# Exact Factorization of Correlations in 2-D Critical Systems

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Support: NSF–DMR and DMS

- Percolation

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- Significance of CFT operators

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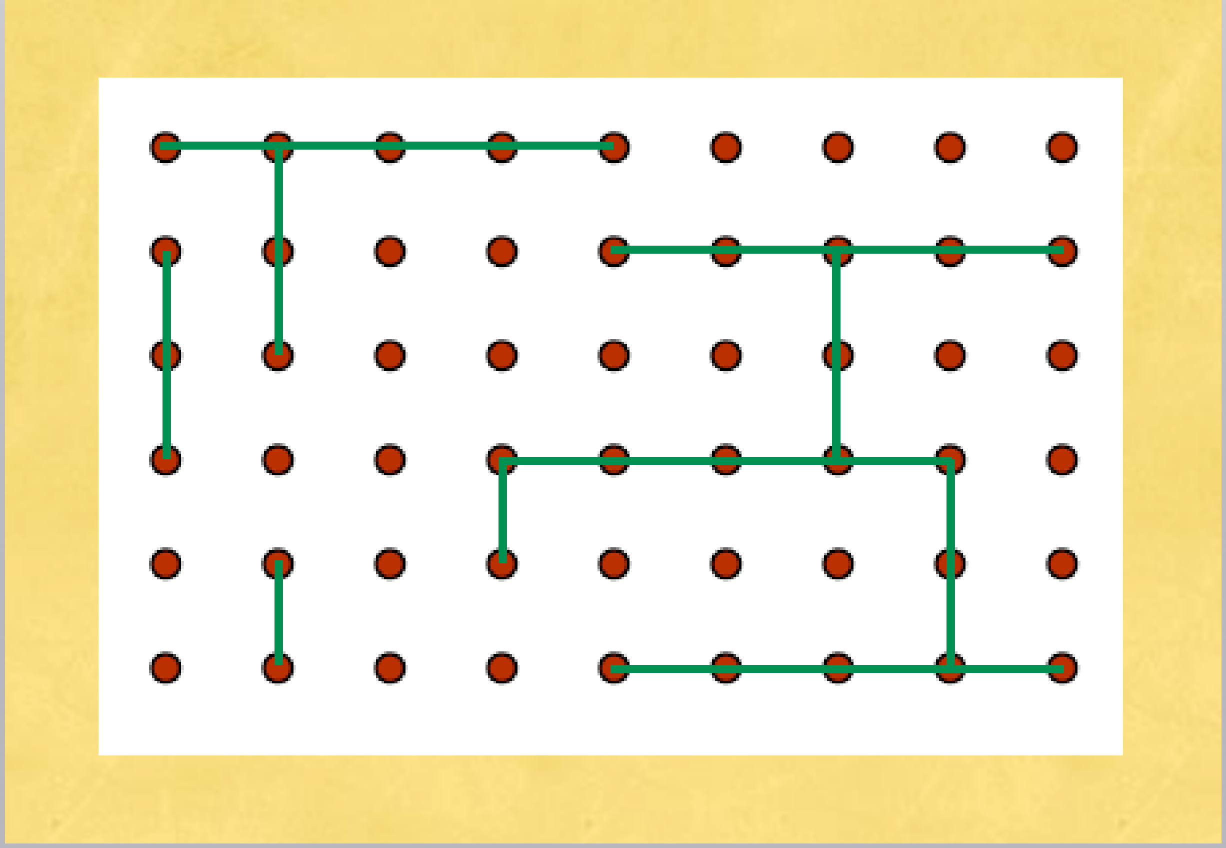

The “thermodynamic” behavior depends on p.
For small p: isolated clusters.
For large p: clusters cross entire system.

At $p = p_c$, there is a phase transition.
Conformal Field Theory applies there (with $c = 0$), in the continuum limit. CFT gives differential equations whose solutions describe various quantities, for example the crossing probability (or the density of a cluster).
We use CFT in the upper half–plane.

