RG Studies of Critical Casimir Forces

H. W. Diehl

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Collaborators:

- Daniel Grüneberg (UDE)
- Daniel Dantchev, (BAS Sofia, Bulgaria)
- Felix Schmidt (UDE)
- Mykola Shpot (ICMP, Lviv, Ukraine)

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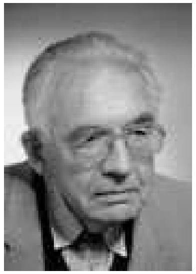
Recent Papers:

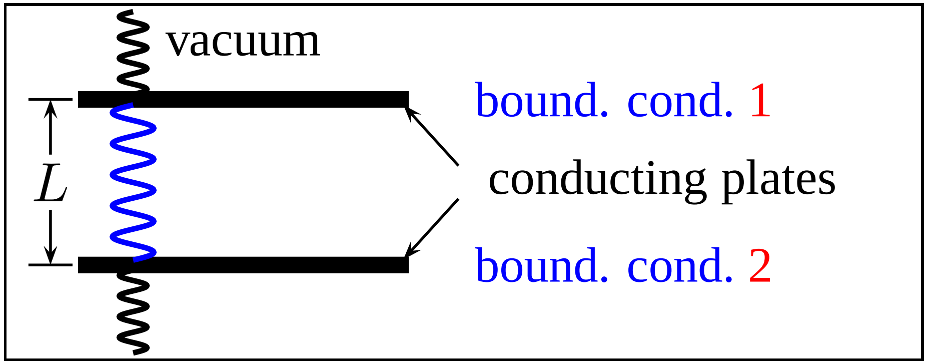
Dantchev, HWD & Grüneberg: PRE **73**, 016131 (2006)
HWD, Grüneberg & Shpot: EPL **75**, 241 (2006)
Grüneberg & HWD: PRB **77**, 115409 (2008)
Schmidt and HWD PRL **101**, 100601 (2008)

Casimir effect in QED



July 15, 1909 – May 4, 2000

1948:
vacuum
fluctuations



- normal modes of electromagnetic field between plates:

$$\omega_{\boldsymbol{q}} = c\,|\boldsymbol{q}|; \quad \boldsymbol{q} = (q_x, q_y, q_z = m\,\pi/L)\,, \quad m \in \mathbb{N}$$

- ground-state energy:

$$E(L) = \frac{1}{2}\sum_{\boldsymbol{q},\mu} \hbar\omega_{\boldsymbol{q}} = \underbrace{C_\Lambda V + C_\Lambda^s A}_{\text{“infinities”}} - \underbrace{\Delta_{\text{QED}}^{(1,2)}(d)}_{\text{universal}}\, \frac{\hbar c}{L}\, \frac{A}{L^{d-1}}\,, \qquad \boxed{\Delta_{\text{QED}}^{(\text{D,D})}(3) = \frac{\pi^2}{720}\,.}$$

- (fluctuation induced) force: $\boxed{\mathcal{F}_C(L) = -\frac{\partial E}{\partial L} = -A\,\frac{\hbar c}{L^{d+1}}\, d\Delta_{\text{QED}}^{(1,2)}(d)}$

reviews: Bordag *et al*, Milton, Mostepanenko & Trunov, Elizalde & Romeo; Golestanian & Kardar …

Casimir effect in QED

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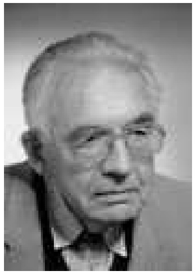

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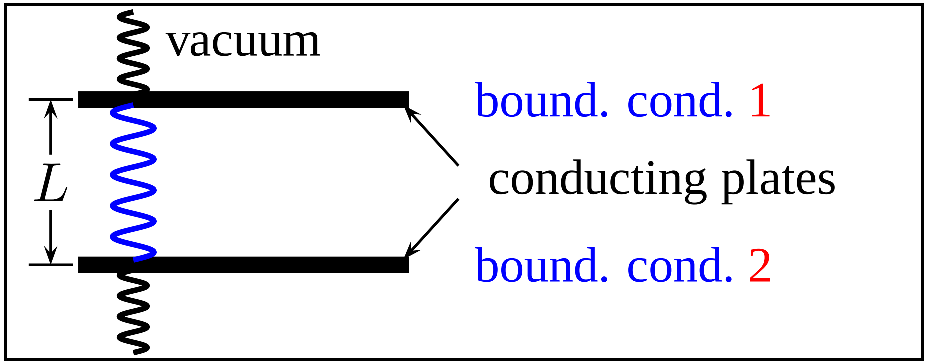

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- (fluctuation induced) force:

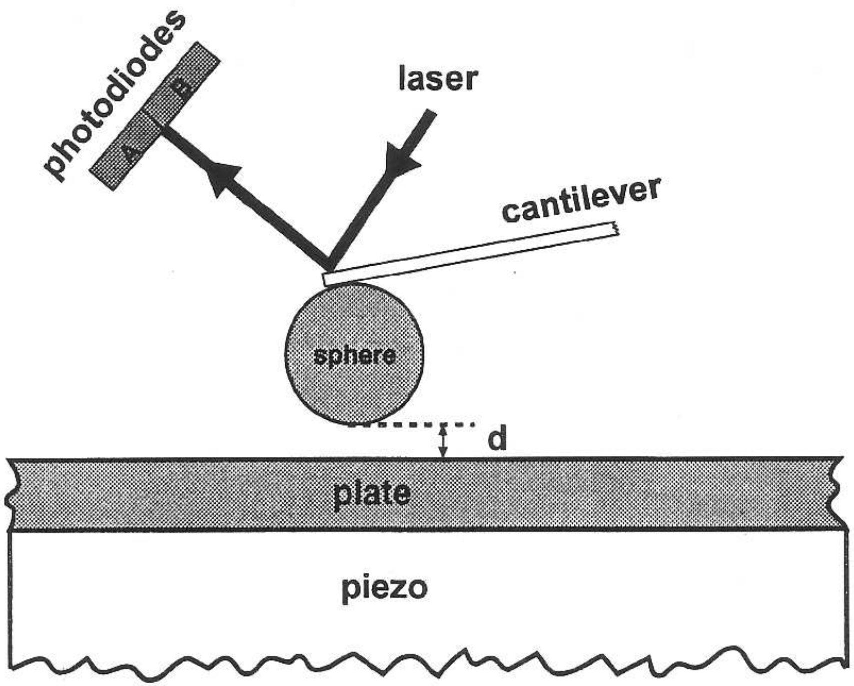
$$\boxed{\mathcal{F}_C(L) = -\frac{\partial E}{\partial L} = -\frac{0.013}{(L/\mu\text{m})^4}\,\frac{\text{dyn}}{\text{cm}^2}\,A}$$

reviews: Bordag *et al*, Milton, Mostepanenko & Trunov, Elizalde & Romeo; Golestanian & Kardar . . .

Experimental Verification

S. Lamoreaux, PRL **87**, 5 (1997);

U. Mohideen and A. Roy, PRL **81**, 4549 (1998); parallel plates: G. Bressi *et al*, PRL **99**, 041804 (2002)



- polystyrene sphere (∅ 196 μm) and sapphire plate coated with Au
- plate-sphere separations from 0.1 to 0.9 μm

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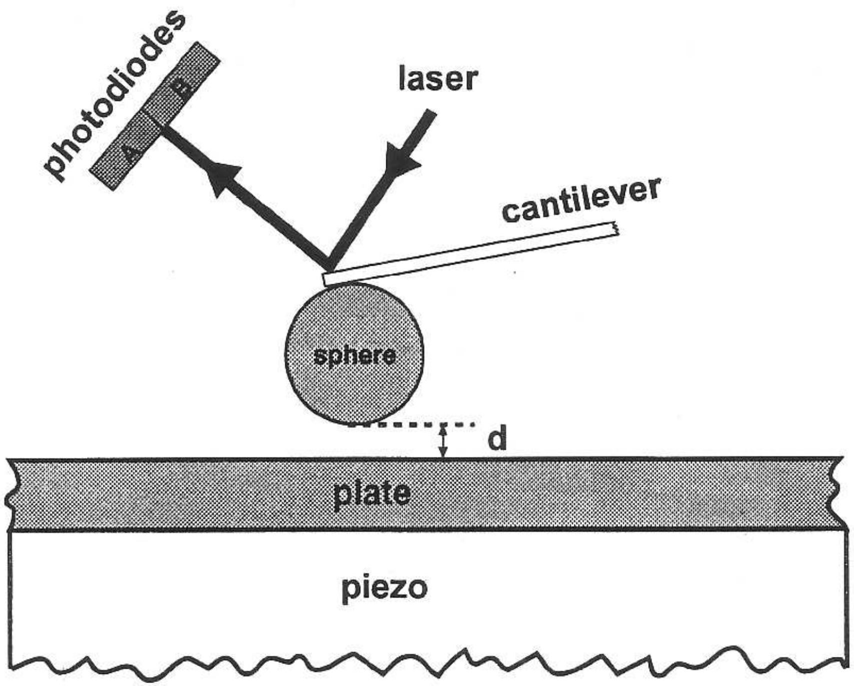

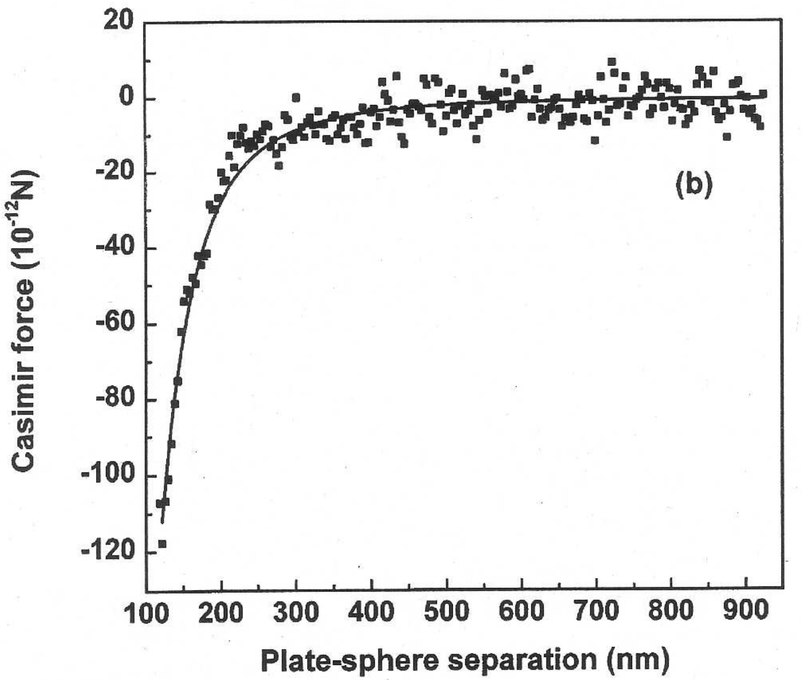

- polystyrene sphere (∅ 196 μm) and sapphire plate coated with Au
- plate-sphere separations from 0.1 to 0.9 μm

solid line: Casimir force for plate-sphere geometry including corrections due to

- finite conductivity
- surface roughness
- finite temperatures

Important Properties of Casimir Force (QED)

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Casimir force (QED)

- is *independent* of microscopic details (**“universal”**)
- depends on *gross features* of
  - medium: space dimension d, dispersion relation, scalar / vector field, geometry, ...
  - boundaries: **boundary conditions**, geometry, curvature, ...
- usually is described by noninteracting (effective Gaussian) field theory – coupling to matter field: only through boundary conditions

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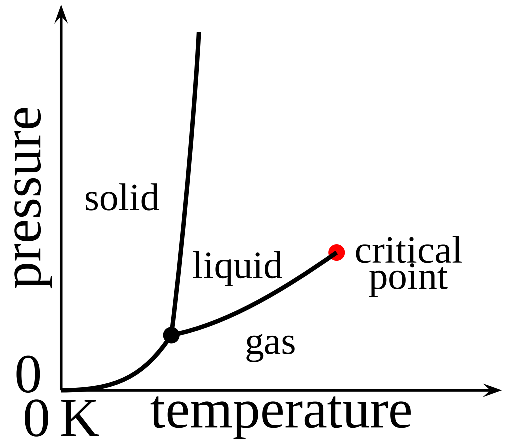
Interacting Field Theories?

Yes, for condensed matter systems at critical points!

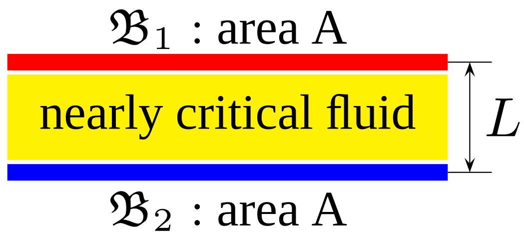
space dimension $d < 4$: Ginzburg criterion fails as $T \to T_c$!

