Nonplanar objects in a fluctuating fluid out of equilibrium

self-forces, self-torques, and violation of the action-reaction principle

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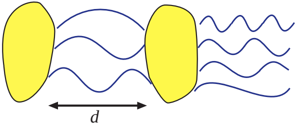
Casimir-like forces at equilibrium

15 JULY 1909 · 4 MAY 2000

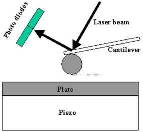
- Casimir (1948): EM vacuum fluctuations
- EM thermal fluctuations (Lifshitz)
- Other fluctuating systems:
  - critical fluids
  - broken continuous symmetry
  - proteins on cell membranes
  - ...



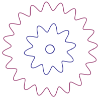
Long-range correlations of the fluctuating medium

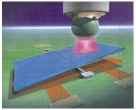
Usually: strong `geometry` dependence

Experiments / Applications

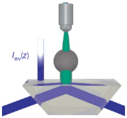


- Lamoreaux (1997), Mohideen & Roy (1998)
- MEMS, nanostructures
  Non-contact gears





- Helium films

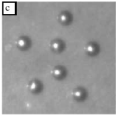


- Direct measurement critical Casimir force
- Agregation/Segregation of colloidal particles

Casimir-like forces out of equilibrium

Long-range fluctuations also appear in nonequilibrium:

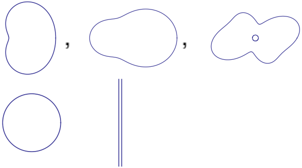
- Couette flow, Rayleigh-Bénard...
- granular fluids
- reaction–diffusion
- EM Casimir driven out of equilibrium, ...



No thermodynamic (equilibrium) potential

Nonequilibrium + nonplanar ⇒ new effects

- more control parameters: sign reversal, noise reduction
- "self-forces" on asymmetric objects
- violation of action–reaction principle



A simple nonequilibrium fluid model

- Reaction–diffusion (w/o detailed balance) $A + B \xrightarrow{k_1} 2B, \quad B \xrightarrow{k_2} A$
- Rate equations for A and B:

$$\frac{\partial}{\partial t}\rho_A = D\nabla^2\rho_A - \frac{k_1}{\rho_{\text{tot}}}\rho_A\rho_B + k_2\rho_B$$
$$\frac{\partial}{\partial t}\rho_B = D\nabla^2\rho_B + \frac{k_1}{\rho_{\text{tot}}}\rho_A\rho_B - k_2\rho_B$$

- Homogeneous stationary solution: $\rho_A^{\text{o}} = \frac{k_2}{k_1}\rho_{\text{tot}}, \quad \rho_B^{\text{o}} = \rho_{\text{tot}} - \rho_A^{\text{o}}$

- Density fluctuations: $\Phi(\mathbf{r}, t) = \rho(\mathbf{r}, t) - \rho^{\text{o}}, \quad \rho^{\text{o}} \equiv \langle \rho(\mathbf{r}, t) \rangle$

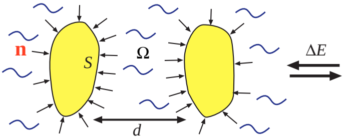
$$\boxed{\frac{\partial \Phi}{\partial t} = -\nabla \cdot (-D\nabla\Phi + \boldsymbol{\xi}_{\text{c}}) - \gamma\Phi + \xi_{\text{nc}}}, \qquad \gamma = k_1 - k_2 > 0$$

$$\left\langle \boldsymbol{\xi}_{\text{c}}{}^{\mu}(\mathbf{r}, t)\boldsymbol{\xi}_{\text{c}}{}^{\nu}(\mathbf{r}', t') \right\rangle = \Gamma_{\text{c}}\, \delta_{\mu\nu}\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'),$$
$$\left\langle \xi_{\text{nc}}(\mathbf{r}, t)\xi_{\text{nc}}(\mathbf{r}', t') \right\rangle = \Gamma_{\text{nc}}\, \delta(\mathbf{r} - \mathbf{r}')\delta(t - t'),$$

