CONICAL DEFECTS IN A SHEET OF PAPER

Bending with local geometrical constraints

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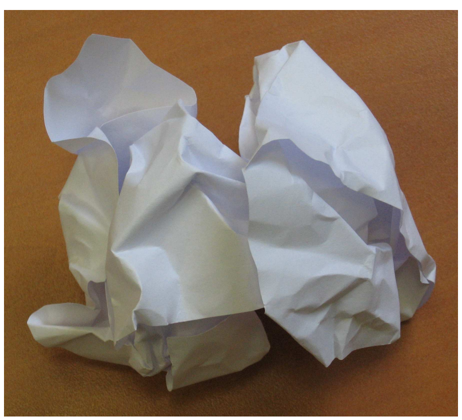
Jemal GUVEN

Universidad Nacional Autónoma de México



How paper folds is rather subtle

It bends without stretching almost everywhere

Local constraint $\Rightarrow$ defects

Simplest local defect: a cone

Remarkable point: conical defect organizes how the sheet folds

OUTLINE

- Introduce fully non-linear framework to describe sheet in this approximation
- First review situation without the local constraint
- Energy spectrum: ground states and (stable) excited states
- Self contacts
- Conical stability vs instability without the center
- Bending surfaces of constant -ve Gauss curvature

SINGLE COMPONENT FLUID MEMBRANE

- On mesoscopic scales relevant degrees of freedom are geometrical

Parametrized surface $\qquad (u, v) \mapsto \mathbf{X}(u, v)$

Two surface tensors encode geometry:

- Induced metric: $g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b$
- Extrinsic curvature: $K_{ab} = \mathbf{e}_a \cdot \partial_b \mathbf{n} = K_{ba}$

  Tangents $\mathbf{e}_a = \partial \mathbf{X} / \partial u^a$, $a = 1, 2$; $\quad$ Normal $\mathbf{n}$

- Covariant derivative ∇_a compatible with metric:
  $[\nabla_a, \nabla_b] \neq 0 \Rightarrow$ Intrinsic Curvature

- Principal curvatures are eigenvalues of $K^a{}_b = g^{ac} K_{cb}$

$$K^a{}_b v^b{}_I = C_I v^a{}_I \qquad I = 1, 2$$

- Free energy constructed out of symmetric scalars

  Mean Curvature $K = g^{ab} K_{ab} = C_1 + C_2$

  Gaussian Curvature $K_G = \mathrm{det} K^a{}_b = C_1 C_2$

FLUID MEMBRANE

Energy cost associated with bending:

- Simplest energy quadratic in curvatures

$$H[\mathbf{X}] = \frac{1}{2}\kappa \int dA\,(C_1 + C_2)^2$$

- $H \sim \int dA\, C_1 C_2$ plays role through boundaries/interfaces
- Global constraints are familiar: A, V, perhaps $M = \int dA\, K$
- Remarkably good description of equilibrium if there is no penalty associated with tangential deformations

ACCOMMODATING A PENALTY ON TANGENTIAL DEFORMATIONS:

- Continuum modeling is technically difficult; no agreed model
- Limit: surface is unstretchable

  -treat as a local constraint: $g_{ab} = g_{ab}^{(0)}$ 'the memory'
- Distances frozen $\Rightarrow$ penalty is infinite

  -opposite fluid membrane idealization; but also geometrical
- A good approximation almost everywhere
  if the surface is thin (Lord Rayleigh):

  Energy penalty associated with bending (h^3)

  $\ll$ penalty associated with stretching (h)

EGREGIOUS THEOREM

- Metric fixed $\Rightarrow$ Gaussian curvature $G = C_1 C_2$ fixed

  Consequence of integrability condition on
  structure equations: $\nabla_a \mathbf{e}_b = -K_{ab}\,\mathbf{n}$