"Thermodynamic" Casimir Effect



M.E. Fisher & P.-G. de Gennes (1978):

- large-λ modes $\approx$ massless
- consider *confined* nearly critical systems



partition sum: $Z = \sum_{\phi} e^{-\mathcal{H}[\phi]} = \int \mathcal{D}\phi \, e^{-\mathcal{H}[\phi]} = \exp\left[-F_{L,A}(T)/k_B T\right]$

$$\frac{F_{L,A}(T)}{k_B T} = \underbrace{L A \, f_{\text{bk}}(T)}_{\text{bulk contribution}} + \underbrace{A \left[f_{s,1}(T,\ldots) + f_{s,2}(T,\ldots)\right]}_{\text{surface contributions}} + \underbrace{A \, f_{\text{res}}(T, L, \ldots)}_{\text{residual}}$$

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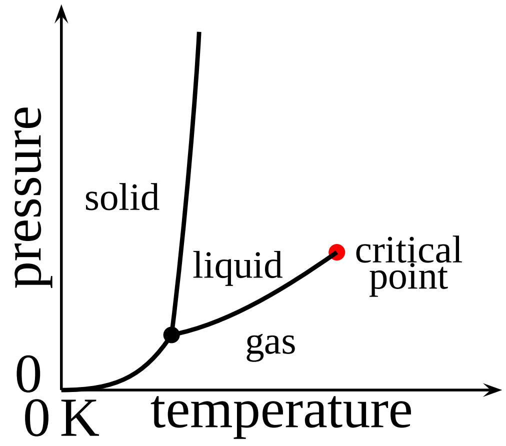

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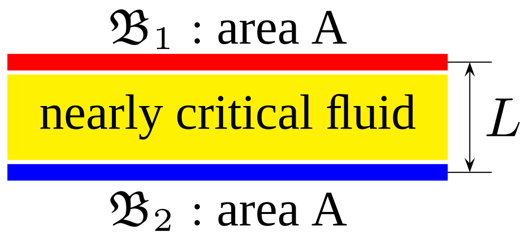

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- Casimir force per area: $$\mathcal{F}_C(T,L,\ldots)/A = -k_B T\,\frac{\partial f_{\mathrm{res}}}{\partial L}$$
- finite size scaling (*only* short-range interactions):

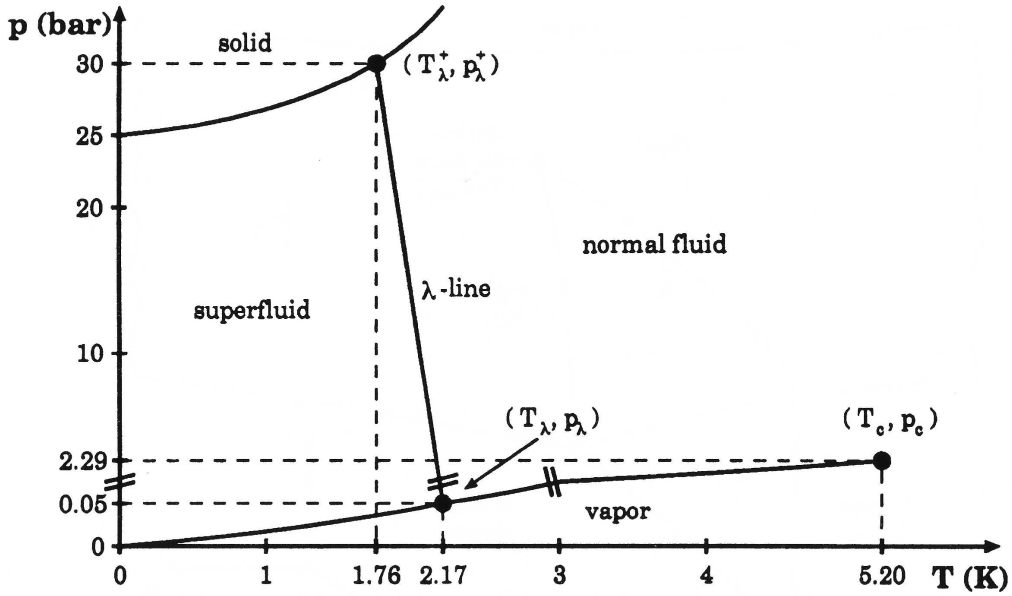
$$f_{\mathrm{res}}(T,L,\ldots) \approx L^{-(d-1)}\,\underbrace{Y}_{\text{universal}}(L/\xi_\infty,\ldots)$$

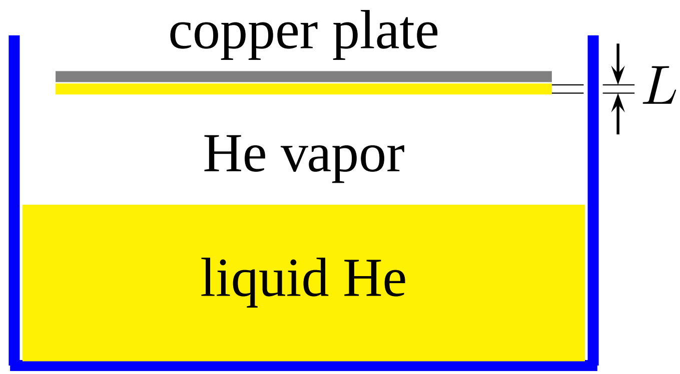
$$\text{at } T_{c,\infty}:\quad f_{\mathrm{res}} \approx \underbrace{\Delta_C}_{\text{Casimir amplitude}}\, L^{-(d-1)}$$

Experimentall Verification: ^{4}He wetting films

- indirect: wetting experiments
  - Garcia & Chan (Helium)
  - Fukuto, Yano & Pershan (binary liquids)
  - Rafai, Bonn & Meunier (binary liquids)
- direct: Hertlein, Helden, Gambassi, Dietrich & Bechinger (binary liquids)

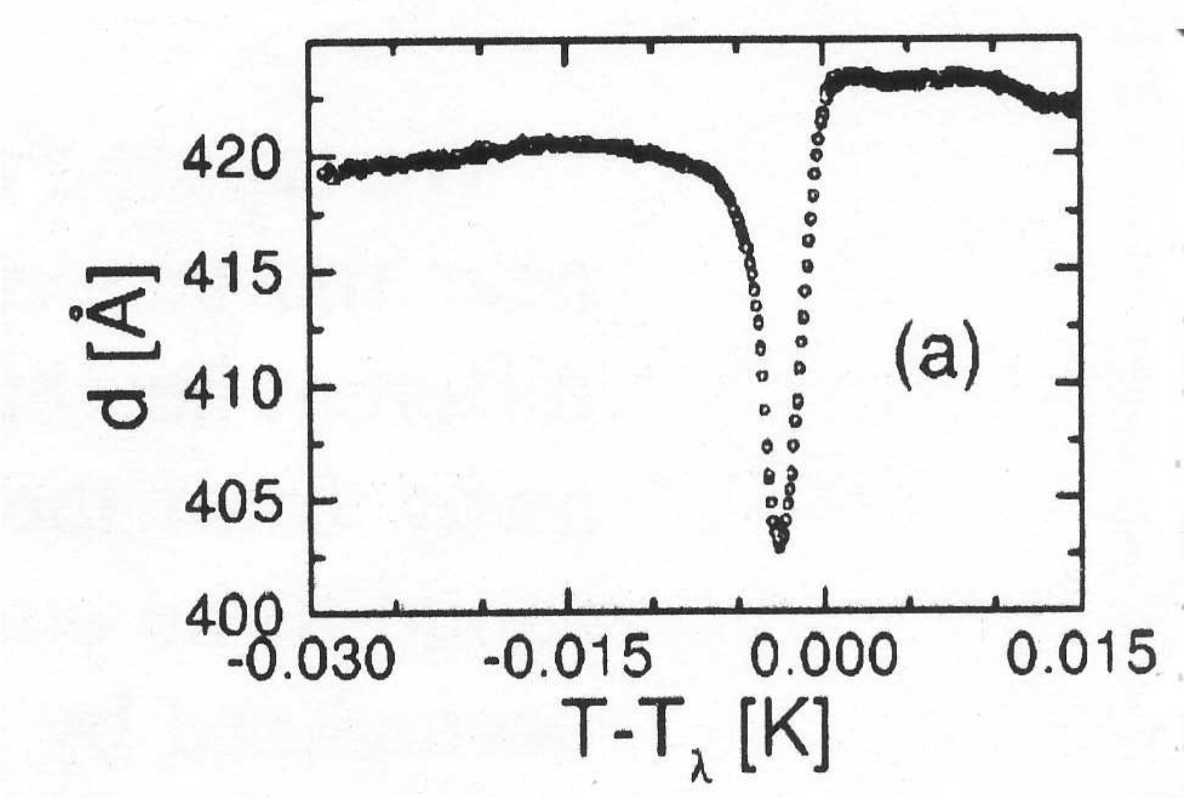
Experimentall Verification: ^{4}He wetting films





$$\underbrace{mgh}_{\text{gravitation}} = \underbrace{\frac{\gamma_{\text{vdW}}}{L^3}}_{\text{van der Waals}} \underbrace{\frac{1}{1+L/L_{1/2}}}_{\text{retardation}} + \underbrace{v\, k_B T\, \frac{\Xi(L/\xi)}{L^3}}_{\text{Casimir}}$$

Thinning of ^{4}He films near T_λ



Experiment:
Garcia & Chan, *PRL* **83**, 1187 (1999)

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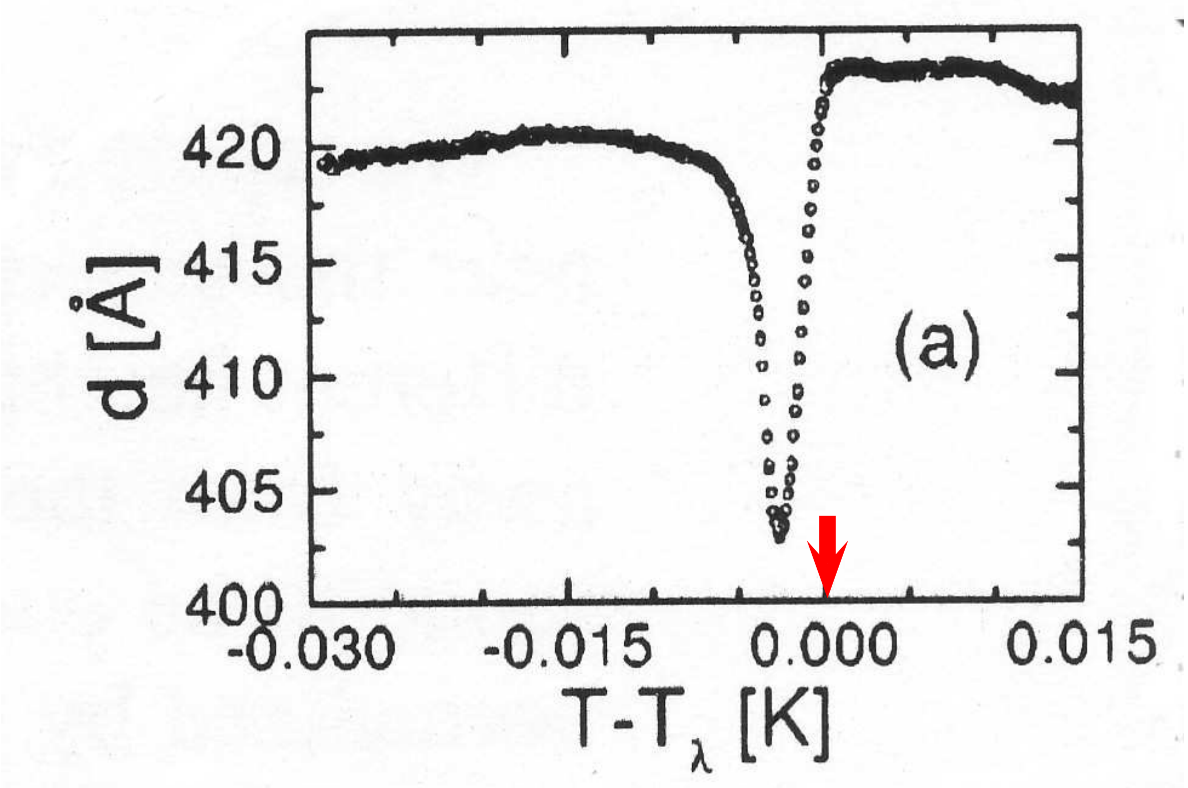

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- Theory: in rather modest state
  - Krech & Dietrich 1991/92: $T \geq T_\lambda$
  - Li & Kardar 1991: $T \ll T_\lambda$ (Goldstone modes)
  - Zandi, Rudnick & Kardar 2004: interface fluctuations
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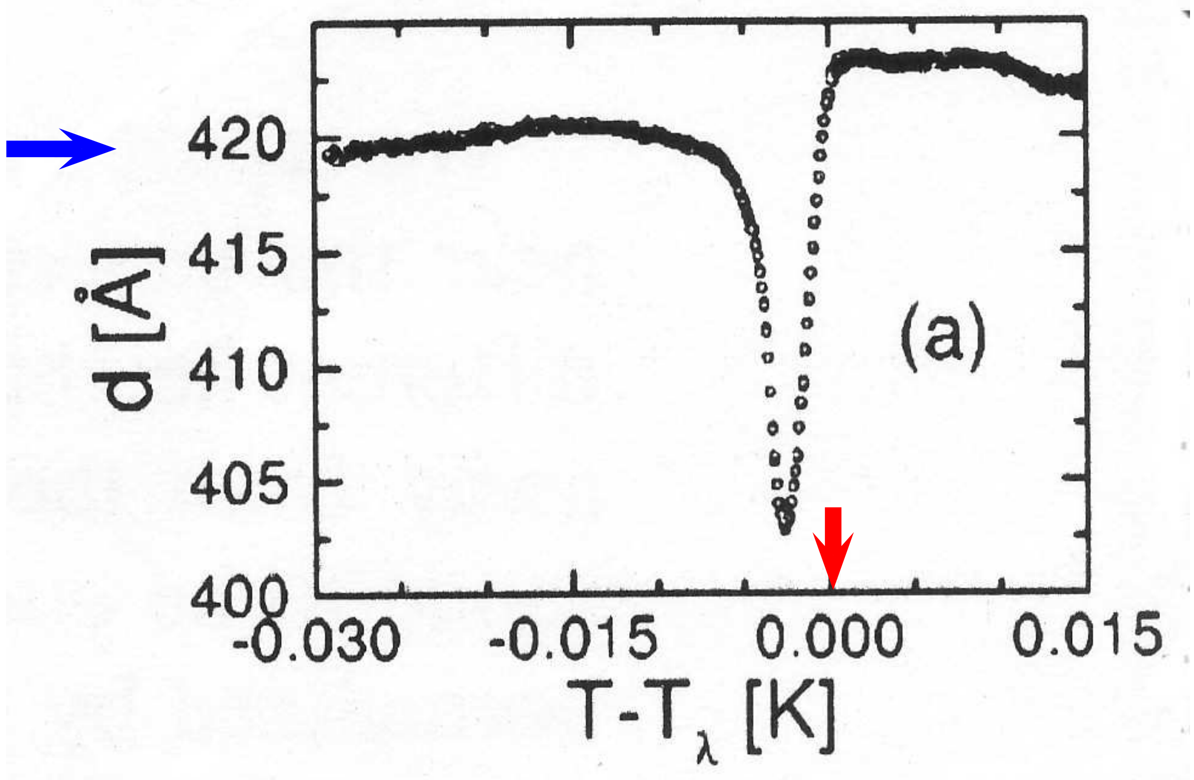