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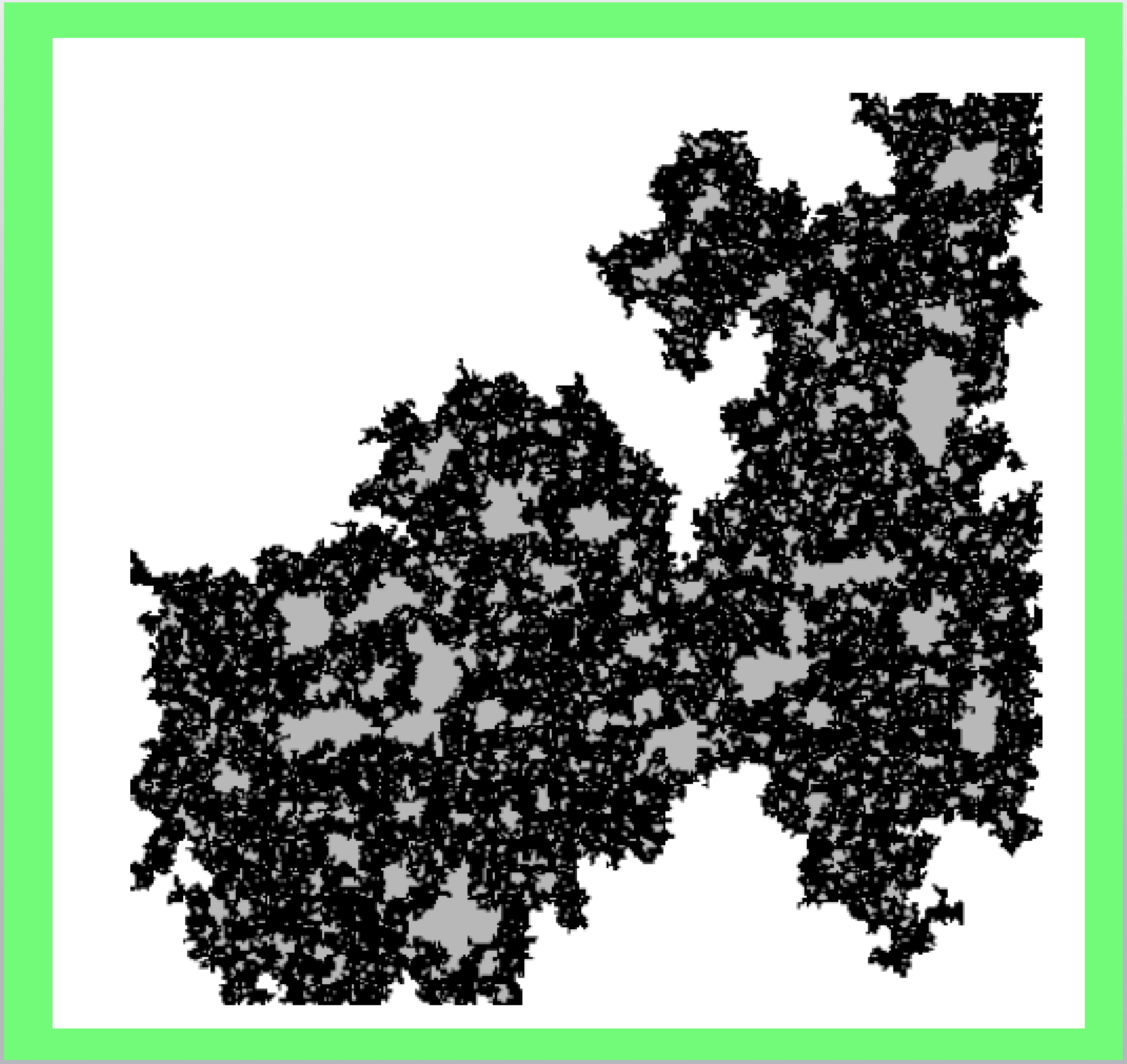
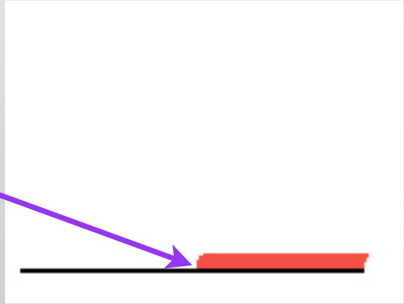
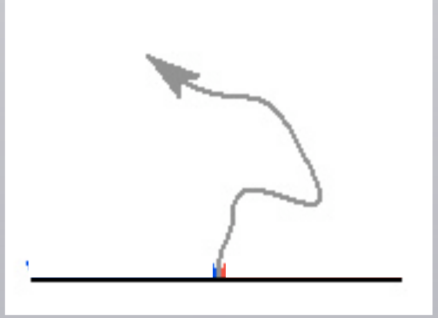


# Useful conformal operators:

1.) $\phi_{(1,2)}(x)$, which implements a change from fixed to free boundaries at x.



2.) $\phi_{(1,3)}(x)$, which creates a cluster anchored on a free boundary at x.

