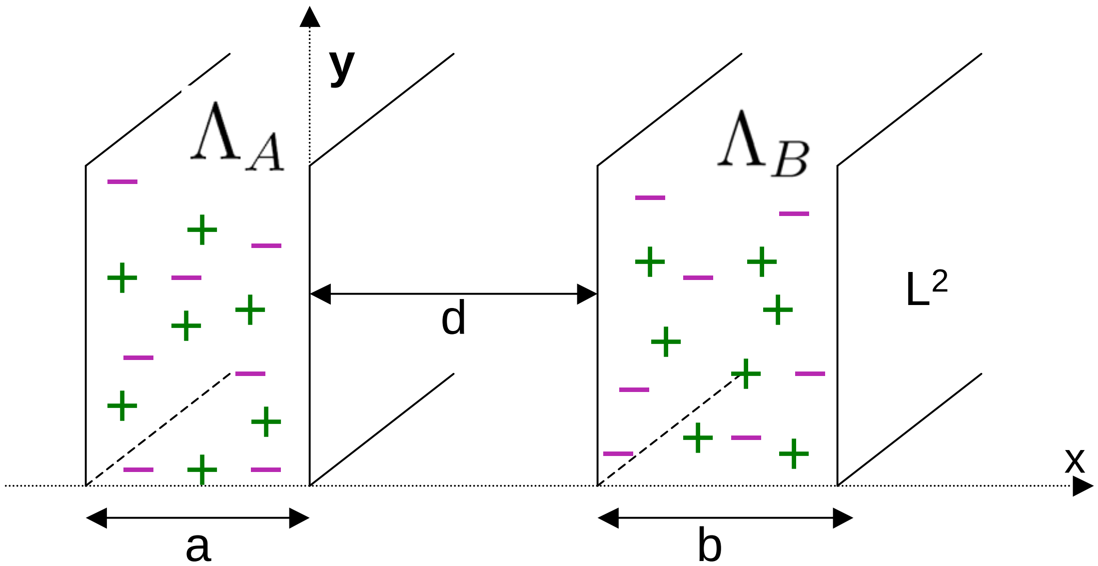
Subdominant terms depend on h and c and contain non universal contributions

Particle fluctuations inside the conductors account for reducing the force by a factor 1/2.

Calculations of the Casimir force based on macroscopic boundary conditions are not correct when the temperature is different from zero.

Statistical theory of the classical Casimir effect

Particles are classical, no photons



P. Buenzli and Ph. A. Martin
J. Stat. Phys. 2005

Forrester, Jancovici and Téllez
Jancovici and Téllez, 1996

Each slab contains classical charges of various species e_γ

Each slab is globally neutral

Particles interact by Coulomb potential :

$$V(\gamma,\gamma',|\mathbf{r}-\mathbf{r}'|) = e_\gamma e_{\gamma'} v(\mathbf{r}-\mathbf{r}') + v_{\mathrm{SR}}(\gamma,\gamma',|\mathbf{r}-\mathbf{r}'|)$$

Coulomb *Short range repulsion*

The total potential energy is $U = U_A + U_B + U_{AB}$

Particles in Λ_A *Particles in* Λ_B *Pair interactions between* Λ_A *and* Λ_B

The two slabs are in thermal equilibrium at the same temperature T

with Gibbs weight $\exp(-\beta U)$

The force per unit surface is

$$\langle f \rangle := \lim_{L\to\infty} \frac{\langle F_x \rangle_L}{L^2} = -\int_{-a}^{0} \mathrm{d}x \int_{d}^{d+b} \mathrm{d}x' \int \mathrm{d}\mathbf{y}\, \frac{x - x'}{\left[(x-x')^2 + |\mathbf{y}|^2\right]^{3/2}}\, c(x, x', |\mathbf{y}|).$$

$$= \frac{1}{2\pi} \int_{-a}^{0} \mathrm{d}x \int_{d}^{d+b} \mathrm{d}x' \int \mathrm{d}\mathbf{k}\, \mathrm{e}^{-k|x-x'|} S(x, x', \mathbf{k})$$

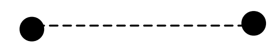
Inhomogeneous static structure function of the two slab system

Central problem : Find the asymptotic form of the charge correlation function between the two slabs for large d.