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Monte Carlo Results

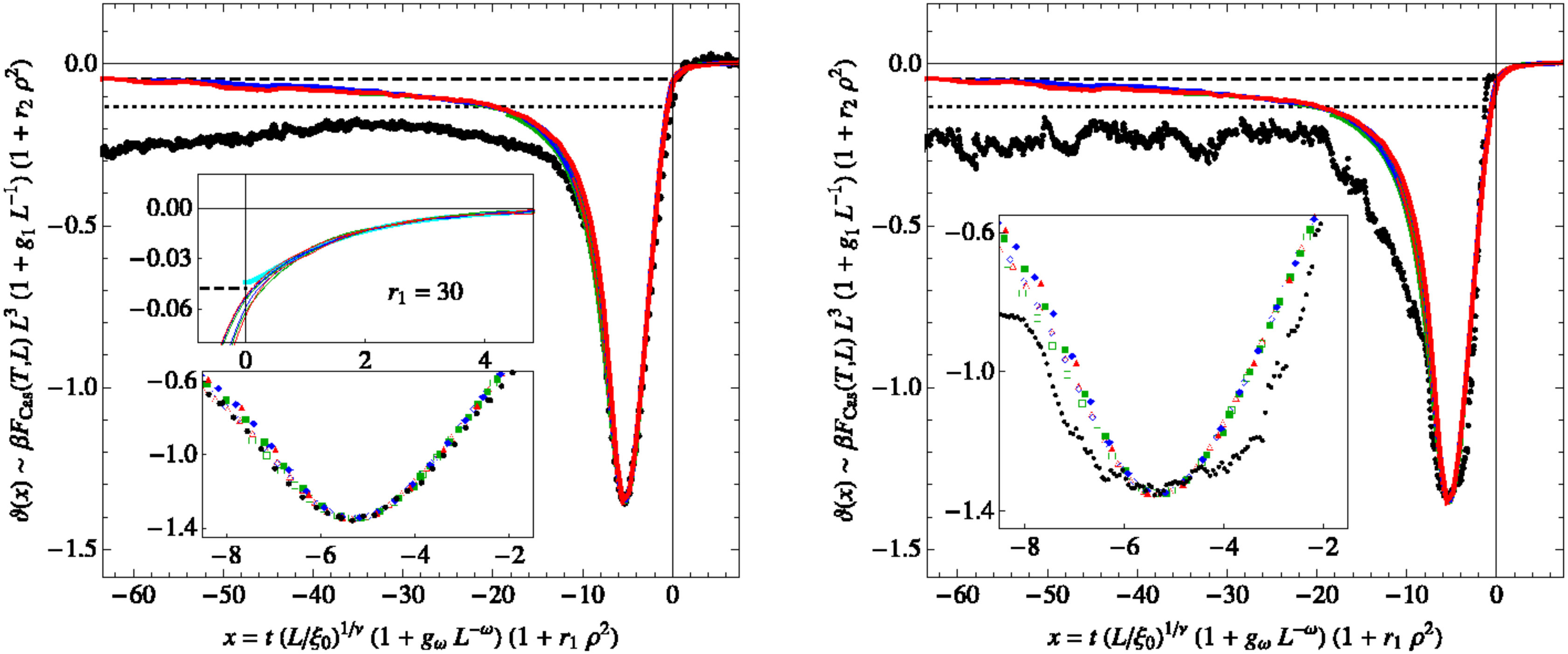
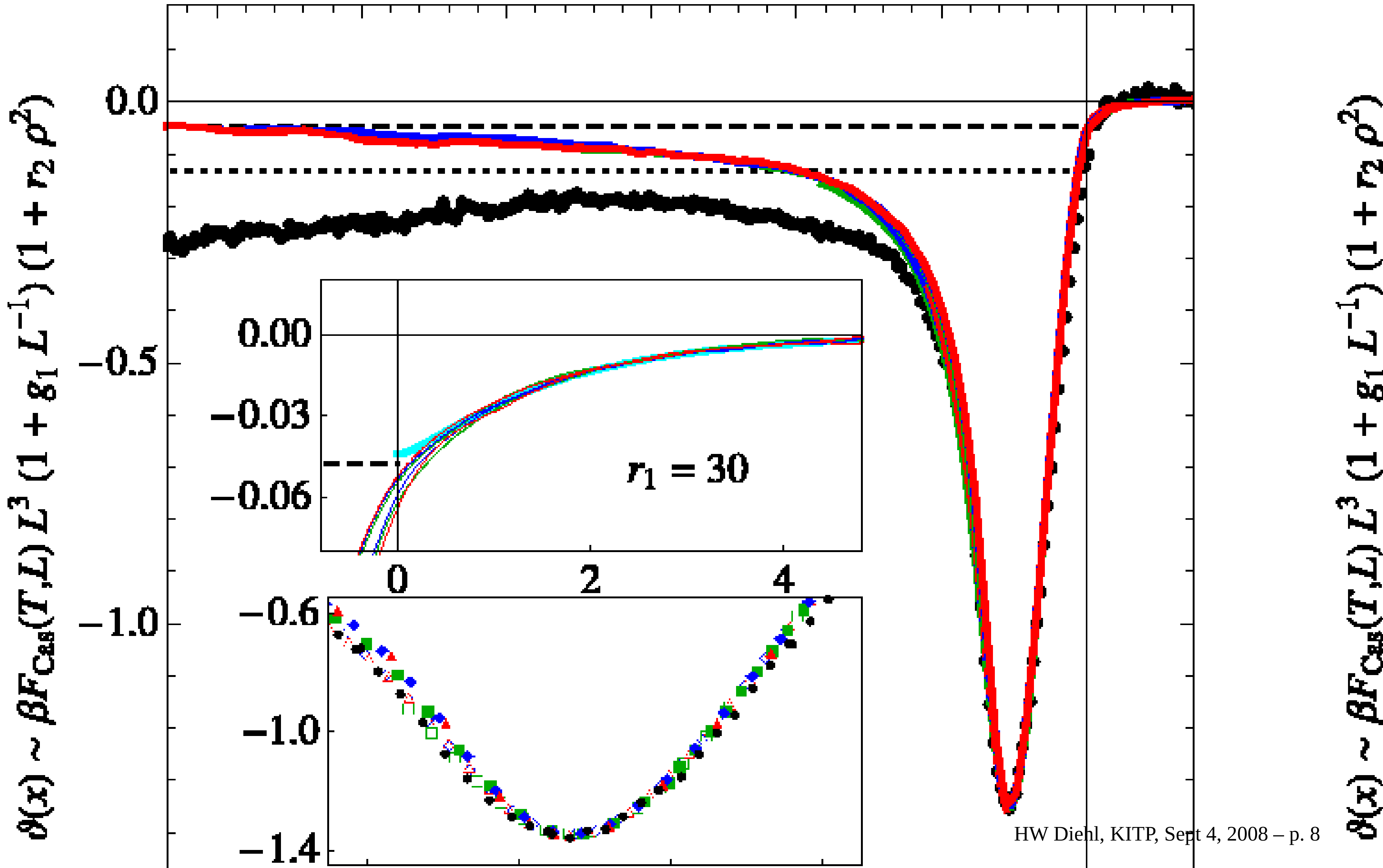


Figure 2: Universal finite-size scaling function $\vartheta(x)$ of the Casimir force, for systems with $L = 8$ (■), $L = 12$ (◆), and $L = 16$ (▲), with aspect ratios $\rho = 1{:}8$ (open) and $\rho = 1{:}16$ (filled), plotted with aspect ratio corrections. The results are compared to the experimental results (•) of Garcia and Chan [5, Cap. 1], rescaled in y-direction to match $\vartheta(x)$ at the minimum (left), as well as with the results of Ganshin *et al.* [7] (right). The capillary waves value $-11\zeta(3)/(32\pi)$ proposed in [4] is shown as dotted line. (color online)

A. Hucht (U. Duisburg-Essen): PRL **99**, 185301 (2007)

Monte Carlo Results



Relevant Issues

- bulk critical behavior at $T_{c,\infty}$ as $L \to \infty$
- confined critical fluctuations, boundary field theory
- finite size and boundary effects
- pseudo-critical or critical behavior in slab at $T_c(L) < T_{c,\infty}$ when $L < \infty$
- dimensional crossover
- low-T Casimir force from confined Goldstone modes
- interface fluctuations

Relevant Issues

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HERE:

- Restriction to
  a) disordered phases,
  b) $T \geq T_{c,\infty}$,
  c) generic non symmetry breaking boundary conditions
  $\Rightarrow$ focus directly fluctuation-induced forces!

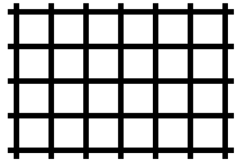
- **Synopsis:**
  (i) breakdown of $\epsilon = 4 - d$ expansion at $T_{c,\infty}$ for some boundary conditions
  (ii) boundary conditions = **scale-dependent** properties!
    - $\Rightarrow$ Neumann boundary condition $\underset{L\to\infty}{\longrightarrow}$ Dirichlet bc!
    - $\Rightarrow$ crossover attractive $\leftrightarrow$ repulsive Casimir forces

  (iii) effects of long-range (van-der-Waals-type) forces
  (see paper with Dantchev)

RG and Bulk Critical Behavior

- microscopic model, e.g. Ising model

$$\mathcal{H} = -\sum_{i\neq j} K_{ij}\, s_i s_j - H \sum_j s_j$$



- mesoscopic model: $\phi =$ order parameter field $\quad (|\boldsymbol{q}| \leq \Lambda)$

$$\mathcal{H} = \int d^d x \left[\frac{1}{2} (\nabla \phi)^2 + \frac{\mathring{\tau}}{2} \phi^2 + \frac{\mathring{u}}{4!} \phi^4 - \mathring{h}\, \phi \right],$$

- behavior for $q \ll \Lambda$: via renormalized field theory: $\Lambda \to \infty$, requires renormalizations
- dimensionless (renormalized) coupling constants $\{g_j = \tau, u, h, \dots\}$

$$\mathring{u} = \mu^{\epsilon} Z_u(u, \Lambda/\mu)\, u\,, \quad \mathring{\tau} - \mathring{\tau}_c = Z_\tau\, \mu^2 \tau\,, \quad \mathring{h} = \mu^{(d+2)/2} Z_\phi^{-1/2} h\,, \quad \phi = Z_\phi^{1/2} \phi_R\,.$$

- $\mu \to \mu\ell \quad \Rightarrow \quad \boxed{g_j \to \bar{g}_j(\ell)}$ running interaction constants

$$\boxed{\ell \frac{d}{d\ell}\, \bar{g}_j(\ell) = \beta_j[\bar{\boldsymbol{g}}(\ell)]}\,, \quad \beta_j(\bar{\boldsymbol{g}}) = \mu\partial_\mu\big|_0\, g_j$$

2-Scale-Factor Universality

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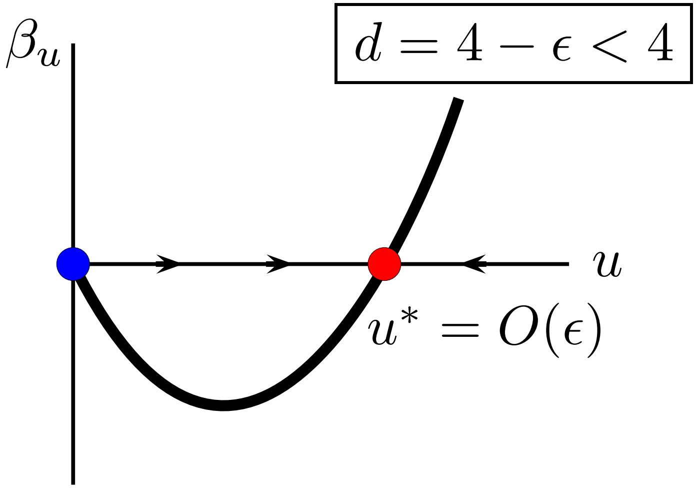

$$\bar{u}(\ell) \underset{\ell\to 0}{\approx} u^* + \text{const}\,(u - u^*)\,\ell^{\omega}$$

$$\bar{\tau}(\ell) \underset{\ell\to 0}{\approx} E_\tau(u)\,\ell^{-\overbrace{\nu}^{\text{universal}}}\,\tau$$

$$\bar{h}(\ell) \underset{\ell\to 0}{\approx} \underbrace{E_h(u)}_{\text{nonuniversal}}\,\ell^{-\Delta/\nu}\,h$$

$$G(\boldsymbol{x}, \ldots; \tau, h, u) \approx \xi^{-d_G - \eta_G}\,\underbrace{E_G(u)}_{\text{powers of } E_h,\, E_\tau}\,\underbrace{G(\boldsymbol{x}/\xi, \ldots; 1, h\,\xi^{\Delta/\nu}, u^*)}_{\text{scaling function}}, \qquad \xi \sim \tau^{-\nu}, \quad \tau = \frac{T - T_{c,\infty}}{T_{c,\infty}}$$

- universality (crit. exponents, scaling functions, amplitude ratios)
- 2-scale-factor universality
- corrections to scaling from terms $\sim (u - u^*)\,\xi^{-\omega}$

Finite Size Scaling ($L^d < \infty$)

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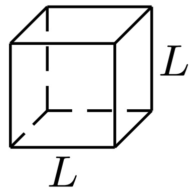
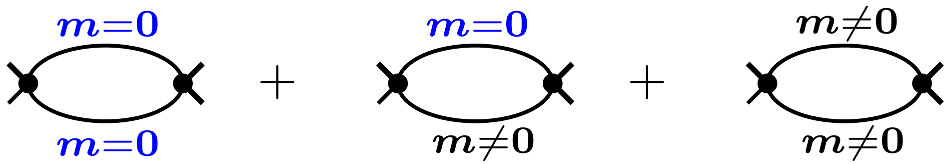
- M.E. Fisher (1971): $\boxed{P_L(\tau)/P_\infty(\tau) = f(\underbrace{L/\xi_\infty}_{\text{relevant ratio}})}$, $\tau = (T - T_{c,\infty})/T_{c,\infty}$
- requires that *no other lengths* matter!
- Privman & Fisher (1984): $\boxed{\frac{F(h,\tau,L)}{k_B T} \approx X(h\,L^{\Delta/\nu}, \tau\,L^{\nu})}$ X universal, but dependent on boundary conditions
- not true for $d > d^* = 4$: $\exists$ 2nd (thermodynamic) length
  - Brézin (1982), $n \to \infty$: $\underbrace{\xi_L}_{\chi_L^{1/2}}(T_{c,\infty}) \sim \begin{cases} \epsilon^{-1/4}\,L\,, & d = 4 - \epsilon \uparrow 4 \\ L\,(\ln L)^{1/4}\,, & d = 4 \\ L\,L^{(d-4)/4}\,, & d > 4 \end{cases}$

RG for Finite Systems: Torus $T_d(L)$

periodic bc



- free propagator: $G_L^{(\text{pbc})}(\boldsymbol{x}_{12}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^d}^{\infty} G_\infty(\boldsymbol{x}_{12} + \boldsymbol{m}L)$

- 

uv singular (bulk) $+$ uv finite $+$ uv finite

- bulk renormalizations (“counter terms”) sufficient to absorb uv singularities

  (Symanzik 81, Brézin 82)

- $\Rightarrow$ bulk RG equations carry over, L not renormalized!

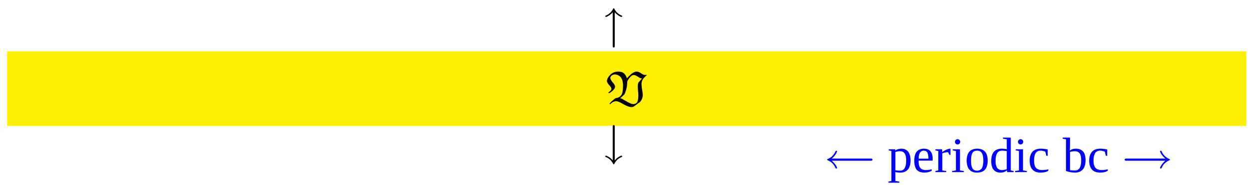
$$G(\boldsymbol{x},\ldots;L,\tau,h,u) \approx \xi^{-d_G-\eta_G}\, E_G(u)\, \underbrace{G(\boldsymbol{x}/\xi, L/\xi\ldots;1,h\,\xi^{\Delta/\nu},u^*)}_{\text{fs scaling function}}$$

- problems: $\lim_{\bar{u}\to u^*} G \to G|_{u^*}$?, computation of scaling function!