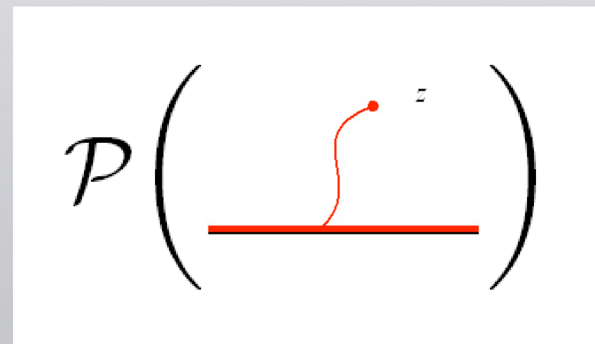


3.) $\phi_{(3/2,3/2)}(z)$, the "magnetization" operator, which measures the density of clusters at z.

Dimensions: $h_{(1,2)} = 0$, $h_{(1,3)} = 1/3$, and $h_{(3/2,3/2)} = 5/96$.

For example, the density of clusters at z which connect to the boundary is

$$\mathcal{P}(z) \propto \langle \phi_{3/2,3/2}(z)\phi_{3/2,3/2}(\bar{z})\rangle \propto \frac{1}{y^{5/48}}$$

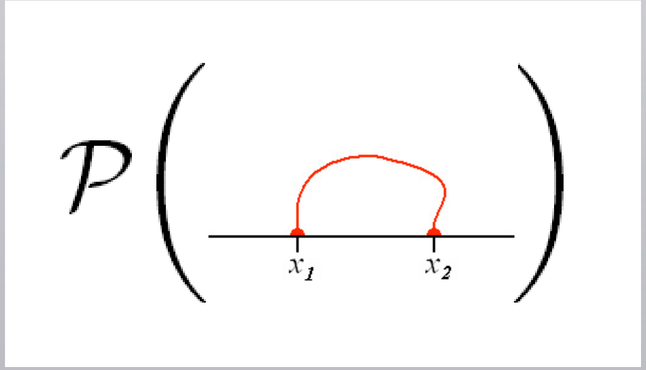


The magnetization operator at the image point appears because the problem is in the half-plane [Cardy].

Here, and below, $\mathcal{P}$ is the probability of a cluster connecting its arguments.

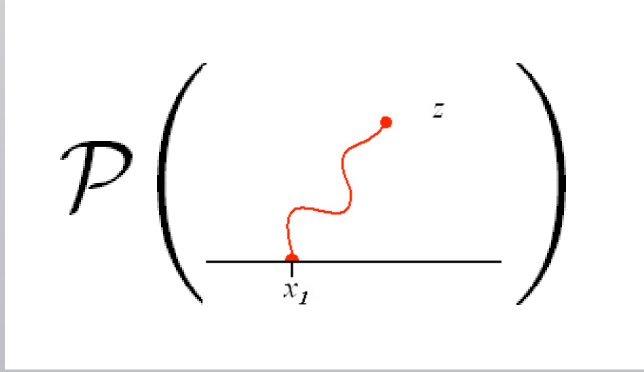
The probability of a cluster connecting $x_1$ and $x_2$ (a limit of Cardy's crossing probability formula):

$$\mathcal{P}(x_1, x_2) \propto \langle \phi_{1,3}(x_1)\phi_{1,3}(x_2)\rangle \propto \frac{1}{(x_2 - x_1)^{2/3}}$$



The probability of a cluster connecting $x_1$ and z:

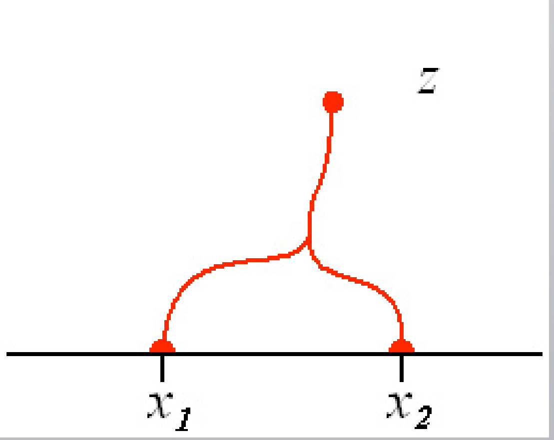
$$\mathcal{P}(x_1, z) \propto \langle \phi_{1,3}(x_1)\phi_{3/2,3/2}(z)\phi_{3/2,3/2}(\bar{z})\rangle \propto \frac{y^{11/48}}{|x_1 - z|^{2/3}}$$




This prediction agrees very well with computer simulations (up to an non-universal, unspecified normalization).