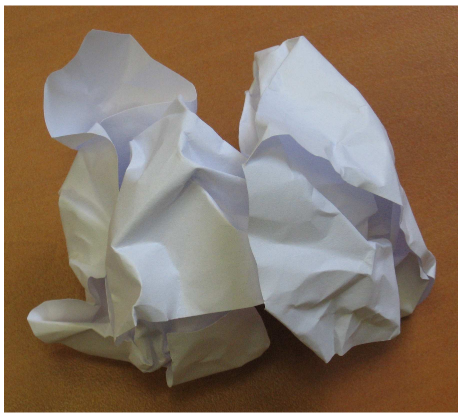
- Flat surface: $G = 0 \Rightarrow C_1$ or $C_2 = 0$

  Unstretchable flat surface remains flat

  Bending occurs in one direction; two implies stretching

Deformed geometry: Cylinder, Cone, Developable surface

DEFECTS



Globally deformed surface generally is not a cylinder, a cone, or developable

Forces applied to surface may oblige it to fold along more than one direction somewhere

Energy effective way to adjust: confine regions where stretching occurs within sharp peaks and ridges (T. Witten et al.)

Poke a circular disc into a circular ring (BenAmar&Pomeau,Cerda&Mahadevan)

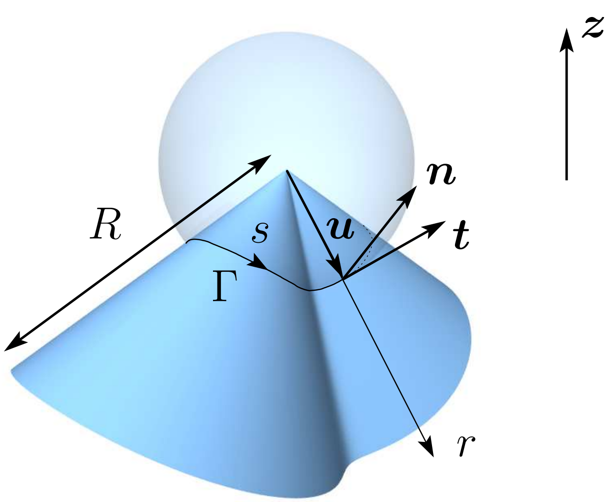
A conical singularity (point defect) is generated

It did not exist previously

Its creation requires external forces

Cones are the most elementary defects: but first studied mid 1990s

A WORD ABOUT CONES



- Closed curve on sphere $\Gamma : s \mapsto \mathbf{u}(s)$

  s arc-length along Γ

  r distance to apex in direction $\mathbf{u}(s)$

  $\Rightarrow (r, s) \mapsto \mathbf{X}(r, s) = r\mathbf{u}(s)$ is a cone

  $L =$ Length of Γ
  $\varphi_{\mathrm{e}} = L - 2\pi$ Angle surplus

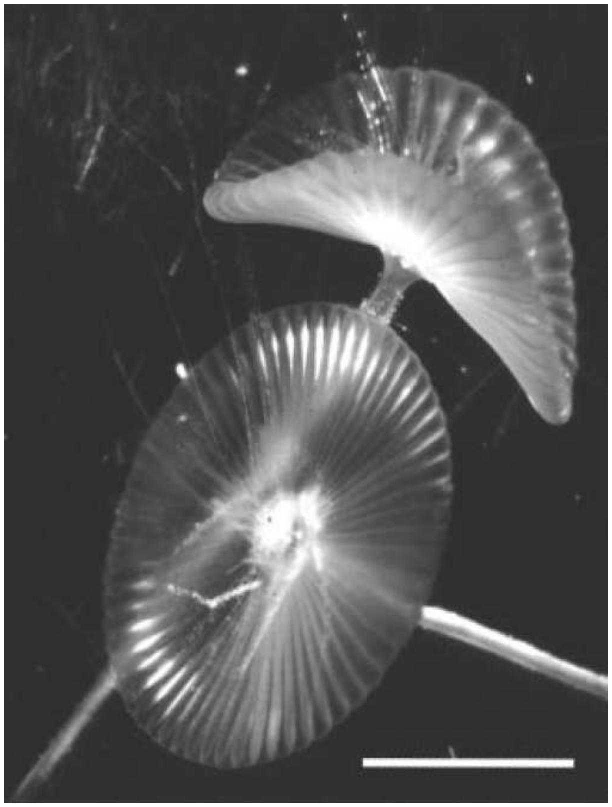
- Two cones with same φ_{e} are isometric