  RG-improved perturbation theory? $\exists$ ϵ-expansion? $1/n$ expansion?

$L \times \infty^{d-1}$ Slabs

- n-component ϕ^4-model, $\mathfrak{V} \equiv \mathbb{R}^{d-1} \times [0, L]$



$$\mathcal{H}[\phi] = \int_{\mathfrak{V}} d^d x \left[\frac{1}{2} (\nabla\phi)^2 + \frac{\mathring{\tau}}{2} \phi^2 + \frac{\mathring{u}}{4!} \phi^4 \right]$$

- antiperiodic bc: $\boxed{\phi(\boldsymbol{x}) = \pm\phi(\boldsymbol{x} + L\,\hat{\boldsymbol{z}})}$

no new counter terms

L not renormalized $\Rightarrow$ dependence on L/ξ !

Lattice and Continuum Models

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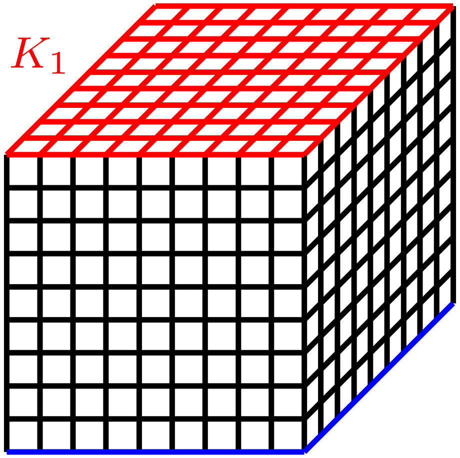
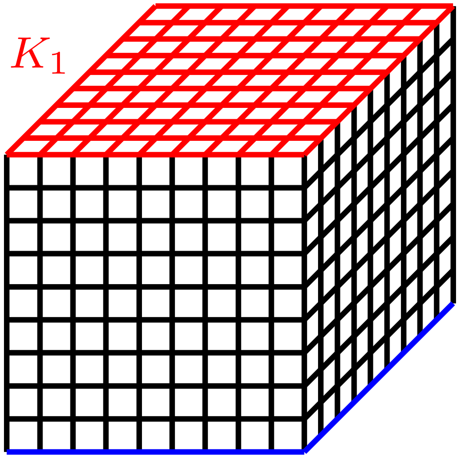


$$\mathcal{H}_{\text{lat}} = -K \sum_{\langle i,j\rangle \notin \mathfrak{B}_1 \cup \mathfrak{B}_2} s_i s_j - K_1 \sum_{\langle i,j\rangle \in \mathfrak{B}_1} s_i s_j - K_2 \sum_{\langle i,j\rangle \in \mathfrak{B}_2} s_i s_j$$

Lattice and Continuum Models



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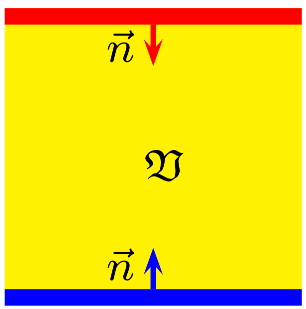

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$$\mathring{c}_j = 1 - 2(d-1)[K_j/K - 1]/a$$

Lattice and Continuum Models

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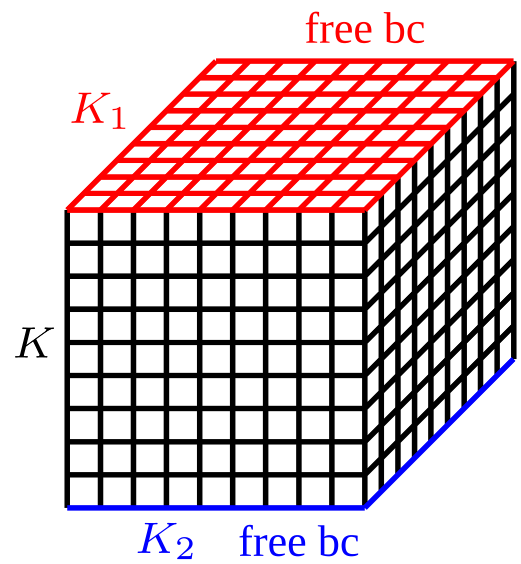

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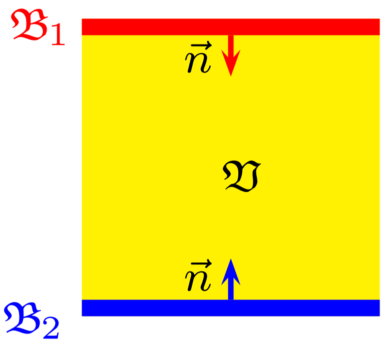

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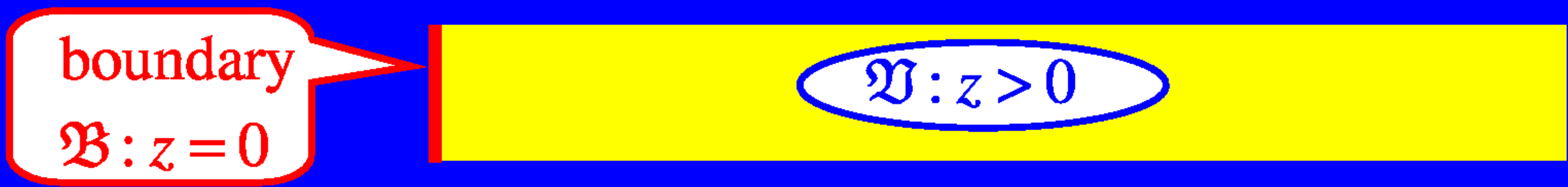
$$\mathring{c}_j = 1 - 2(d-1)[K_j/K - 1]/a$$

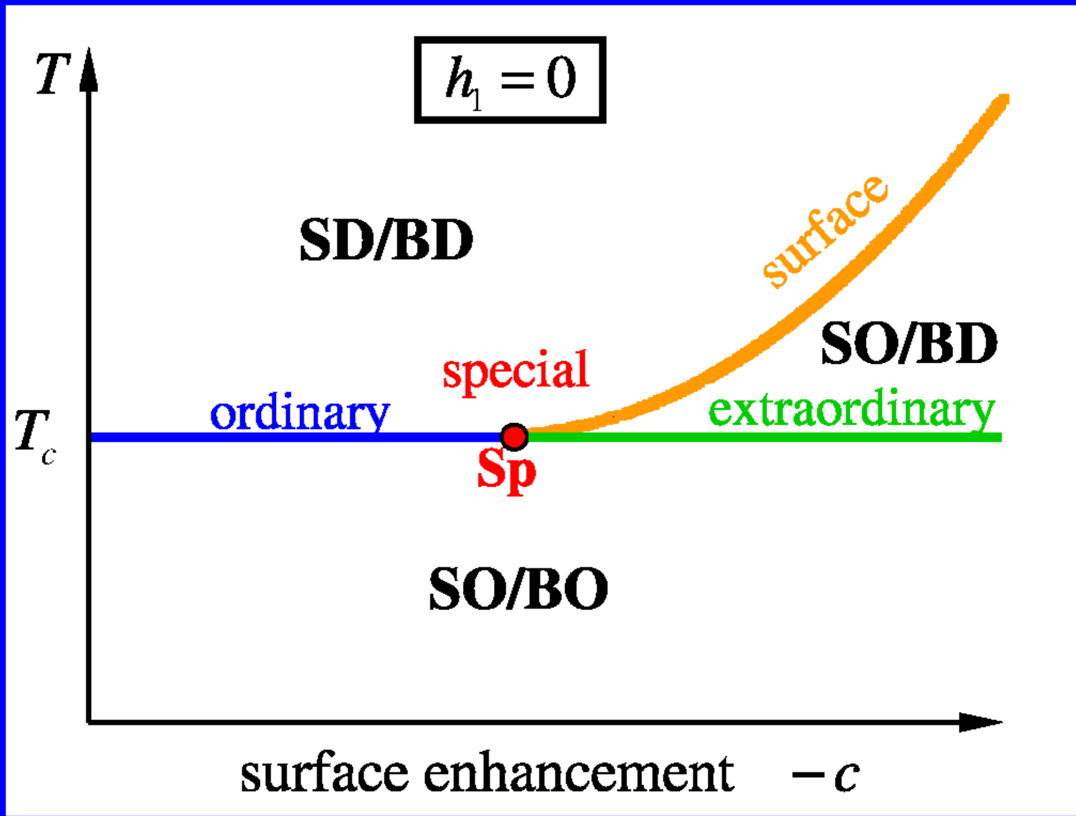
(fluctuating) boundary condition: $\boxed{\partial_n \phi = \mathring{c}_j\, \phi}$

$$\mathring{c}_j = \begin{cases} \infty & : \text{Dirichlet} \\ 0 & : \text{Neumann} \end{cases}$$

Semi-infinite n-Vector Model



$$\mathcal{H} = \int_{\mathfrak{V}=\mathbb{R}^d_+} \mathrm{d}V \left[\frac{1}{2}(\nabla\phi)^2 + \frac{\tau_0}{2}\phi^2 + \frac{u_0}{4!}\phi^4\right] + \int_{\mathfrak{B}} \mathrm{d}A \left[\frac{c_0}{2}\phi^2\right]$$



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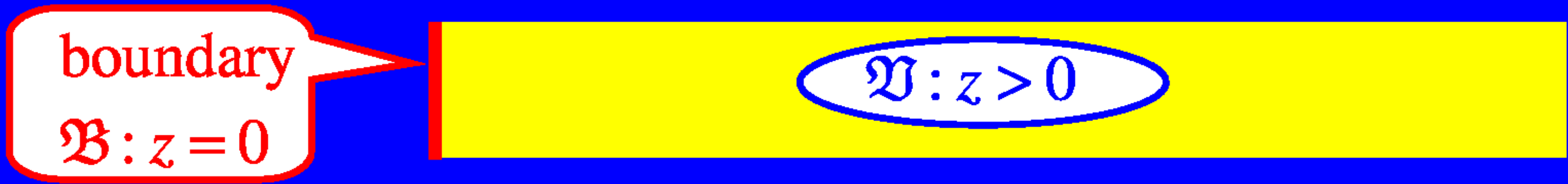

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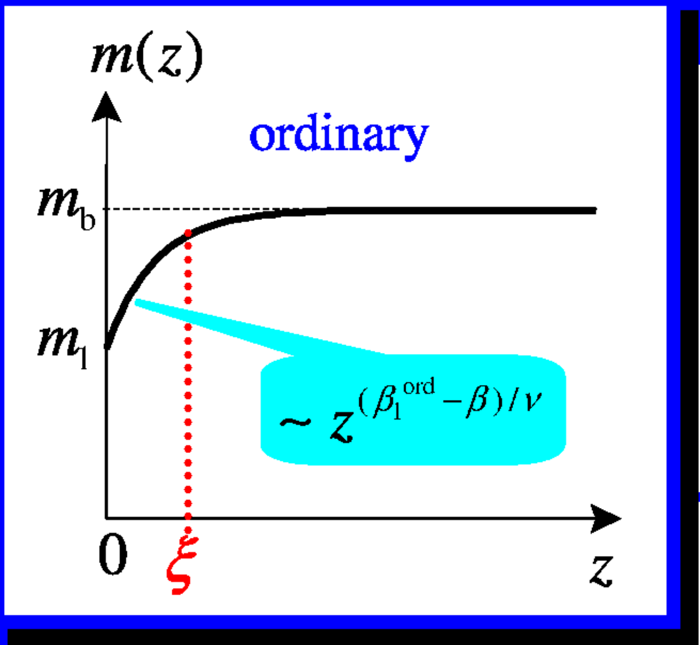

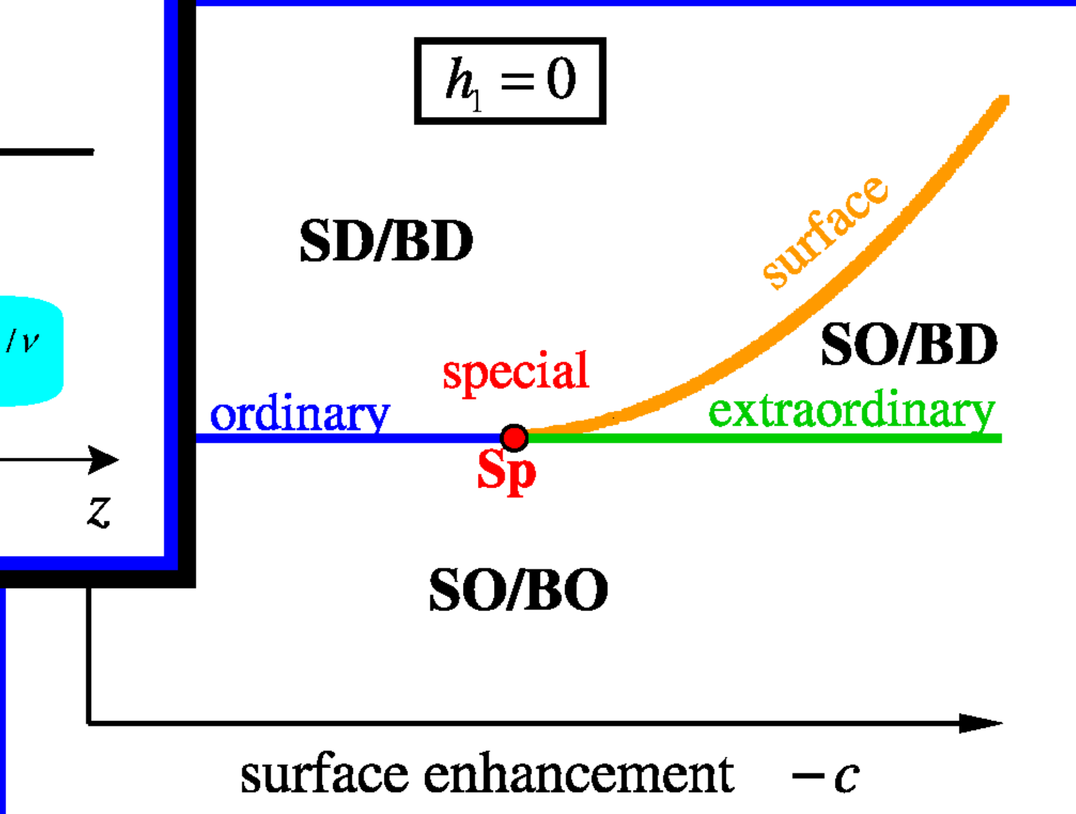

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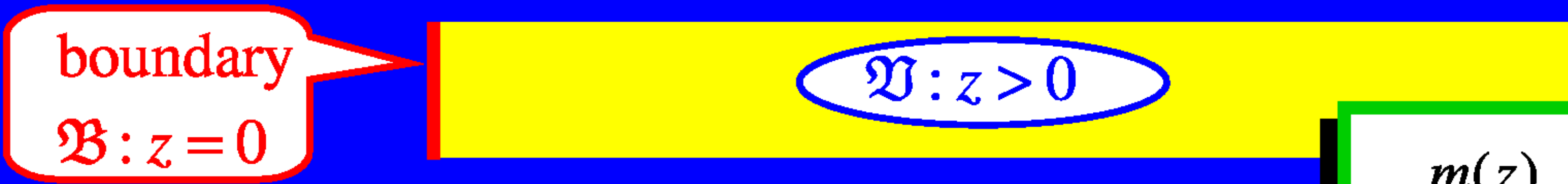