The probability of a cluster connecting $x_1$, $x_2$ and z:

$$\mathcal{P}(x_1, x_2, z) \propto \langle \phi_{1,3}(x_1)\phi_{1,3}(x_2)\phi_{3/2,3/2}(z)\phi_{3/2,3/2}(\bar{z})\rangle$$
$$\propto y^{-5/48}(x_2 - x_1)^{-2/3}F(\eta)$$

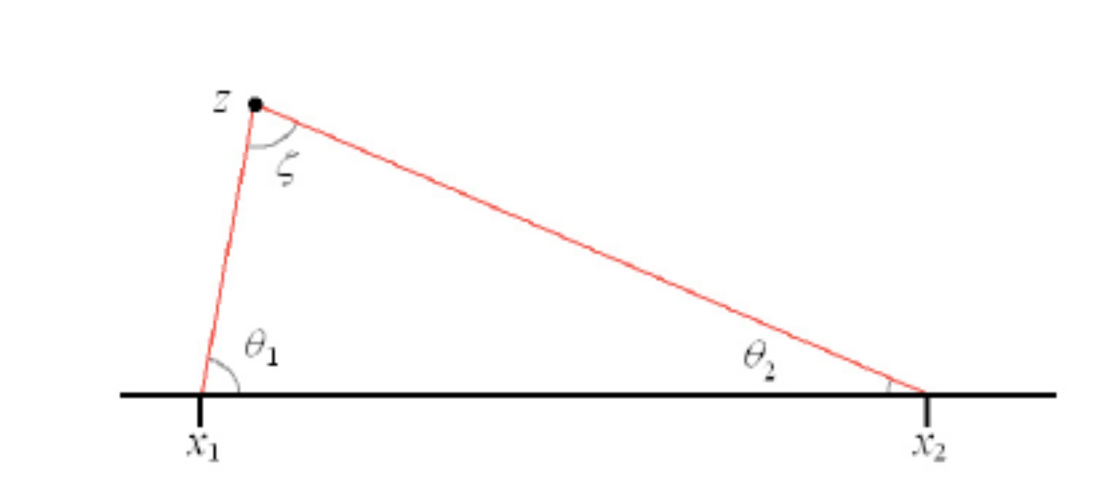



$$\mathcal{P}(x_1, x_2, z)$$

$$\eta = \frac{(z - x_2)(\bar{z} - x_1)}{(\bar{z} - x_2)(z - x_1)}$$

Because of the $\phi_{(1,3)}(x_i)$, $\mathcal{P}(x_1, x_2, z)$ satisfies a third-order differential equation. By considering the asymptotic behavior as $x_1 \rightarrow x_2$, one can identify solutions physically.

It is useful to use
a new variable:

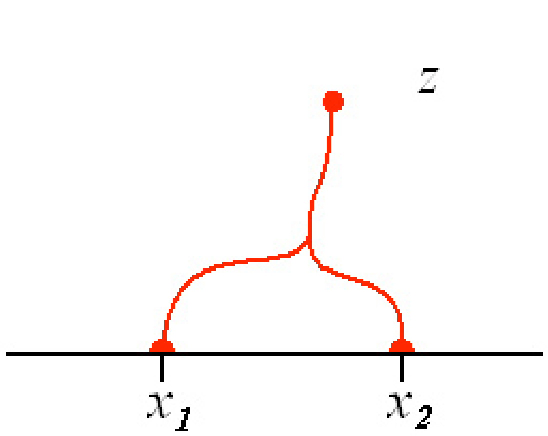



$$\eta = e^{-2i(\theta_1+\theta_2)} = e^{2i\zeta}$$

One solution gives

$$F(\zeta) = \sin^{1/3}(\zeta).$$

$$\mathcal{P}(x_1, x_2, z) \propto y^{-5/48}(x_2 - x_1)^{-2/3}\sin^{1/3}(\zeta)$$

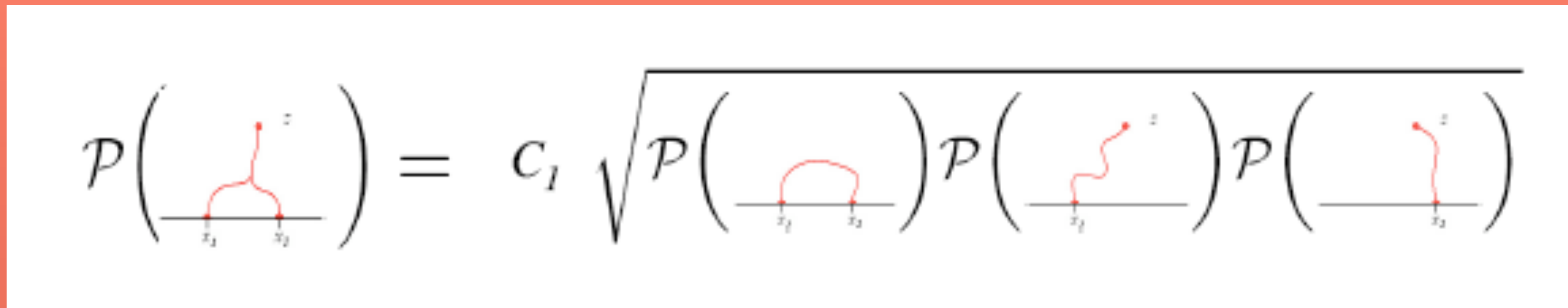



$$\mathcal{P}(x_1, x_2, z)$$

Putting all this together...