(1) Isometric deformation of planar disc: $L = 2\pi$, $\varphi_{\mathrm{e}} = 0$
- $L = 2\pi \Rightarrow \Gamma$ lies in one hemisphere
- Ground state a great circle
- Requires external force to generate conical singularity

(2) Inflated Corner of paper bag: $\varphi_{\mathrm{e}} = -\pi$ a deficit
- Familiar circular ice cream cone

(3) Cone with surplus angle: (Muller,BenAmar,JG 2008)
- will show $\exists$ non-trivial ground state; stable excited states
- No external forces are necessary!

THE MERMAID'S WINEGLASS



- *Acetabularia acetabulum*

-unicelular algae, Hammerling's experiment

-base, stalk, conical cap, 0.5 - 10 cm

- Growing disc: $R \propto \ell$
  circumference $2\pi R$, geodesic radius ℓ

  $R = \ell$ embedded as planar disc
  $R < \ell$ circular cone
  $R > \ell$ a surplus angle

Cannot be embedded as an axially symmetric geometry

Nor is it a very good wineglass!

THE CONE AS A CONSTRAINED EQUILIBRIUM

• Bending energy of a conical disc of radius R

$$B = \frac{1}{2} \ln(R/r_0) \int ds\kappa^2$$

$\kappa = -\mathbf{n} \cdot \mathbf{t}'$ geodesic curvature on sphere; s arc-length on S^2

Energy density $\sim 1/r$

• A cone is not a critical point of bending energy without local constraints

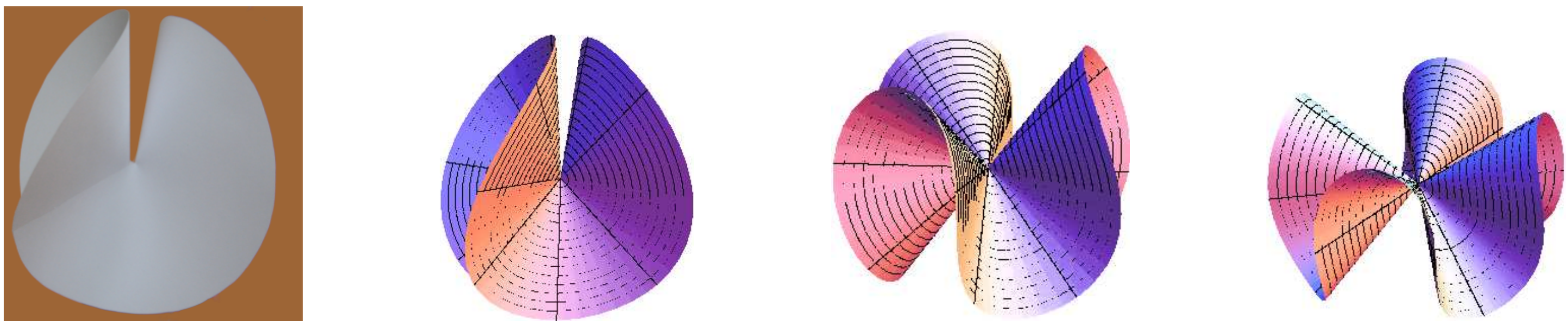
Poor man's treatment: treat as Euler-Elastica on sphere

Implement unstretchability by fixing length

Shortcomings: relies on touch of luck, equations are missing; no idea what stresses are; impossible to determine if configurations are stable