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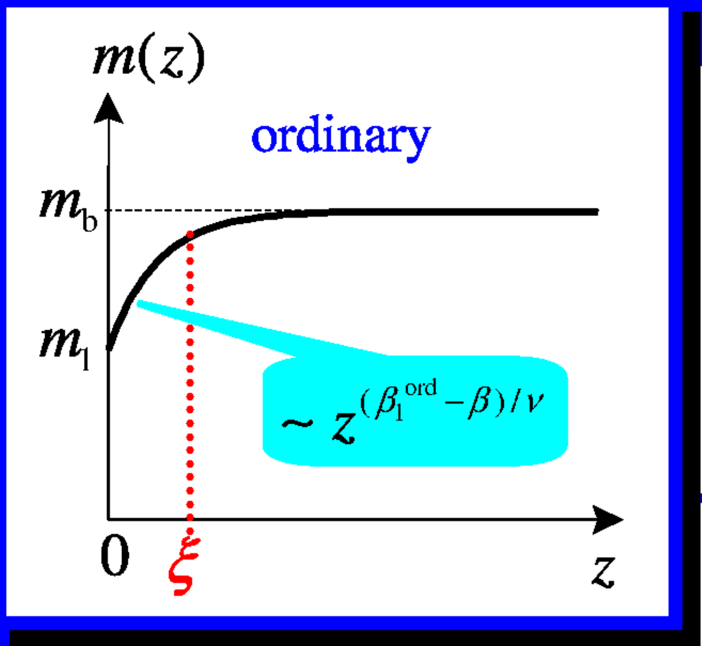

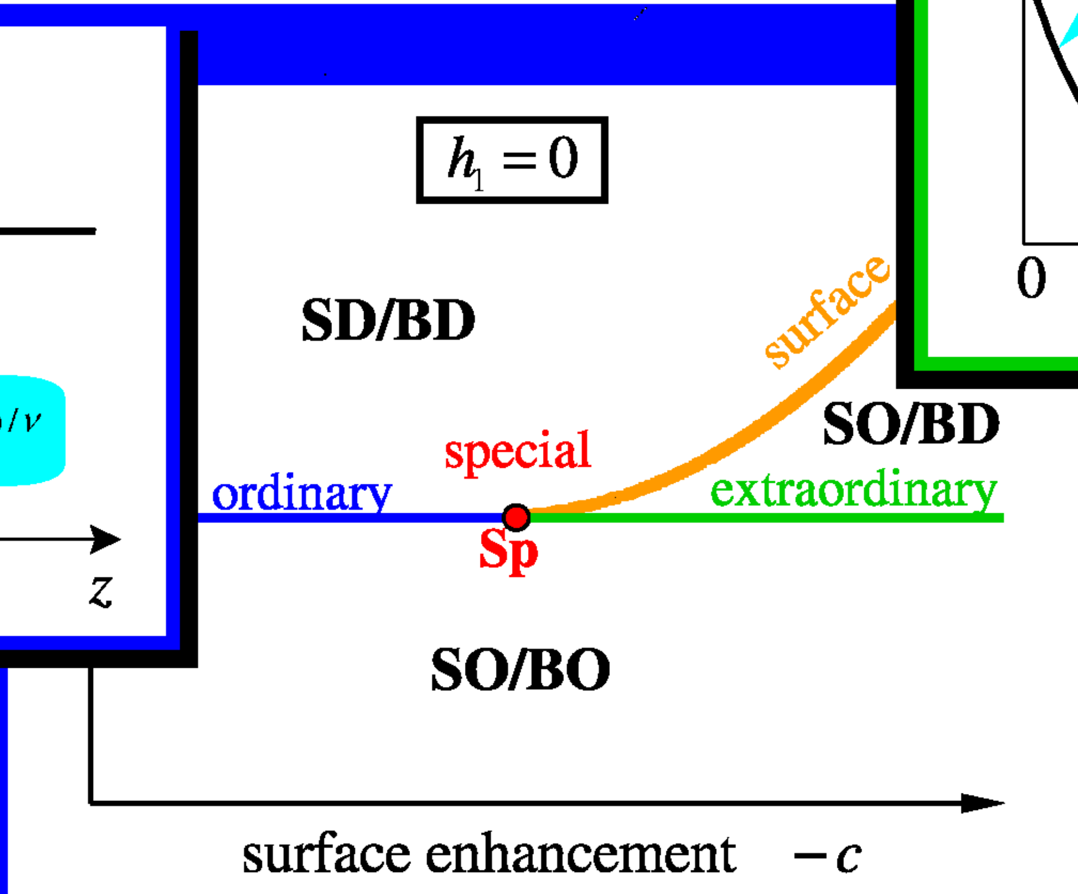

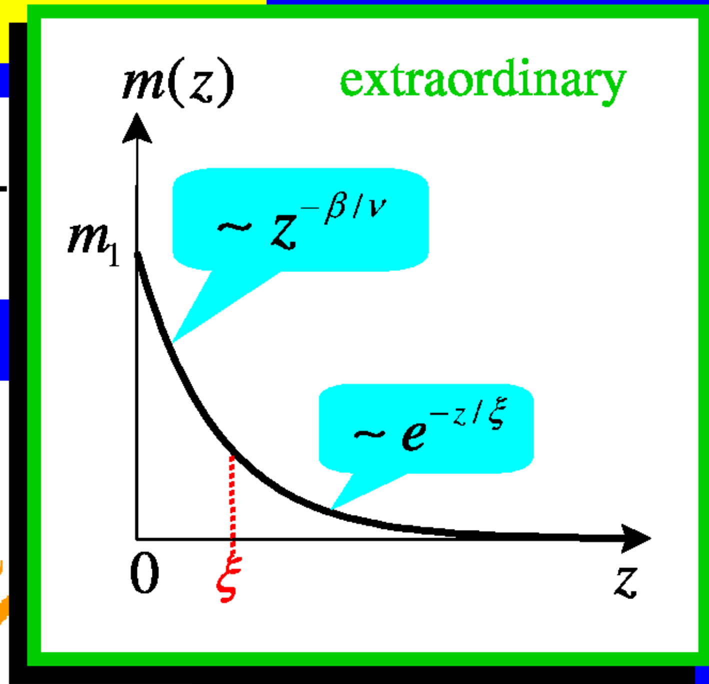

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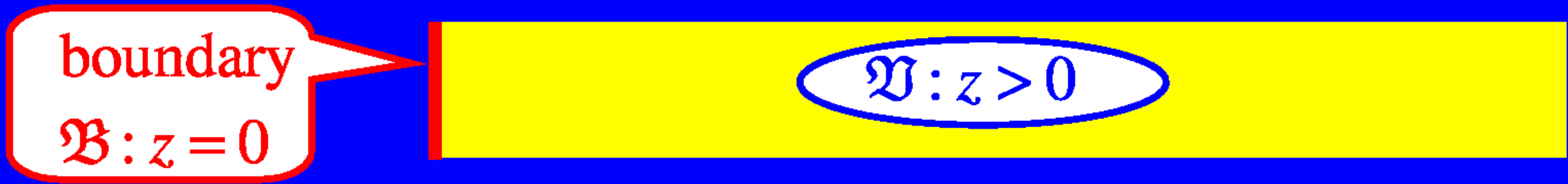

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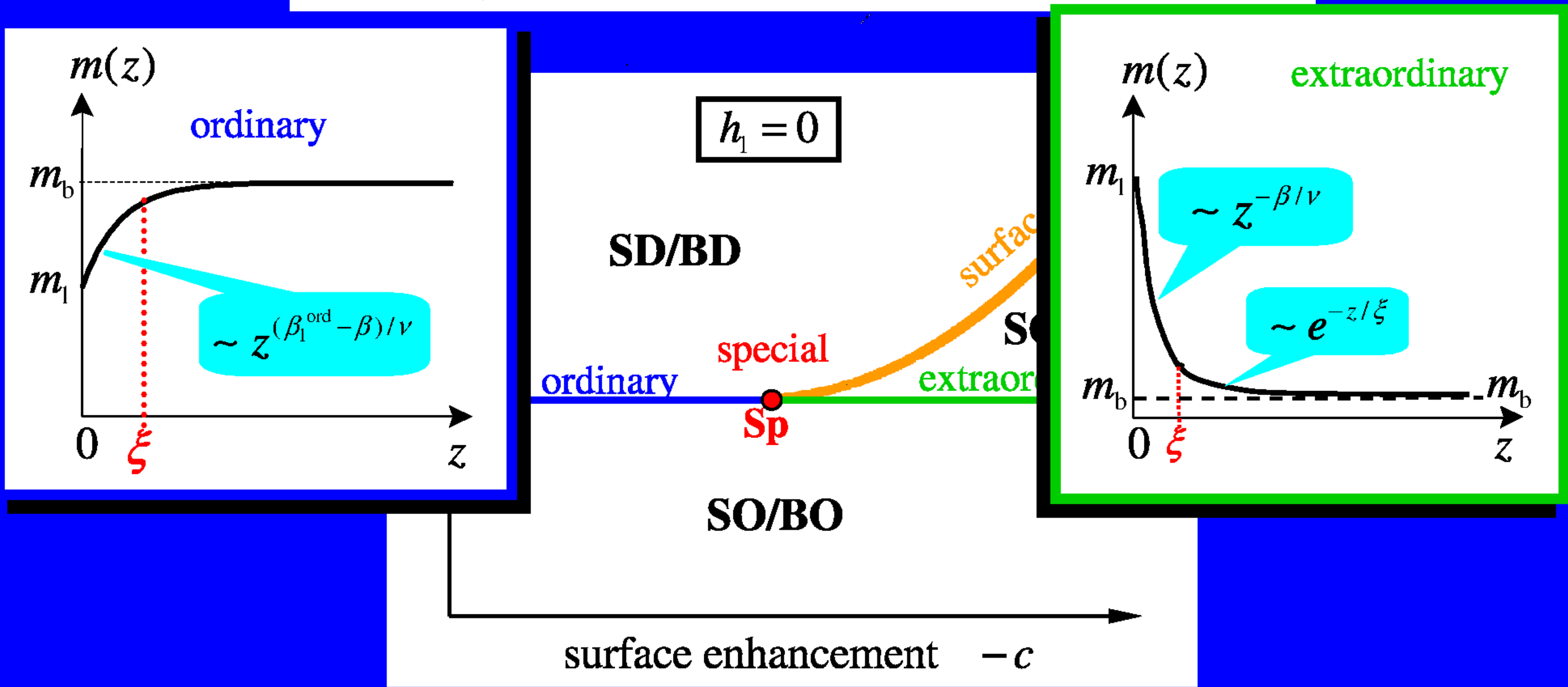

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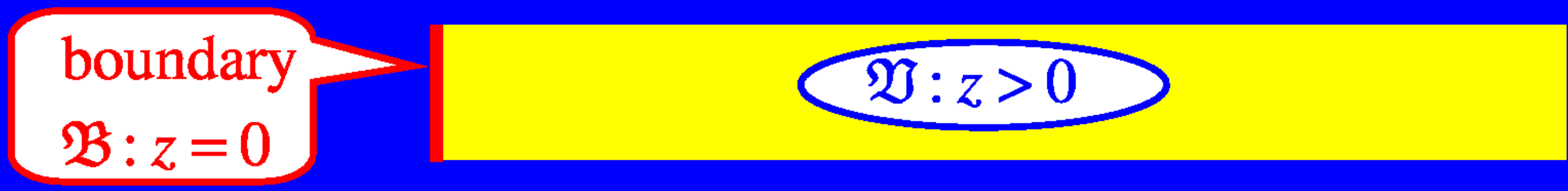

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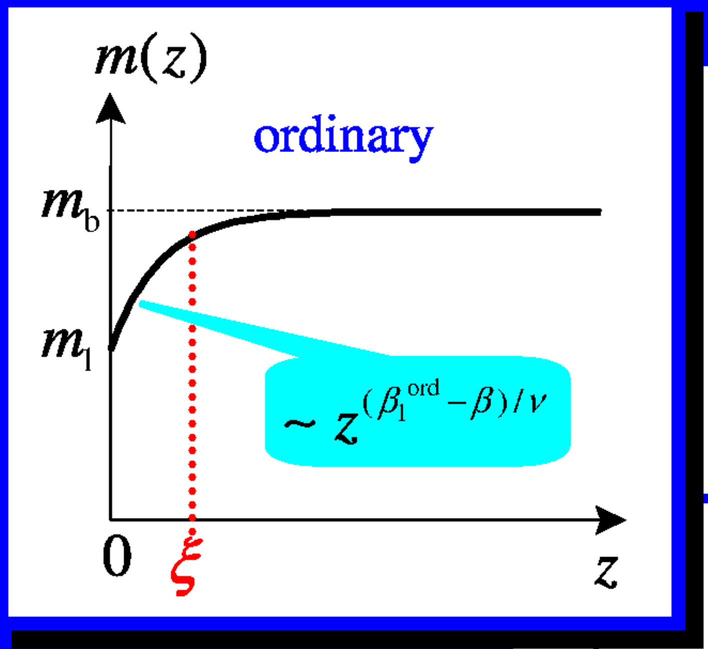