Study the Ursell function by the techniques of Mayer graphs and integral equations for Coulomb fluids

Ursell function $h(i,j) := \dfrac{\rho(i,j)}{\rho(i)\rho(j)} - 1$

Mayer bond $f(i,j) = e^{-\beta V(i,j)} - 1$

Screened potential : chain resummations $-\beta v(i,j)$ = 

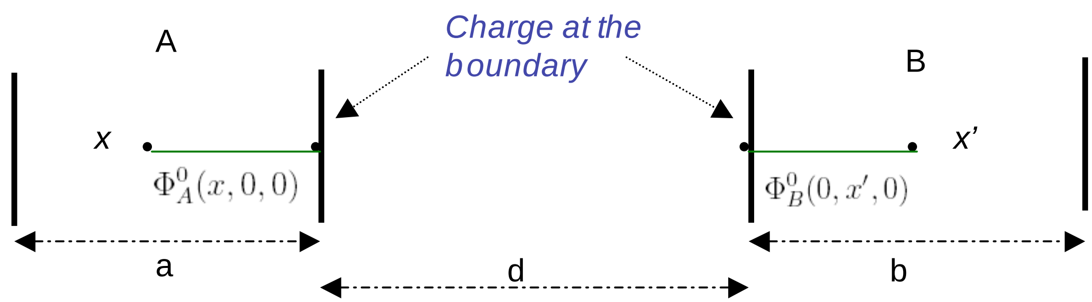
$$\Phi(\mathbf{r},\mathbf{r}') = v(\mathbf{r}-\mathbf{r}') - \frac{1}{4\pi}\int \mathrm{d}\mathbf{r}_1\, \kappa^2(\mathbf{r}_1)\; v(\mathbf{r}-\mathbf{r}_1)\; \Phi(\mathbf{r}_1,\mathbf{r}') = \Phi(\mathbf{r}',\mathbf{r})$$

$$\Delta\Phi(\mathbf{r},\mathbf{r}') - \kappa^2(\mathbf{r})\Phi(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}')$$ ← *Inhomogeneous Debye-Hückel equation*

where $\kappa(\mathbf{r}) := \left(4\pi\beta \sum e_\gamma^2 \rho(\gamma\,\mathbf{r})\right)^{1/2}$ is the local inverse Debye screening length $\ell_D(\mathbf{r})$

Asymptotic potential

$$\Phi_{AB}(x, x', \tfrac{q}{d}) \sim \frac{q}{4\pi d \sinh q} \Phi_A^0(x,0,0)\Phi_B^0(0,x',0), \quad d \to \infty$$

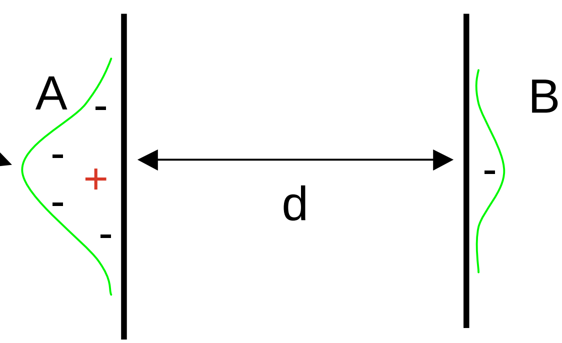


Electroneutrality sum rules

The total charge of the screening cloud around a specified charge in the system compensates it exactly

Interpretation in the slab geometry :

Screening cloud

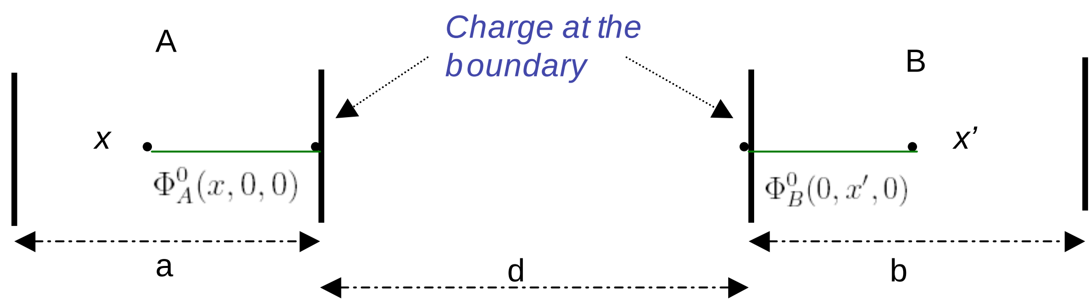


$$\int d\mathbf{r} \sum_{\gamma} e_\gamma \rho(\gamma, \mathbf{r}) h(\gamma\, \mathbf{r}, \gamma'\, \mathbf{r}') = -e_{\gamma'}$$

charge density at $\mathbf{r}$ conditionned by the presence of a charge $e_{\gamma'}$ at $\mathbf{r}'$