SURFACE SHAPES

In absence of external forces, conical geometry is completely determined by surplus angle φ_e and a quantum number n



$$\varphi_\mathrm{e} = 2\pi, \; n = 2, 3, 4$$

Two or more folds (four-vertex theorem, symmetry $\kappa \leftrightarrow -\kappa$)

φ_e small, $\exists$ equilibrium solution without self-contacts for all $n \geq 2$

KISSING or physics in the ruff

Increase $\varphi_e \Rightarrow$ conical geometry packs more densely

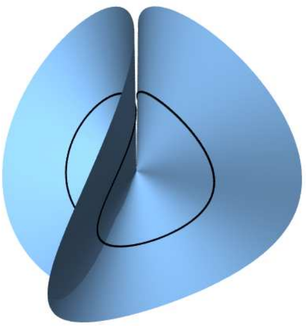
At some point, mathematical surface will self-intersect, first with 2-fold

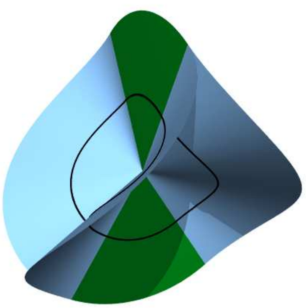
The physical surface does not

n	2	3	4	5	10	50	$\to \infty$
$\varphi_e^{(n)}$	7.08	13.30	17.78	21.12	29.38	34.92	35.23

$\varphi_e > 35.23$:
all states exhibit self-contact

SELF CONTACT





L = 2 Pi + 8.38

- Contact $\Rightarrow$ normal forces
  Two regions deform accordingly

- Stress on contact $\neq$ stress on free cone

- Two boundary conditions associated with forces pressing two regions together:

  (i) continuity of tangent plane
  (ii) discontinuity of curvature

$$\Delta\kappa = \sqrt{\Delta\,\text{tension}}$$

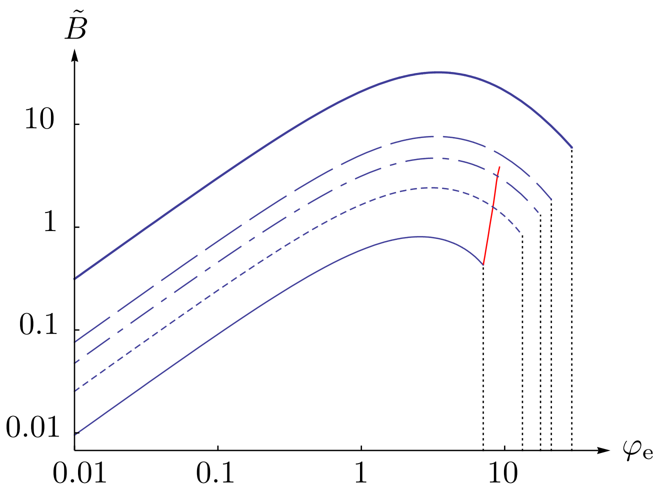
  - analogous to adhesion

- Contact length L_{contact}
  Energy minimization w.r.t L_{contact} $\Rightarrow$ equilibrium shape

- Minimum breaks two-fold symmetry

Increase $\varphi_e \Rightarrow$ crowding, higher curvatures

THE ENERGY SPECTRUM



Scaled energy density B for $n = 2, 3, 4, 5, 10$.

$\varphi_{\mathrm{e}} < \varphi_{\mathrm{e}}^{(2)} = 7.08$: $n = 2$ is ground state

Red curve: touching 2-fold

$\varphi_{\mathrm{e}} \geq \varphi_{\mathrm{e}}^{(2)}$: continuity $\Rightarrow$ initially $n = 2$ remains ground state

$\varphi_{\mathrm{e}} > 8.27$: $B_3 < B_2$ $n = 3$ becomes the ground state

DEFORMING THE GEOMETRY

(1)Identify geometries minimizing H subject to local constraint
(2)What is the stress associated with a given geometry?