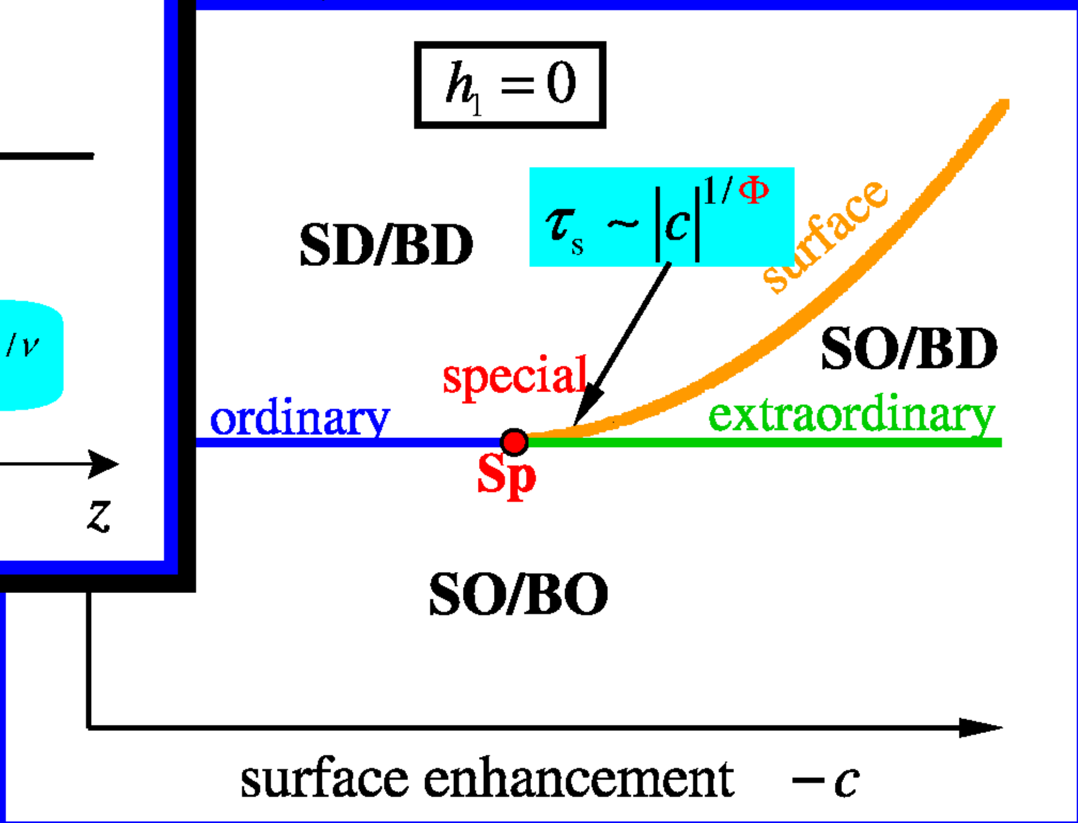

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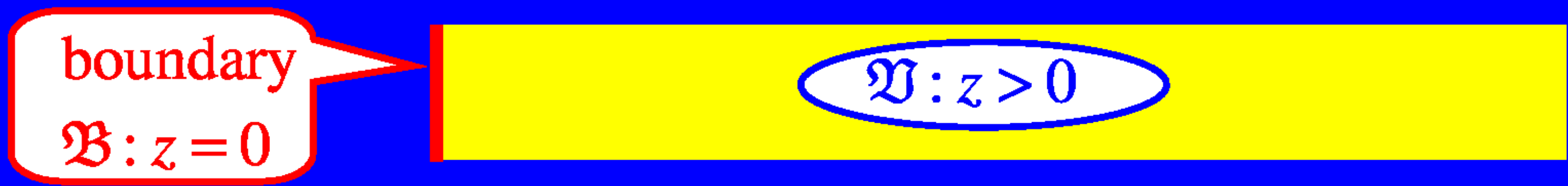

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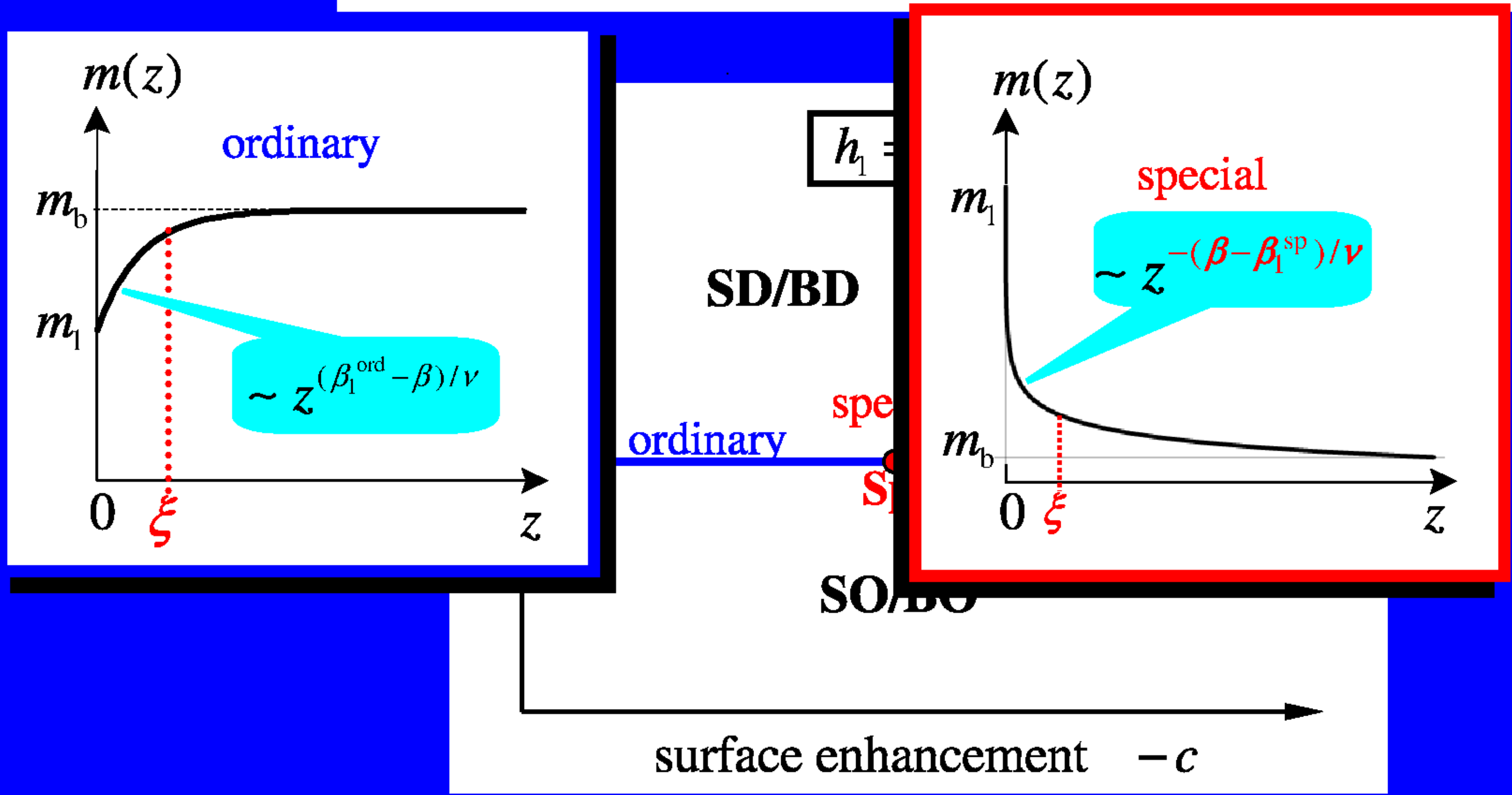

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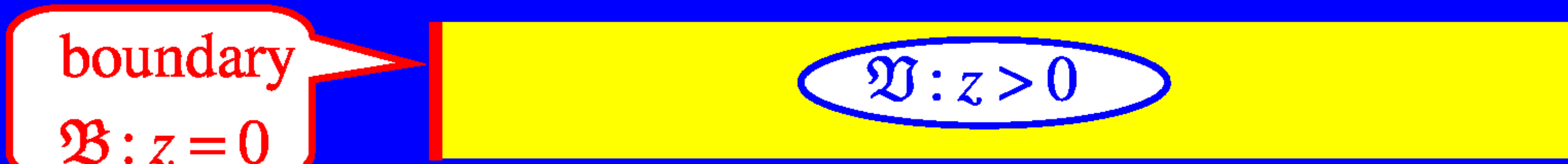

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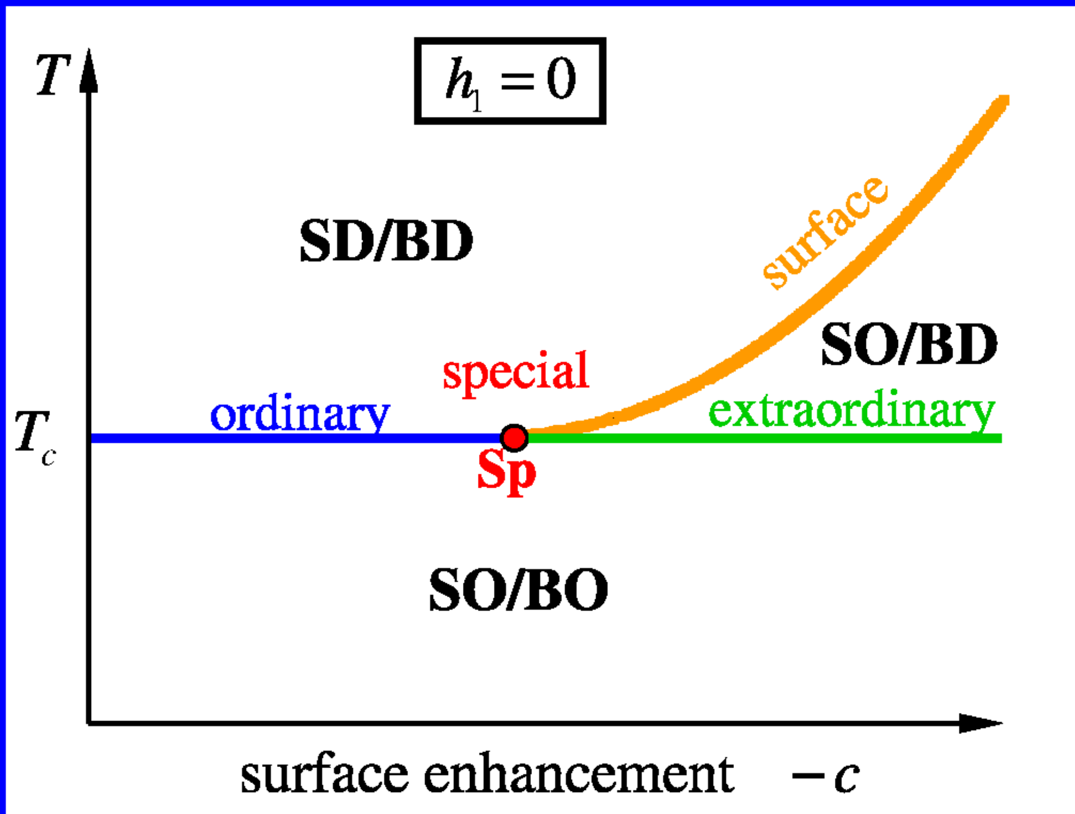

Boundary Conditions $\overset{?}{\longleftrightarrow}$ Universality

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Boundary conditions $\overset{?}{\Longleftrightarrow}$ universal Δ_C, Y etc?

- various scales
  1. **microscopic:** lattice ($a_{\text{min}} = a$)
     - free bc; $K_j \neq K$ in general
  2. **mesoscopic:** continuum theory ($a_{\text{min}} \gtrsim \pi/\Lambda$)
     - $\partial_n \phi = \mathring{c}_j \, \phi$; $d = 3$: $\boxed{\mathring{c}_j a = 1 - 4(K_j/K - 1)}$ *(nonuniversal fctn)*
     - $\mathring{c}_j = \infty \quad \Leftrightarrow$ Dirichlet
     - $\mathring{c}_j = 0 \quad \Leftrightarrow$ Neumann

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   - $\mathring{c}_j = \mathring{c}_{\text{sp}} \Leftrightarrow$ (MC, $d = 3$) $\boxed{K_j/K \simeq 1.5}$

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3. **large length scales:** $a \ll z \lesssim \xi$

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   b) $c = 0$ (critical, special transition): $\phi(\boldsymbol{x}) \approx \underbrace{C_{\text{sp}}(z_j)}_{\sim z_j^{-(\beta - \beta_1^{\text{sp}})/\nu} \to \infty} \phi|_{\mathfrak{B}_j}$

   c) Neumann bc: *no* (stable or unstable) fixed pt associated!

Previous Field Theory RG Results

RG analysis in $d = 4 - \epsilon$ dimensions:

- Symanzik 1981: $\boxed{\Delta_C^{(\mathrm{bc})}(\epsilon, n)/n = a_0^{(\mathrm{bc})} + a_1^{(\mathrm{bc})}(n)\,\epsilon}$

  for *Dirichlet-Dirichlet (D-D)* boundary conditions ($\mathring{c}_1 = \mathring{c}_2 = \infty$)

- Krech & Dietrich 1991, 1992: $\Delta_C^{(\mathrm{bc})}(\epsilon, n)$ and $Y^{(\mathrm{bc})}$ for $T \geq T_{c,\infty}$ to $O(\epsilon)$

  for bc $=$ periodic, antiperiodic, $\overbrace{\text{D-D, D-sp, sp-sp}}^{\text{free bc}}$

  $$\Delta_C^{(\mathrm{bc})} \begin{cases} < 0 & \mathrm{bc} = \text{per, D-D, sp-sp} \\ > 0 & \mathrm{bc} = \text{aper, D-sp} \end{cases}$$

"sp" $=$ "special" $=$ "critically enhanced"

$$c = 0 \quad \Leftrightarrow \quad \mathring{c}_{\mathrm{sp}} = \mathring{c}_{\mathrm{sp}}(\Lambda) \neq 0$$

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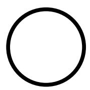
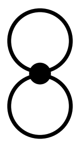
- Loop expansion: $\dfrac{-F}{k_B T} =$  $+$  $+ \mathcal{O}(\text{3-loops})$

-  $= G_L^{(\mathrm{bc})}(\boldsymbol{x}; \boldsymbol{x}') = \displaystyle\int \frac{d^{d-1}p}{(2\pi)^{d-1}} \sum_m \frac{\langle z|m\rangle\langle m|z'\rangle}{p^2 + k_m^2 + \mathring{\tau}}\, e^{i\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}')}$

Free Propagator

$$G_{\rm bc}^{(L)}(\boldsymbol{x};\boldsymbol{x}') = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \sum_m \frac{\langle z|m\rangle\langle m|z'\rangle}{p^2+k_m^2+\mathring{\tau}}\, e^{i\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}')}$$

- periodic boundary conditions: $\exists$ zero mode at $\mathring{\tau}=0 \quad (T=T_{c,\infty})$

$$\langle z|m\rangle = L^{-1/2}\, e^{ik_m z}\,, \quad k_m = 2\pi m/L\,, \quad m = 0, \pm 1, \pm 2, \ldots$$

- antiperiodic boundary conditions: no zero mode
- Dirichlet-Dirichlet bc: no zero mode