$$\boxed{\mathcal{P}(x_1, x_2, z) = C_1 \sqrt{\mathcal{P}(x_1, x_2)\, \mathcal{P}(x_1, z)\, \mathcal{P}(x_2, z)}}$$

$$C_1 = \frac{2^{7/2}\, \pi^{5/2}}{3^{3/4}\, \Gamma(1/3)^{9/2}} = 1.0299268\ldots$$



This result is universal, as well as exact. Letting z go to the real axis shows that $C_1$ is a (boundary) operator product expansion coefficient. Further, this factorization holds in any simply–connected region (with the same $C_1$).

(The formula for $C_1$ arises from the transformation properties of certain hypergeometric functions that solve the DE arising from conformal field theory.)

Note that our results for $\mathcal{P}$ are exact, and apply to a fluid. We are not aware of any similar exact formulas.

We have tested this equation extensively by computer simulation. We find $C_1 = 1.030 \pm 0.001$, so the agreement is excellent:

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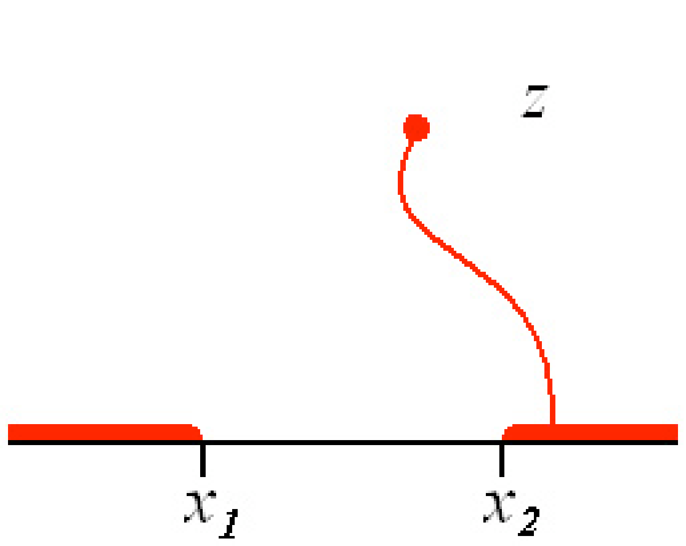
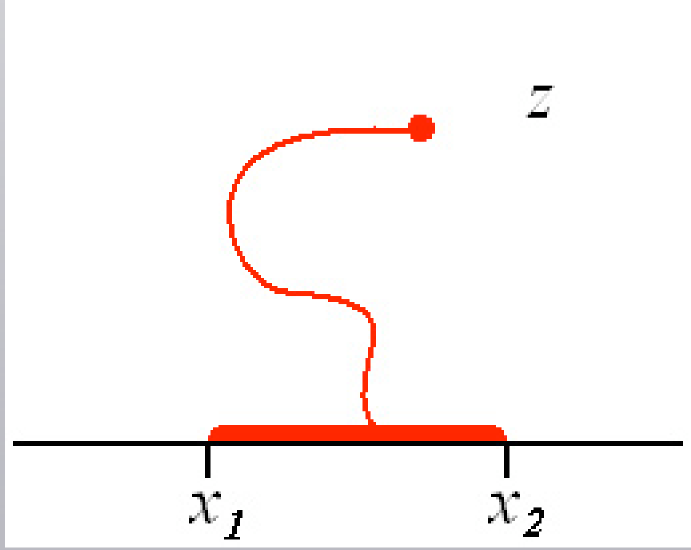
But the question as to why this factorization occurs remains unanswered...

More is possible... using the correlation function

$$< \phi_{(1,2)}(x_1)\ \phi_{(1,2)}(x_2)\ \phi_{(3/2,3/2)}(z)\ \phi_{(3/2,3/2)}(z^*) >$$

We can calculate interval connection probabilities:



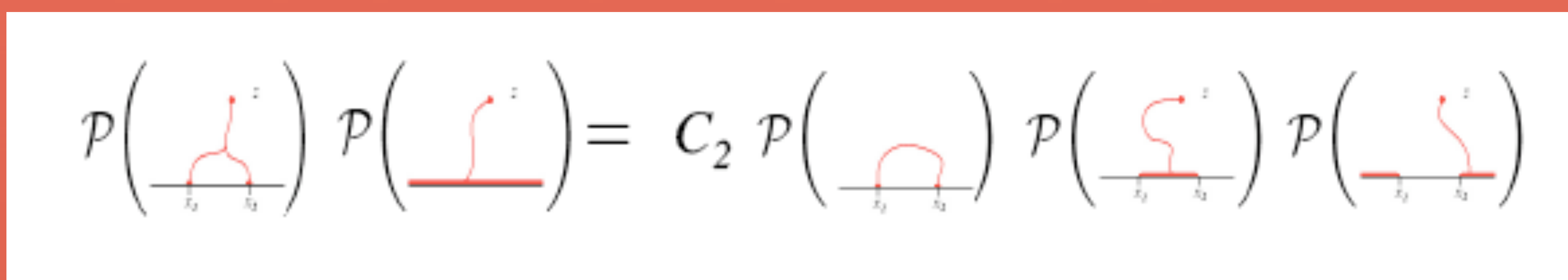

$$\mathcal{P}\left(\overline{(x_1, x_2)}, z\right) \propto y^{-5/48} \cos^{1/3}(\zeta/2)$$

$$\mathcal{P}((x_1, x_2), z) \propto y^{-5/48} \sin^{1/3}(\zeta/2)$$

# This implies factorizations involving the interval functions

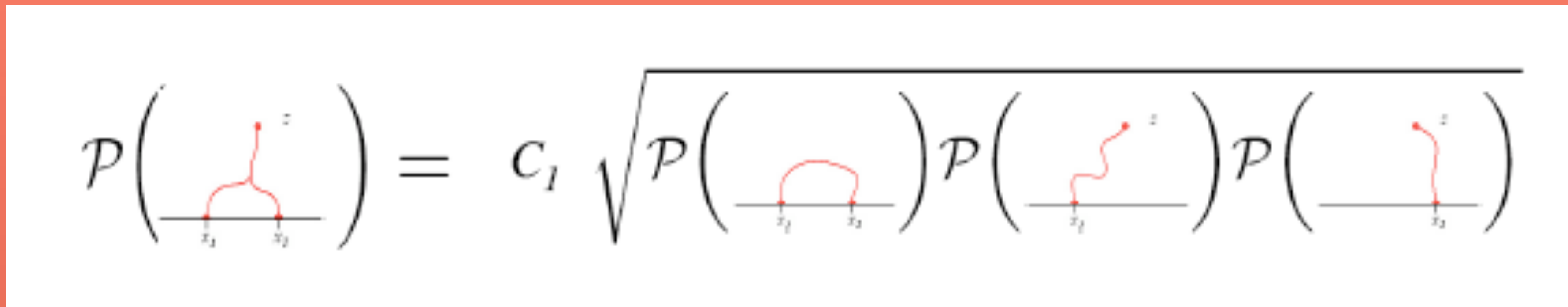
$$\mathcal{P}(x_1, x_2, z)\,\mathcal{P}(z) = C_2\,\mathcal{P}(x_1, x_2)\,\mathcal{P}((x_1, x_2), z)\,\mathcal{P}\Big(\overline{(x_1, x_2)}, z\Big)$$