Treat first without metric constraint

How does energy change under a small deformation

$$\mathbf{X}(u) \to \mathbf{X}(u) + \delta\mathbf{X}(u)$$

H depends on $\mathbf{X}$ only through g_{ab} and K_{ab}

$$H[\mathbf{X}] = \int dA\, \mathcal{H}(g_{ab}, K_{ab})\,, \quad dA = \sqrt{g} d^2u$$

- g_{ab} and K_{ab} depend on $\mathbf{X}$ through the tangent vectors $\{\mathbf{e}_a, \mathbf{n}\}$

$$g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b\,, \quad K_{ab} = \mathbf{e}_a \cdot \partial_b \mathbf{n}$$

$$\mathbf{e}_a = \partial_a \mathbf{X}\,, \quad \mathbf{e}_a \cdot \mathbf{n} = 0\,, \quad \mathbf{n}^2 = 1$$

- Enforce structural constraints using Lagrange multipliers

Treat $\mathbf{X}$,$\mathbf{e}_a$,$\mathbf{n}$, g_{ab} and K_{ab} as independent variables

$$
\begin{aligned}
H_C &= H[g_{ab}, K_{ab}] + \int dA\ \mathbf{f}^a \cdot (\mathbf{e}_a - \partial_a \mathbf{X}) \\
&\quad + \int dA \Big[\lambda_\perp^a (\mathbf{e}_a \cdot \mathbf{n}) + \lambda_n (\mathbf{n}^2 - 1) \Big] \\
&\quad - \int dA \Big[\mathcal{H}^{ab} (K_{ab} - \mathbf{e}_a \cdot \partial_b \mathbf{n}) - \frac{1}{2} \mathcal{T}^{ab} (g_{ab} - \mathbf{e}_a \cdot \mathbf{e}_b) \Big]
\end{aligned}
$$

• $\mathbf{X}$ appears only in tangency constraint (translation invariance) $\Rightarrow$

$$\frac{\delta H_C}{\delta \mathbf{X}} = \nabla_a \mathbf{f}^a$$

• Equilibrium $\mathbf{f}^a$ conserved: $\nabla_a \mathbf{f}^a = 0$ is the shape equation (JG 2004)

$$\mathbf{f}^a = (\mathcal{T}^{ab} - \mathcal{H}^{ac} K_c{}^b)\, \mathbf{e}_b - \nabla_b \mathcal{H}^{ab}\, \mathbf{n}$$

• Given $H[g_{ab}, K_{ab}]$: $\quad \mathcal{T}^{ab} = -2\delta H / \delta g_{ab}$, $\quad \mathcal{H}^{ab} = \delta H / \delta K_{ab}$

–Physical stress completely determined by geometry

STRESS ASSOCIATED WITH BENDING

$\mathcal{H} = (C_1 + C_2)^2/2 + \sigma \Rightarrow \mathbf{f}^a = f^{ab}\,\mathbf{e}_b + f^a\,\mathbf{n}\,,$

$$f^{ab} = K(K^{ab} - \frac{K}{2}g^{ab}) - \sigma g^{ab}\,; \quad f^a = -\nabla^a K$$

f^{ab} local, quadratic in K_{ab}

Projections of conservation law $\perp$ and $\|$ to surface:

- Shape equation: $\nabla_a f^a - K_{ab}f^{ab} = 0$ is

$$-\nabla^2 K + \frac{1}{2}K(K^2 - 2K_{ab}K^{ab}) + \sigma K = 0$$

- Bianchi Identities: $\nabla_a f^{ab} + K^{ab} f_a = 0$

- Consequence of tangential defo $\equiv$ reparametrization on $\mathbf{X}$
- not so if geometry is constrained or
  freedom there are material degrees of living on surface

- Sphere without constraints: $K_{ab} = g_{ab}K/2$, K constant

  Constraints set up stresses

INCLUDE METRIC CONSTRAINT

Deformation preserves metric: construct

$$H_C[\mathbf{X}] = H[\mathbf{X}] - \frac{1}{2}\int dA\, T^{ab}\,(g_{ab} - g_{ab}^{(0)})$$