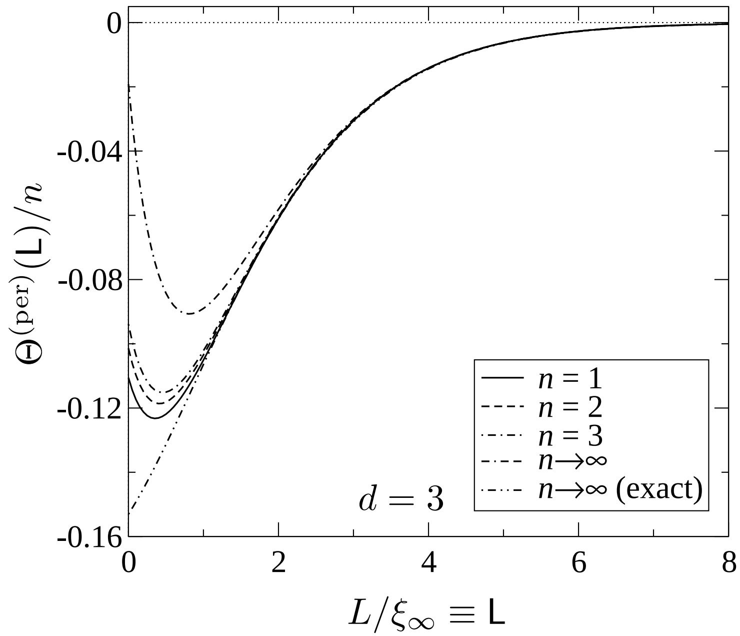
$$\langle z|m\rangle = \sqrt{\frac{2}{L}}\,\sin(k_m z)\,, \quad k_m = \pi m/L\,, \quad m = 1, 2, \ldots$$

- Dirichlet-Neumann bc: no zero mode
- Neumann-Neumann bc: $\exists$ zero mode at $\mathring{\tau}=0 \quad (T=T_{c,\infty})$

$$\langle z|m\rangle = \sqrt{\frac{2-\delta_{m,0}}{L}}\,\cos(k_m z)\,, \quad k_m = \pi m/L\,, \quad m = 0, 1, 2, \ldots$$

Problems: for periodic and sp-sp bc

- n dependence of scaling function (periodic bc)



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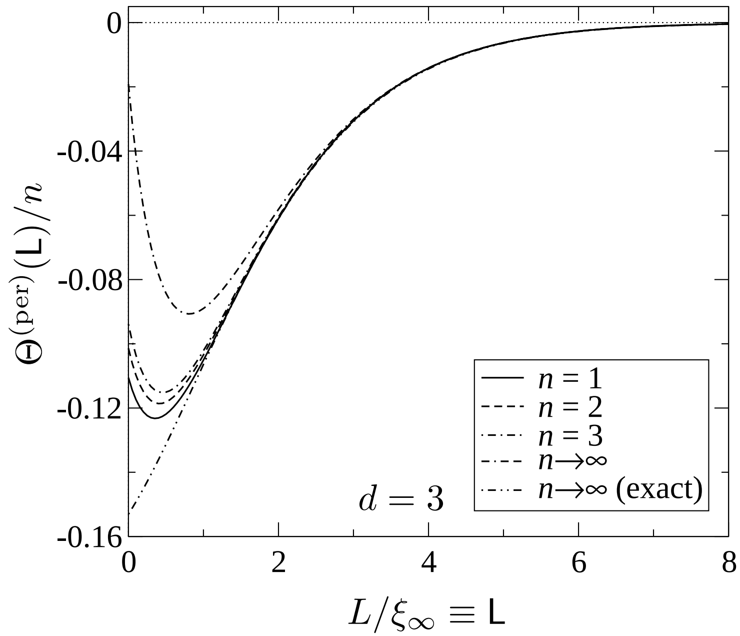
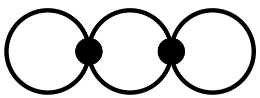
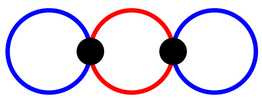


- $\Rightarrow$ theory ill-defined beyond 2 loops
  periodic and sp-sp boundary conditions involve *zero modes* at $T_{c,\infty}$!

-  $= \ldots +$  $+ \ldots =$ infrared singular at $T_{c,\infty}$

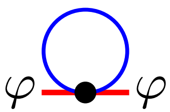
- violation of analyticity requirements at $T_{c,\infty}$ when $L < \infty$

Revised Field Theory

- $\phi(\boldsymbol{r}, z) = \underbrace{\varphi(\boldsymbol{r})}_{\text{0-mode contribution}} + \psi(\boldsymbol{r}, z)\ , \qquad \int_0^L dz\, \psi(\boldsymbol{r}, z) = 0\ .$

- effective $(d-1)$-dimensional FT: $\boxed{e^{-\mathcal{H}_{\text{eff}}[\varphi]} \equiv \text{Tr}_\psi\, e^{-\mathcal{H}[\varphi+\psi]}}$

$$\mathcal{H}_{\text{eff}}[\varphi] = F_\psi + \mathcal{H}[\varphi] - \ln \left\langle e^{-\mathcal{H}_{\text{int}}[\varphi,\psi]} \right\rangle$$

- 

gives shift $\boxed{\mathring{\tau} \longrightarrow \mathring{\tau}_{\text{bc}}^{(L)} = \mathring{\tau} + \delta\mathring{\tau}_{\text{bc}}^{(L)}}$

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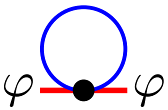
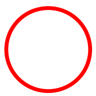
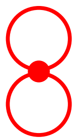
-  gives shift $\boxed{\mathring{\tau} \longrightarrow \mathring{\tau}_{\rm bc}^{(L)} = \mathring{\tau} + \delta\mathring{\tau}_{\rm bc}^{(L)}}$

- $F = F_\psi +$  $+$  $+ \ldots$

  $\sim (u^*)^{(3-\epsilon)/2}$

  - RG improved perturbation theory infrared *well-behaved* at $T = T_{c,\infty}$!
  - …not at $T_{c,L} \Rightarrow$ must require $T \geq T_{c,\infty}$ (as for nonzero-mode bc)
  - $\Rightarrow \epsilon^{3/2},\ \epsilon^{5/2},\ \epsilon^{5/2}\ \ln\epsilon,\ \ldots$ contributions to Casimir amplitudes (in conformity with exact $n\to\infty$ solution)

Results: Casimir Amplitudes

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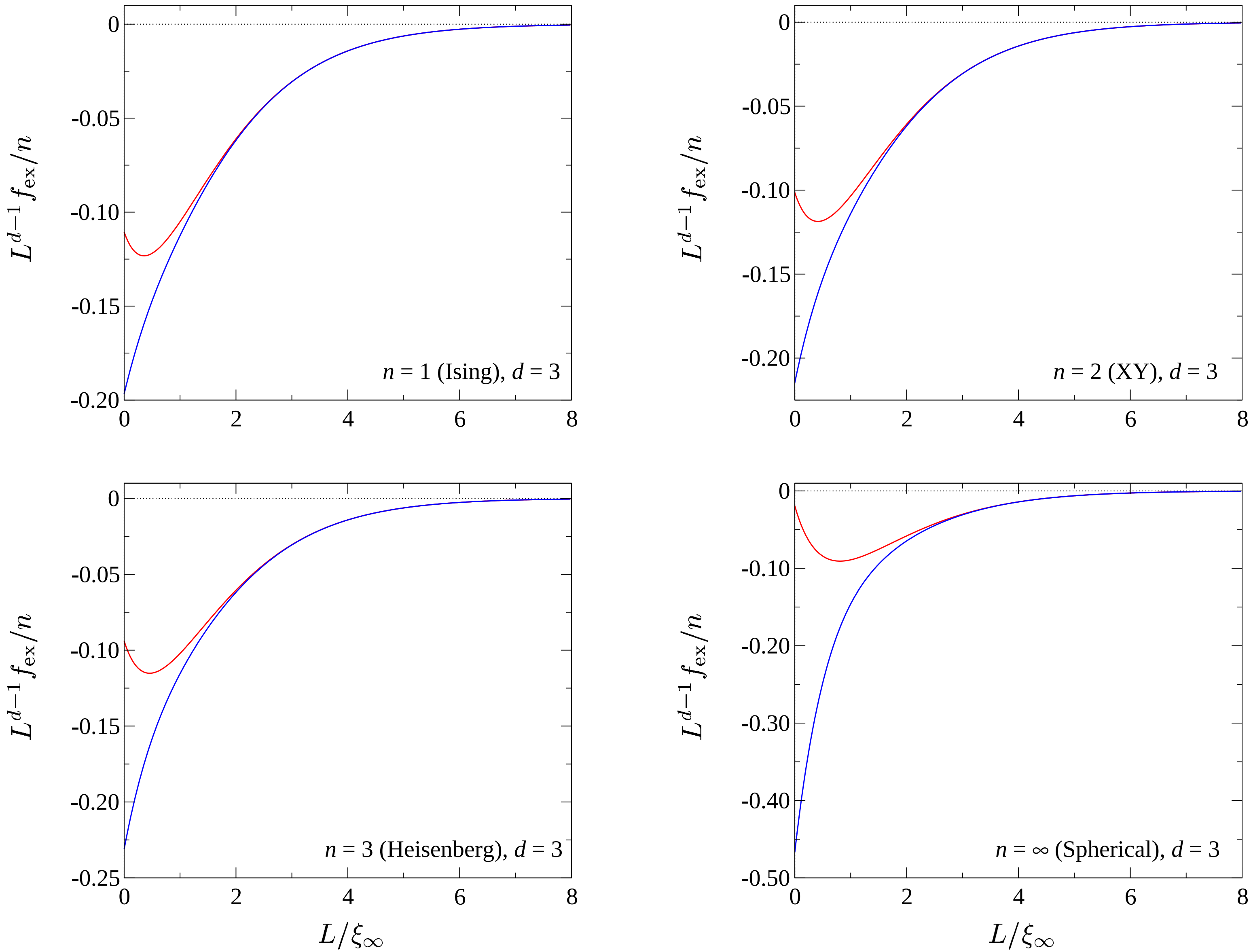
$$\frac{\Delta_C^{\rm per}}{n} = -\frac{\pi^2}{90} + \frac{\pi^2\epsilon}{180}\left[1 - \gamma - \ln\pi + \frac{2\zeta'(4)}{\zeta(4)} + \frac{5}{2}\frac{n+2}{n+8}\right] - \frac{\pi^2}{9\sqrt{6}}\left(\frac{n+2}{n+8}\right)^{3/2}\epsilon^{3/2} + O(\epsilon^2)$$

$$\frac{\Delta_C^{\rm sp,sp}}{n} = -\frac{\pi^2}{1440} + \frac{\pi^2\epsilon}{2880}\left[1 - \gamma - \ln(4\pi) + \frac{5}{2}\frac{n+2}{n+8} + \frac{2\zeta'(4)}{\zeta(4)}\right] - \frac{\pi^2}{72\sqrt{6}}\left(\frac{n+2}{n+8}\right)^{3/2}\epsilon^{3/2} + O(\epsilon^2)\,.$$

extrapolated ϵ-expansion results: periodic boundary conditions



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Grüneberg & HWD: PRB **77**, 115409 (2008) cond-mat/ 0710.4436

Two-Loop Calculations: $0 \le c_j < \infty$

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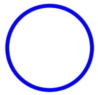
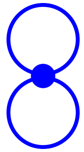
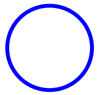
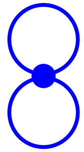
- Loop expansion: $\frac{-F}{k_B T} =$  $+$  $+ \mathcal{O}(\text{3-loops})$
-  $= G_L^{(\mathrm{bc})}(\boldsymbol{x};\boldsymbol{x}') = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \sum_m \frac{\langle z|m\rangle\langle m|z'\rangle}{p^2 + k_m^2 + \mathring{\tau}}\, e^{i\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}')}$
  - $-\partial_z^2 |m\rangle = k_m^2 |m\rangle$
  - $\langle z|m\rangle = A_m(L,\mathring{c}_1,\mathring{c}_2)\cos[k_m(L,\mathring{c}_1,\mathring{c}_2)z + \vartheta_m(k_m,\mathring{c}_1)]$
  - k_m from bc: $\Rightarrow$ $\boxed{R(k_m;L,\mathring{c}_1,\mathring{c}_2) \stackrel{!}{=} 0}$ *transcendental equation!*

Two-Loop Calculations: $0 \le c_j < \infty$

- Loop expansion: $\dfrac{-F}{k_B T} =$  $+$  $+ \mathcal{O}(\text{3-loops})$
-  $= G_L^{(\mathrm{bc})}(\boldsymbol{x};\boldsymbol{x}') = \displaystyle\int \frac{d^{d-1}p}{(2\pi)^{d-1}} \sum_m \frac{\langle z|m\rangle\langle m|z'\rangle}{p^2 + k_m^2 + \mathring{\tau}}\, e^{i\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}')}$
  - $-\partial_z^2 |m\rangle = k_m^2 |m\rangle$
  - $\langle z|m\rangle = A_m(L,\mathring{c}_1,\mathring{c}_2)\cos[k_m(L,\mathring{c}_1,\mathring{c}_2)z + \vartheta_m(k_m,\mathring{c}_1)]$
  - k_m from bc: $\Rightarrow$ $\boxed{R(k_m;L,\mathring{c}_1,\mathring{c}_2) \overset{!}{=} 0}$ *transcendental equation!*
- evaluation of sums: complex integration (Abel-Plana techniques)

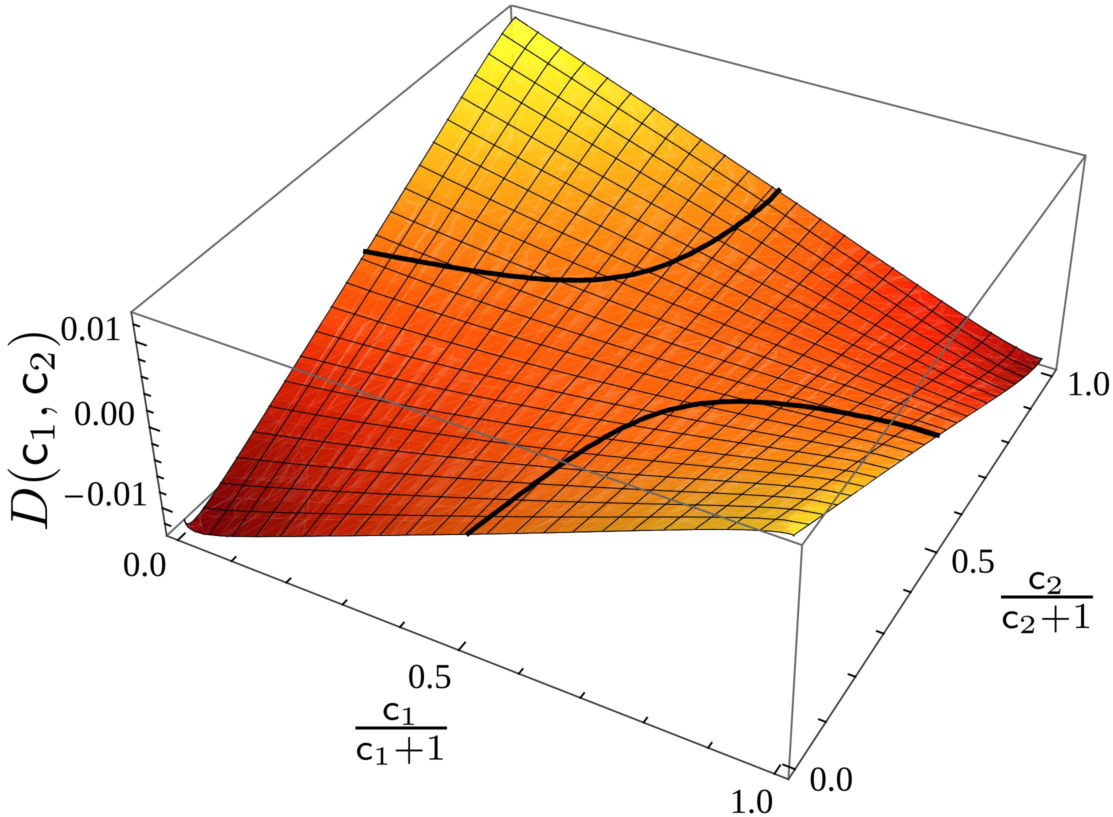
$$\sum_m g(k_m)(k_m^2+b)^a = \sum_m \mathrm{Res}[g(k)\,(k^2+b)^a\, f(k)]_{k=k_m}$$

$$= \int_{\mathcal{C}_1} \frac{dz}{2\pi i}\, g(k)\,(k^2+b)^a\, f(k)$$

deform contour

Scaling function: residual free energy

$$f_{\text{res}}(L)/n \approx L^{-(d-1)}\, D\big(c_1 L^{\Phi/\nu}, c_2 L^{\Phi/\nu}\big)$$