$$C_2 = \frac{8\pi^2}{3}\frac{1}{\Gamma(1/3)^3} = 1.36893\ldots$$



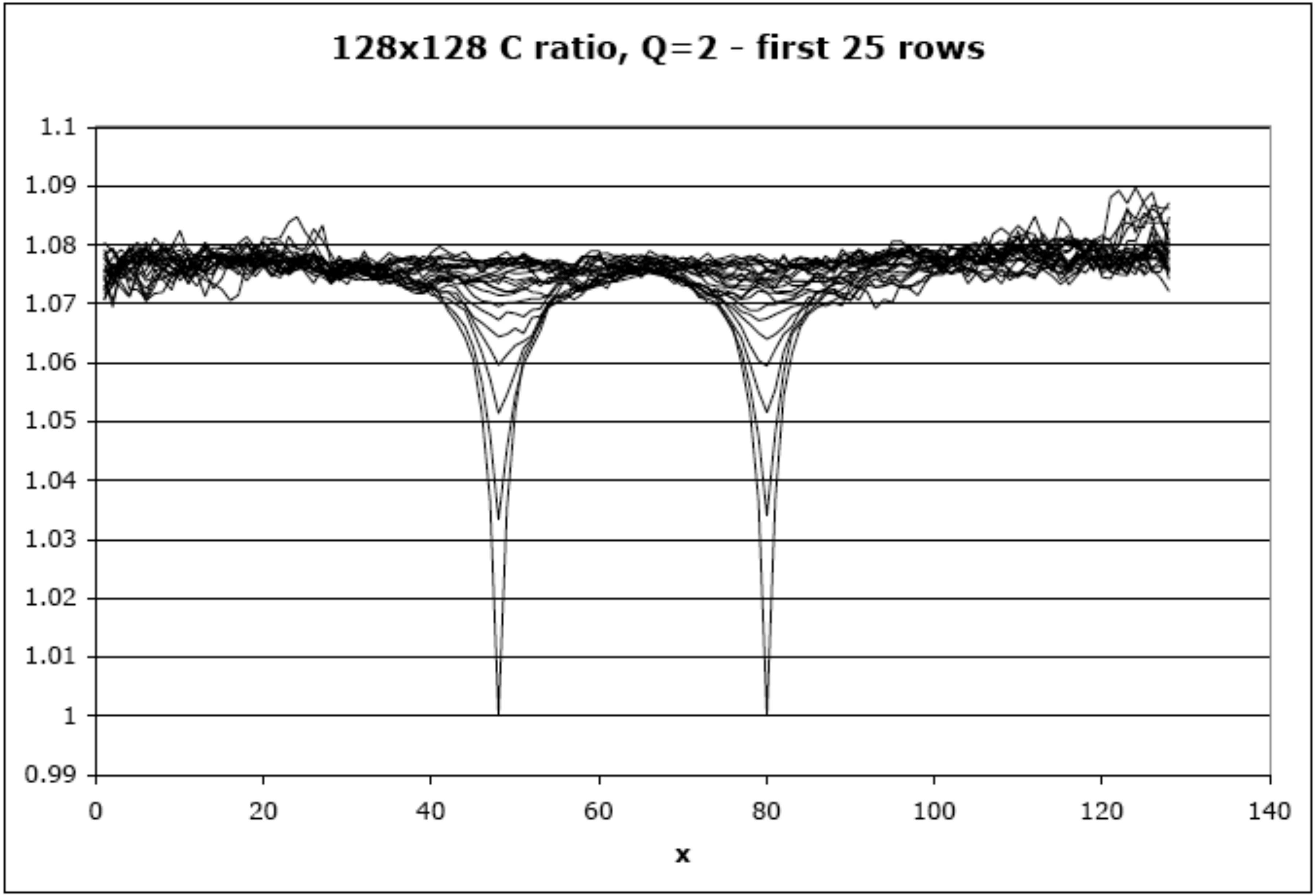
(Additionally, one may eliminate $\mathcal{P}(x_1, x_2, z)$ and express the interval functions in terms two– and three–point functions.)

Consider the first factorization:

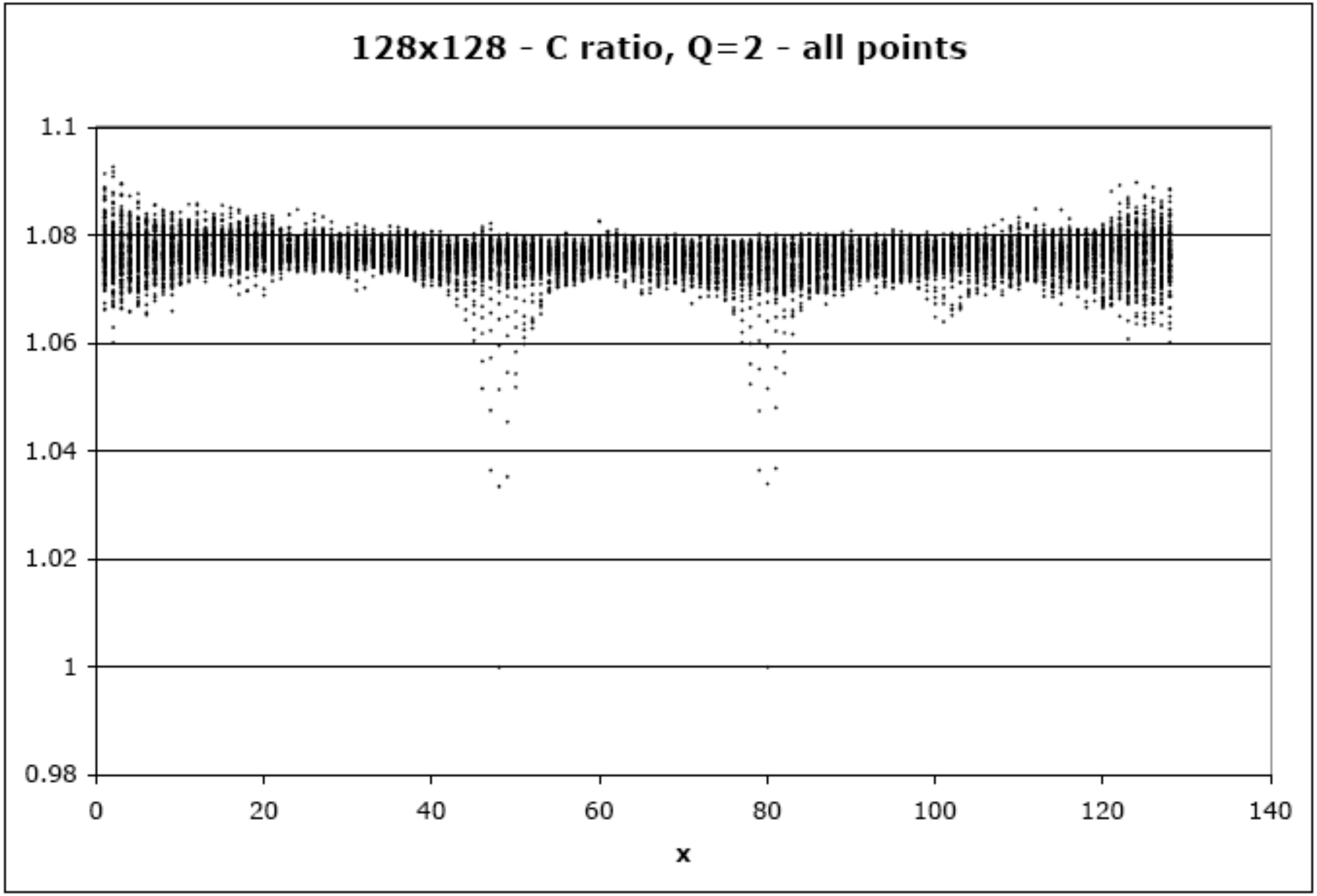


Numerics show this still holds (with same $C_1$), but asymptotically (for points sufficiently far apart), when $x_1$ or $x_2$ (or both) are off the real axis.

Numerical and theoretical evidence that this factorization (with different $C_1$) for connection probabilities of Fortuin-Kastelyn clusters in the critical q-state Potts models:



128x128 C ratio, Q=2 - first 25 rows
1.1
1.09
1.08
1.07
1.06
1.05
1.04
1.03
1.02
1.01
1
0.99
0
20
40
60
80
100
120
140
x


128x128 - C ratio, Q=2 - all points
1.1
1.08
1.06
1.04
1.02
1
0.98
0
20
40
60
80
100
120
140
x

We also have numerical evidence for this factorization at the 3-D percolation point (within ±3%)

Further, in 2-D we have found that the (original) factorization holds for any central charge c if one uses the operators $\phi_{(1,3)}(x)$ and $\phi_{(1/2,0)}(x)$:

$$< \phi_{(1,3)}(x_1)\, \phi_{(1,3)}(x_2)\, \phi_{(1/2,0)}(z, z^*) > =$$

$$C\, \frac{\sqrt{< \phi_{(1,3)}(x_1)\, \phi_{(1,3)}(x_2) > < \phi_{(1,3)}(x_1)\, \phi_{(1/2,0)}(z, z^*) >}}{< \phi_{(1,3)}(x_2)\, \phi_{(1/2,0)}(z, z^*)>}$$

Further, these operators are the only possible choice giving this factorization. (We have no general physical interpretation as yet.)

Finally, consider 2-D percolation again. In a rectangle with fixed bc on the vertical ends, we define the quantities $P_L(z)$, $P_R(z)$ and $P_{LR}(z)$ as the density of clusters that touch the left, right, and both sides respectively, and $\pi_h$ be the horizontal crossing probability. Then consider the ratio

$$C(z) = \frac{P_{LR}(z)}{\sqrt{P_L(z)P_R(z)\Pi_h}}$$

We find, numerically, that C(z) is
1. constant to within a few % everywhere in the rectangle
2. a function of x only (ie it is independent of the vertical coordinate).

Guided by these observations, we have used CFT to show that in a semi-infinite strip one has

$$C(x) = C_0 \frac{{}_2F_1(-1/2, -1/3, 7/6, e^{-2\pi x})}{\sqrt{{}_2F_1(-1/2, -2/3, 5/6, e^{-2\pi x})}}$$

and we also have expressions for $C(x)$ in an arbitrary rectangle (assuming that there is no y-dependence).

# Conclusions:

Recent results (from conformal field theory) give exact and universal factorizations of connection probabilities in critical 2-D percolation. These results generalize to other 2-D systems and 3-D percolation.

Peter Kleban, Jacob J. H. Simmons, and Robert M. Ziff, "Anchored Critical Percolation Clusters and 2-D Electrostatics", Phys. Rev. Letters **97**, 115702 (2006) [arXiv: cond-mat/0605120].

"Exact factorization of correlation functions in 2-D critical percolation", Jacob J. H. Simmons, Peter Kleban, and Robert M. Ziff [arXiv: 0706.4105].