Stress $\mathbf{f}^a$ a sum of two terms: $\mathbf{f}^a = \mathbf{f}^a_{\text{Bending}} + T^{ab}\,\mathbf{e}_b$

Constraint introduces tangential stress proportional to T^{ab}

Stress no longer depends only on geometry

In equilibrium, $\mathbf{f}^a$ is conserved: $\nabla_a \mathbf{f}^a = 0$

Normal projection: $\mathcal{E}_{\text{Bending}} - K_{ab}T^{ab} = 0$

Tangential projection: $\nabla_a T^{ab} = 0$

Fix metric $\Rightarrow$ $\delta\mathbf{X}$ can only change extrinsic geometry

SHAPE EQUATIONS FOR CONE (Müller, JG 2008)

Surface determination:

$$\kappa'' + \frac{1}{2}\kappa^3 + \kappa + \kappa r^2 T_\parallel = 0$$

$T_\parallel, T_\perp, T_{\parallel\perp}$ are projections on tangent $\mathbf{t}$ and radius $\mathbf{u}$

Consistency $\Rightarrow$

$$T_\parallel = -\frac{C_\parallel(s)}{r}$$

What's left is Euler Elastica on sphere

$\nabla_a T^{ab} = 0 \Rightarrow$

$$\begin{aligned} T_{\parallel\perp} &= \frac{C'_\parallel \ln r}{r^2} + \frac{C_{\parallel\perp}}{r^2} \\ T_\perp &= \frac{C''_\parallel}{r^2}(\ln r + 1) + \frac{C_\parallel}{r^2} + \frac{C'_{\parallel\perp}}{r^2} + \frac{C_\perp}{r} \end{aligned}$$

C's functions of s

Circle: $C_\parallel$ constant; expect $C_{\perp\parallel} = 0$ in absence of external torques

FORCE,TORQUE and BOUNDARY CONDITIONS

Response to $\mathbf{X}(u) \rightarrow \mathbf{X}(u) + \delta\mathbf{X}(u)$ (Capovilla&JG 2002,JG 2004)

$$\delta H = \int dA\, \mathcal{E}\, \mathbf{n} \cdot \delta \mathbf{X} - \int dA\, \nabla_a \left[\mathbf{f}^a \cdot \delta \mathbf{X} + \frac{\partial \mathcal{H}}{\partial K_{ab}} \mathbf{n} \cdot \nabla_b\, \delta \mathbf{X} \right]$$

Equilibrium response is a divergence

Boundary conditions changed wrt fluid membrane: $\delta\mathbf{X}$ needs to be consistent with isometry

Cone $\delta\mathbf{X} = \mathbf{Z}_C + \mathbf{Z}_T$

$$\mathbf{Z}_C = r(\Psi_C(s)\mathbf{t} + \Phi_C(s)\mathbf{n})\,, \quad \Psi_C' + \kappa\Phi_C = 0$$

$$\mathbf{Z}_T = \Psi_T(s)\mathbf{u} + \Psi_T'(s)\mathbf{t} + \Phi_T(s)\mathbf{n}\,, \quad \Psi_T'' + \Psi_T + \kappa\Phi_T = 0$$

Boundary of fixed r:

Φ_T: $C_{\parallel\perp} = 0$

Φ_C: $C_\perp = (\kappa^2/2 + 1)/R$

Limit $R \rightarrow \infty$, $T_\parallel = -T_\perp$.

$$\oint ds \mathbf{f}_\perp = 0$$

STABILITY:

Deform equilibrium cone into another with same surplus

$$\delta^2 B = \int dA\, \Phi \mathcal{L} \Phi\,, \quad \mathbf{\Phi} = \mathbf{n} \cdot \delta \mathbf{X}$$

$\mathcal{L} = (\mathcal{L}_0 - \frac{1}{2}\kappa^2 + C_{\|})^2 + V(\kappa)$ is self-adjoint, V not positive;
$\mathcal{L}_0 = \partial_s^2 + \frac{1}{2}(3\kappa^2 + 1 - C_{\|})$ Not obviously tractable!