Asymptotic potential

$$\boxed{\Phi_{AB}(x,x',\tfrac{q}{d}) \sim \frac{q}{4\pi d \sinh q}\Phi_A^0(x,0,0)\Phi_B^0(0,x',0), \quad d\to\infty}$$



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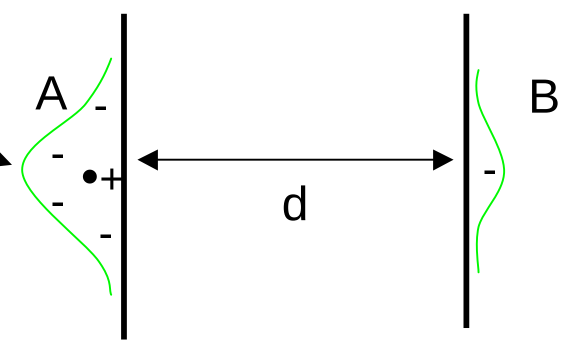

$$\int d\mathbf{r} \underbrace{\sum_\gamma e_\gamma \rho(\gamma,\mathbf{r}) h(\gamma\,\mathbf{r},\gamma'\,\mathbf{r}')}_{} = -e_{\gamma'}$$

charge density at $\mathbf{r}$ conditionned by the presence of a charge $e_{\gamma'}$ at $\mathbf{r}'$

Final result $$\boxed{\langle f \rangle (d) \sim -\frac{\zeta(3)}{8\pi\beta d^3}}$$

Decoupling of classical matter and radiation:

The Bohr-van Leuween theorem

Classical matter in thermal equilibrium decouples from the transverse part of the electromagnetic field

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Proof :

Shift the variable $\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{r}) \longrightarrow \mathbf{p}$ in the momentum integral in the partition function

➔ $\int_\Lambda d\mathbf{r} \int d\mathbf{p} \exp\left[-\beta \frac{\left(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{r})\right)^2}{2m}\right] \exp(-\beta U(\mathbf{r}))$ independant of $\mathbf{A}(\mathbf{r})$!

In the long distance-high temperature limit, the system tends to behave classically. The Casimir effect becomes dominated by pure electrostatic forces.

The general quantum model: Use functional integral representation $\longrightarrow$ classical-like formalism

Bosonic functional integral representation of the photon field

Coherent state for mode $(\mathbf{k}\lambda)$ $\quad a_{\mathbf{k}\lambda}|\alpha_{\mathbf{k}\lambda}\rangle = \alpha_{\mathbf{k}\lambda}|\alpha_{\mathbf{k}\lambda}\rangle \quad |\boldsymbol{\alpha}\rangle = \prod_{\mathbf{k}\lambda}|\alpha_{\mathbf{k}\lambda}\rangle$

leads to the functional integral representation:

$$\langle\boldsymbol{\alpha}|\mathrm{e}^{-\beta H}|\boldsymbol{\alpha}\rangle = \mathrm{e}^{-\beta D_N} \lim_{\eta\to 0_+}\left[\int_{\boldsymbol{\alpha}(0)=\boldsymbol{\alpha}}^{\boldsymbol{\alpha}(1)=\boldsymbol{\alpha}} \mathrm{d}[\boldsymbol{\alpha}(\cdot)]\, \mathrm{e}^{-\int_0^1 \mathrm{d}\tau\left(\boldsymbol{\alpha}^*(\tau)\frac{\partial}{\partial\tau}\boldsymbol{\alpha}(\tau)+\beta H_0^{\mathrm{rad}}(\boldsymbol{\alpha}(\tau))\right)} \mathcal{T}\left[\mathrm{e}^{-\beta\int_0^1 \mathrm{d}\tau H_{\mathbf{A}}(\boldsymbol{\alpha}(\tau))}\right]\right]_\eta$$