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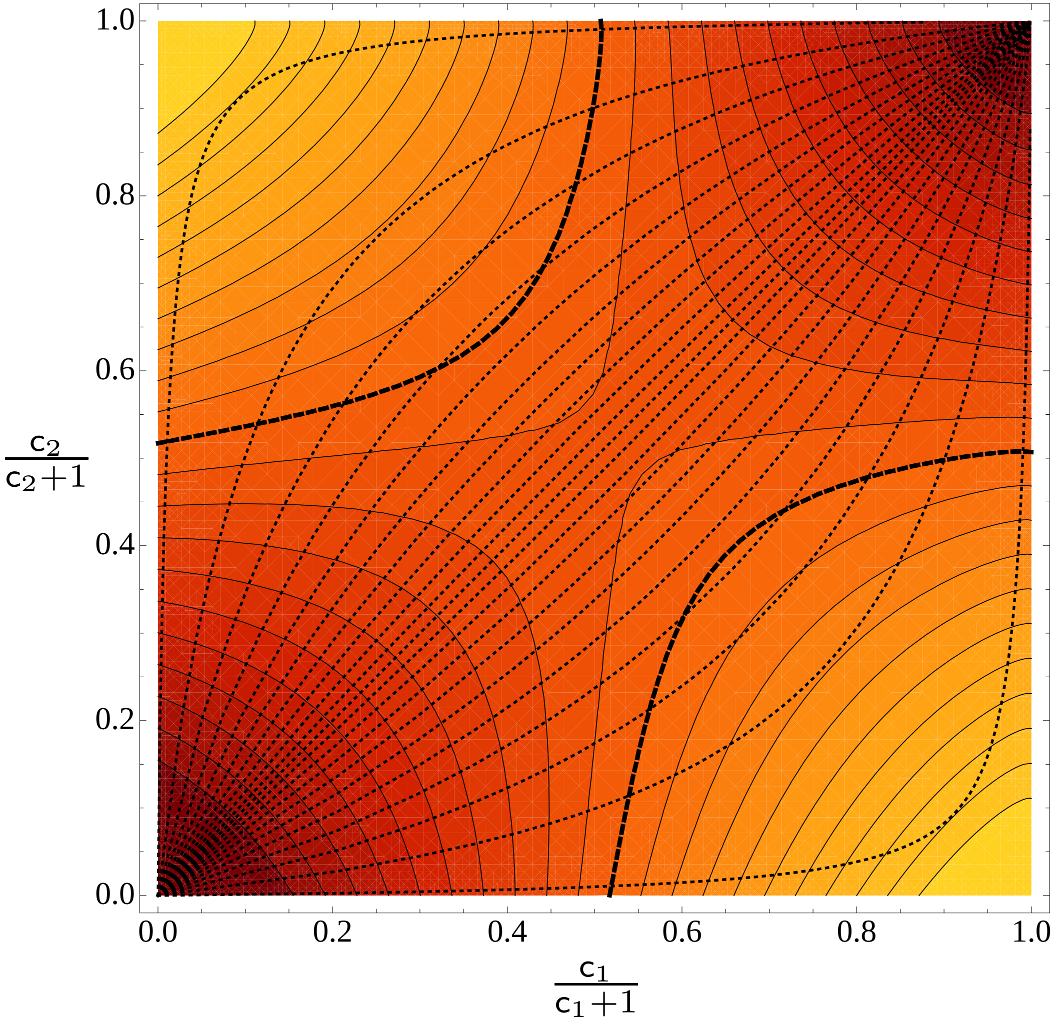
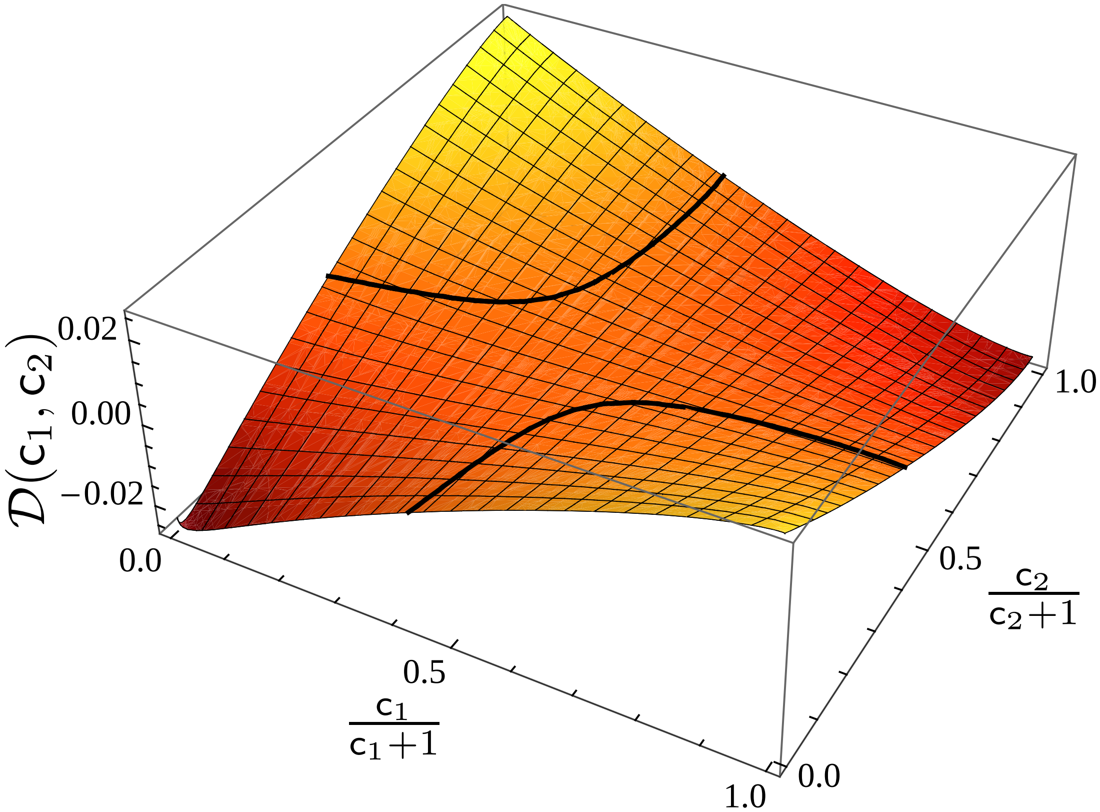


Scaling function: critical Casimir force

$$\frac{\mathcal{F}_C}{nAk_BT} \approx L^{-d}\,\mathcal{D}\big(c_1 L^{\Phi/\nu}, c_2 L^{\Phi/\nu}\big)$$



$$\mathcal{D}(\mathsf{c}_1,\mathsf{c}_2) = \big[d-1+(\Phi/\nu)\big(\mathsf{c}_1\partial_{\mathsf{c}_1}+\mathsf{c}_2\partial_{\mathsf{c}_2}\big)\big]\,D(\mathsf{c}_1,\mathsf{c}_2)$$

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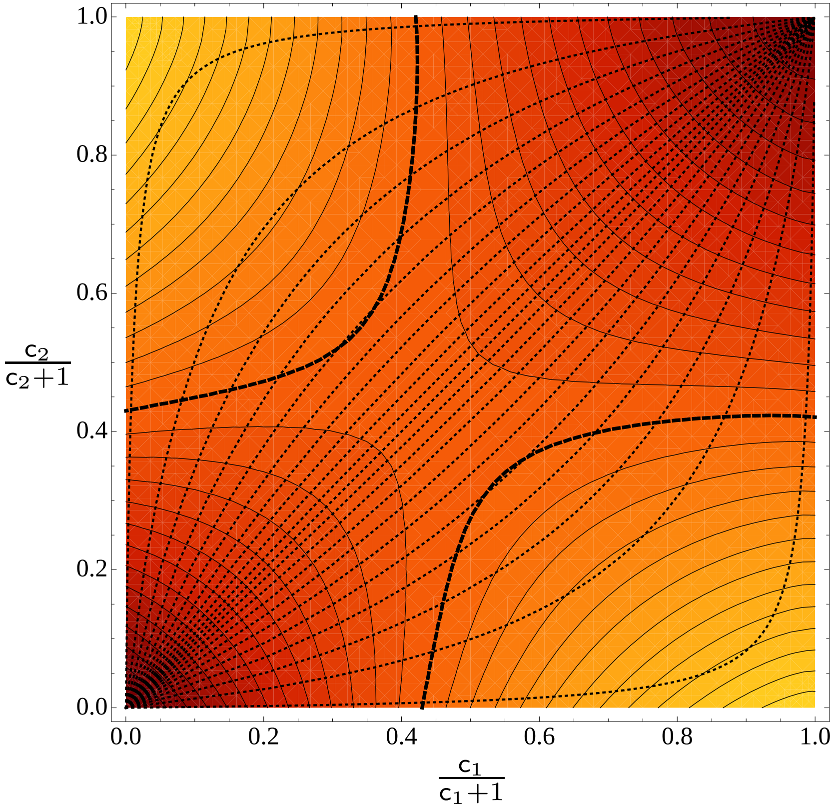
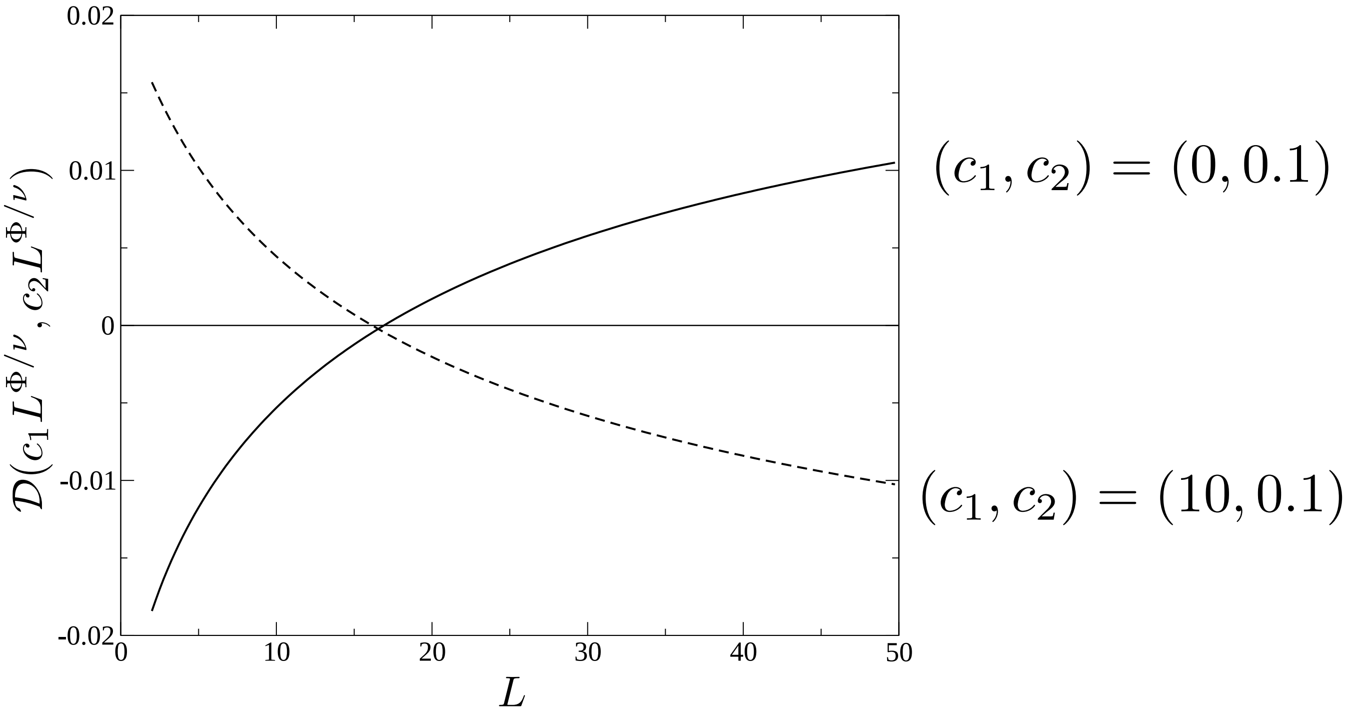


$$\mathcal{D}(\mathsf{c}_1, \mathsf{c}_2) = \big[d - 1 + (\Phi/\nu)\big(\mathsf{c}_1\partial_{\mathsf{c}_1} + \mathsf{c}_2\partial_{\mathsf{c}_2}\big)\big]\, D(\mathsf{c}_1, \mathsf{c}_2)$$

Crossover repulsive ↔ attractive



Scaled Casimir force $$\frac{\mathcal{F}_C}{nAk_BT} L^d = \mathcal{D}\big(c_1 L^{\Phi/\nu}, c_2 L^{\Phi/\nu}\big)$$

F. M. Schmidt & HWD: PRL **101**, 100601 (2008); arXiv:0806.2799, and to be published

Summary / Conclusions

- breakdown of $\epsilon = 4 - d$ expansion for bound. cond's involving zero modes!
- boundary conditions = scale dependent properties
- Neumann bc does not corrspond to fixed point!
- Casimir force can be attractive or repulsive
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- Observable?
  - in Monte Carlo simulations? yes!
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Thank you for your attention!