Isometry: $\Psi' + \kappa\Phi = 0$: $\int ds \kappa \Phi = 0$

Look at $\mathcal{L}\Phi_i = \lambda_i \Phi_i$, $i = 1, 2, 3, \ldots$
Sufficiency: $\lambda_1 = 0 \Rightarrow$ Configurations are stable.

Zero modes of $\mathcal{L}$ associated with rotational invariance: $\mathbf{\Phi} = \mathbf{n} \cdot \mathbf{\Omega} \times \mathbf{X} \approx \mathbf{t} \cdot \mathbf{\Omega}$. One such is κ' (also zero mode of $\mathcal{L}_0$).

In practice: decompose Φ into finite number of Fourier modes,
use Gram-Schmidt to implement orthogonality to κ

$\mathcal{L}$ no longer diagonal, diagonalize to find all $\tilde{\lambda}_i$ are positive

Conical equilibrium states free of self-contacts are stable
wrt isometric deformations preserving cone

WHEN THE CENTER CANNOT HOLD

- There is a surprise:

  Remove disc surrounding apex

  Conical annulus relaxes into some other flat geometry

  Conical ground state is unstable wrt deformation destroying the cone

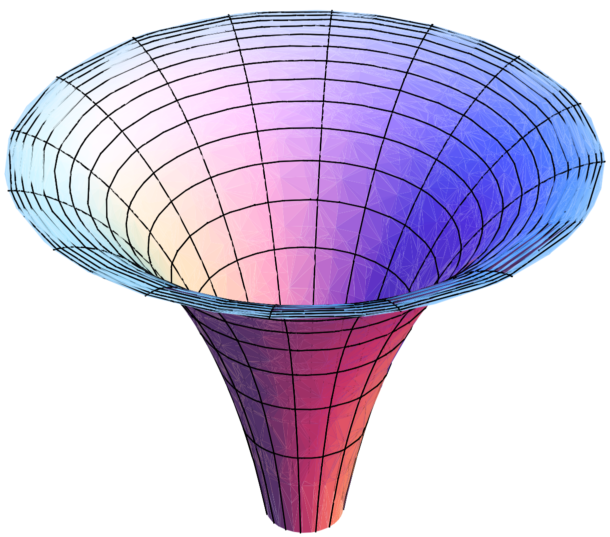
  New surface is tangent developable

Why is this interesting?

Truncated cones can be glued together to model non-flat surfaces:

How does a surface of constant negative Gaussian curvature bend?

PSEUDOSPHERE: candidate axisymmetric ground state



Maximum size $\sim K_G^{-1}$ set by curvature $\Rightarrow \exists$ critical size beyond which axial symmetry is incompatible with curvature

$R = e^{\ell}$, ℓ arc-length along meridian $\Rightarrow$ R' increases exponentially to $R' = 1$ on boundary rim

If surface grows, geometry must ripple

Approximate by a telescope of conical annuli with increasing surplus angles

$K_G \neq 0$ captured by geodesic curvatures

Instability toward tangent developable allows cascade of bifurcations through $n = 2, 3, \ldots$

Consistent with Hilbert: complete Surface $K_G < 0$ cannot be embedded in $\mathbb{R}^3$

ENDNOTE

Interesting new behavior when a surface bends with local constraint on metric

What we learn from conical toy models lays a foundation for understanding more general morphologies

Issues of relevance to biological membranes, viral capsids, ...

Challenge: Describe fluctuating constrained geometries

Papers available on arXiv with references

Coworkers: Martin Müller & Martine Ben Amar(Ecole Normale Superieure), Riccardo Capovilla(CINVESTAV), Markus Deserno(Carnegie-Mellon), Pablo Vázquez(student at UNAM)

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