Imaginary time propagator for the particles in the **time dependent classical vector potential**

$$\mathbf{A}(\mathbf{r},\boldsymbol{\alpha}(\tau)) = \left(\frac{4\pi\hbar c^2}{R^3}\right)^{1/2}\sum_{\mathbf{k}\lambda} g(k)\frac{\mathbf{e}_{\mathbf{k}\lambda}}{\sqrt{2\omega_{\mathbf{k}}}}\left(\alpha_{\mathbf{k}\lambda}^*(\tau)\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}} + \alpha_{\mathbf{k}\lambda}(\tau)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}\right)$$

Feynman-Kac-Ito path integral representation

Particle in an external potential

Normalized conditional Wiener measure for closed path $\boldsymbol{\xi}(s)$

$$\langle \mathbf{r}|\exp\left(-\beta\frac{\left(\mathbf{p}-\frac{e_{\gamma_i}}{c}\mathbf{A}(\mathbf{r}\;)\right)^2}{2m}-\beta V^{\mathrm{ext}}\right)|\mathbf{r}\rangle=\left(\frac{1}{2\pi\lambda^2}\right)^{3/2}\int D(\boldsymbol{\xi})$$

Thermal wave length

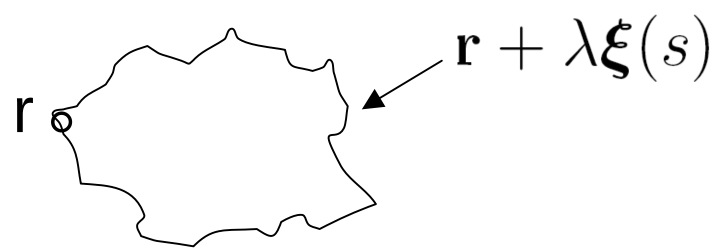
$$=\quad \exp\left(-i\sqrt{\frac{\beta e^2}{mc^2}}\int_0^1 \mathbf{A}(\mathbf{r}+\lambda\xi(s))\cdot d\xi(s)\right)\exp\left(-\beta\int_0^1 ds V^{\mathrm{ext}}((\mathbf{r}+\lambda\boldsymbol{\xi}(s))\right)$$

Flux of the magnetic field *Action of the potential*

Classical-like structure

Point quantum charge ➔ Random charged filament

q• ➔



$\boldsymbol{\xi}(s),\; 0\le s<1,\; \boldsymbol{\xi}(0)=\boldsymbol{\xi}(1)=0$ *Brownian path* → *internal degree of freedom*

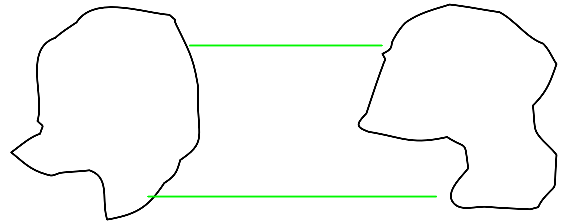
q ➔ (r, $\boldsymbol{\xi}$) *Enlarged phase space*

Many particle system

Pair Coulomb interaction

$$V(\mathbf{r}_i,\boldsymbol{\xi}_i,\mathbf{r}_j,\boldsymbol{\xi}_j)=\int_0^1 ds\frac{1}{|\mathbf{r}_i+\lambda_{\gamma_i}\boldsymbol{\xi}_i(s)-\mathbf{r}_j-\lambda_{\gamma_j}\boldsymbol{\xi}_j(s)|}$$

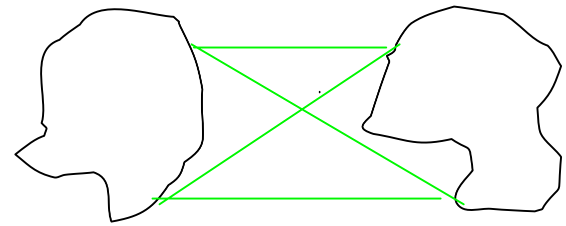
Equal times



Important observation

Decompose $V(i,j)=V_{\mathrm{elec}}(i,j)+W_c(i,j)$

$$V_{\mathrm{elec}}(i,j)=\int_0^1 ds_1\int_0^1 ds_2\frac{1}{|\mathbf{r}_i+\lambda_{\gamma_i}\boldsymbol{\xi}_i(s_1)-\mathbf{r}_j-\lambda_{\gamma_j}\boldsymbol{\xi}_j(s_2)|}$$



genuine classical electrostatic potential between two charged wires :

$$W_c(i,j)=\int_0^1 ds_1\int_0^1 ds_2(\delta(s_1-s_2)-1)\frac{1}{|\mathbf{r}_i+\lambda_{\gamma_i}\boldsymbol{\xi}_i(s_1)-\mathbf{r}_j-\lambda_{\gamma_j}\boldsymbol{\xi}_j(s_2)|}$$

part of $V(i,j)$ due to intrinsic quantum fluctuations

$W_c(i,j)$ *is asymptotically dipolar :*

$$W_c(\mathbf{r}_1,\boldsymbol{\xi}_1,\mathbf{r}_2,\boldsymbol{\xi}_2)\sim\int_0^1 ds_1\int_0^1 ds_2(\delta(s_1-s_2))-1)\,(\lambda_1\boldsymbol{\xi}(s_1)\cdot\nabla_{\mathbf{r}_1})\,(\lambda_2\boldsymbol{\xi}(s_2)\cdot\nabla_{\mathbf{r}_2})\,\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|}$$

Fluctuating dipoles

Quantum charge behave as fluctuating multipoles (« structured charges »)

The effective magnetic potential

Averaging on the degrees of freedom of the field yields an effective magnetic interaction between filaments :

$$\langle \exp[-\beta U_{\mathbf{A}}(\mathcal{L}_1,\ldots,\mathcal{L}_n)]\rangle_{\mathrm{rad}} \longrightarrow \begin{array}{l} \left\langle \exp\left[i \int_0^1 d\tau (f(\tau)\alpha^*(\tau) + f^*(\tau)\alpha(\tau))\right]\right\rangle_{\mathrm{rad}} = \\ \exp\left[- \int_0^1 d\tau \int_0^1 d\tau' f^*(\tau) \langle \alpha(\tau)\alpha^*(\tau')\rangle_{\mathrm{rad}} f(\tau')\right] \end{array}$$

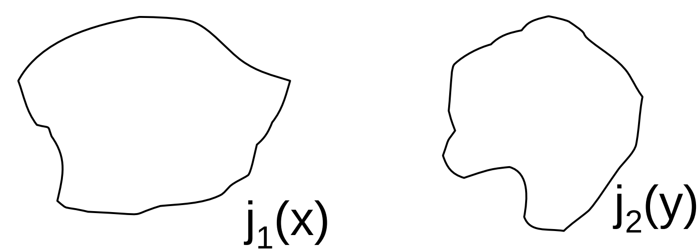
Result

$$e^{-\beta D_N}\langle \exp[-\beta U_{\mathbf{A}}(\mathcal{L}_1,\ldots,\mathcal{L}_n)]\rangle_{\mathrm{rad}} = \prod_{r=1}^{n} \exp\left(-\frac{\beta e_{\gamma_r}^2}{2}\mathcal{W}_{\mathrm{m}}(\mathcal{L}_r,\mathcal{L}_r)\right) \exp\left(-\beta \sum_{r<s}^{n} e_{\gamma_r} e_{\gamma_s} \mathcal{W}_{\mathrm{m}}(\mathcal{L}_r,\mathcal{L}_s)\right)$$

Loop-loop magnetic self energy

Loop-loop magnetic potential

Interpretation of the magnetic potential:
current-current interaction
between currents carried by the loops



$\mathcal{W}_{\mathrm{m}}(i,j)$ is dipolar at large distances

In the space of loops all the techniques of classical statistical mechanics are available, using the basic two body potentials

$$V_{\mathrm{elec}}(i,j) \qquad W_c(i,j) \qquad \mathcal{W}_m(i,j)$$

Proceed as in the analysis of the classical model:
Cluster expansion, Mayer series….

Leads to the same result:

$$f(d) \;=\; -\frac{\zeta(3)}{8\pi\beta d^3} + R(\beta,\hbar,d)\,, \qquad R(\beta,\hbar,d) \;=\; \mathcal{O}(d^{-4})$$

Universality follows from screening sum rules
in the space of loops

Open questions

How to deal with the low temperature-short distance regime within the microscopic model ?

Is the standard Casimir force formula modified by quantum charge fluctuations in the ground state of the metals ?

Corrections to the leading asymptotic term ?

Make explicit connections with Lifshitz theories

Related results: retardation effects in the theory of van der Waals forces (work in progress)