Force on a surface S

- Non-flux b.c.: $\mathbf{n}\cdot(-D\nabla\Phi+\boldsymbol{\xi}_{\mathrm{c}})|_{\mathbf{r}\in\partial\Omega}\equiv 0$
- Local pressure: $p=p(\rho(\mathbf{r},t)) \quad\Longrightarrow\quad \langle p\rangle \approx p_0+\frac{p_0''}{2}\left\langle\Phi^2\right\rangle$
- Average force on S

$$\mathbf{F}_S=\frac{p_0''}{2}\int_S \mathrm{d}\sigma\,\mathbf{n}\,\left\langle\Phi^2\right\rangle$$



Stationary state

After characteristic time $\approx \mathrm{O}(\gamma^{-1})$:

$$\Phi_{\mathrm{st}}(\mathbf{r},t)=\int\mathrm{d}t'\int_\Omega \mathrm{d}\mathbf{r}'\,G(\mathbf{r},\mathbf{r}',Dt-Dt')\,(-\nabla\cdot\boldsymbol{\xi}_{\mathrm{c}}+\xi_{\mathrm{nc}})(\mathbf{r}',t')$$
$$+\int\mathrm{d}t'\int_{\partial\Omega}\mathrm{d}\sigma(\mathbf{r}')\,G(\mathbf{r},\mathbf{r}',Dt-Dt')\,\mathbf{n}(\mathbf{r}')\cdot\boldsymbol{\xi}_{\mathrm{c}}(\mathbf{r}',t')$$

- Green function $G(\mathbf{r},\mathbf{r}',\tau)$

$$(-\nabla^2+\kappa^2-i\omega)G(\mathbf{r},\mathbf{r}',\omega)=\delta(\mathbf{r}-\mathbf{r}'), \qquad \kappa^{-1}=[\tfrac{\gamma}{D}]^{-\frac{1}{2}}$$

$$\mathbf{n}(\mathbf{r})\cdot\nabla G(\mathbf{r},\mathbf{r}',\omega)|_{\mathbf{r}\in\partial\Omega}\equiv 0 \qquad = \text{corr. length}$$

mesosc. scale

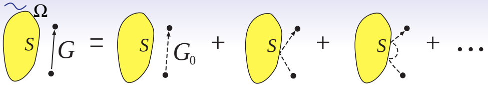
- Static structure factor

$$\left\langle \Phi_{\mathrm{st}}(\mathbf{r},t)\Phi_{\mathrm{st}}(\mathbf{r}',t)\right\rangle = \frac{\Gamma_{\mathrm{c}}}{2D}\delta(\mathbf{r}-\mathbf{r}')+\frac{\Gamma}{2D}G(\mathbf{r},\mathbf{r}',\omega=0), \qquad \Gamma=\Gamma_{\mathrm{nc}}-\frac{\gamma}{D}\Gamma_{\mathrm{c}}$$

- At thermal equilibrium: fluctuation–dissipation theorem

$$\Gamma_{\mathrm{c}}=2k_{\mathrm{B}}TD, \quad \Gamma_{\mathrm{nc}}=2k_{\mathrm{B}}T\gamma \quad \Longrightarrow \begin{cases}\Gamma=0\\ \text{Only microscopic corr.}\end{cases}$$

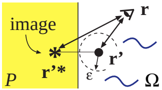
- Multiple scattering



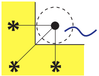
$$G(\mathbf{r},\mathbf{r}') = G_0(\mathbf{r}-\mathbf{r}') - \int_S \mathrm{d}\sigma_1 \; G(\mathbf{r},\mathbf{r}_1)\, \mathbf{n}_1 \cdot \nabla_1 G_0(\mathbf{r}_1 - \mathbf{r}'),$$

Short-range divergencies

- 3D: $G_0(\mathbf{r}-\mathbf{r}') = \frac{\mathrm{e}^{-\kappa|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$; 2D: $G_0(\mathbf{r}-\mathbf{r}') = \frac{\mathrm{K}_0(\kappa|\mathbf{r}-\mathbf{r}'|)}{2\pi}$
- 1 plate: $G_P(\mathbf{r},\mathbf{r}') = G_0(\mathbf{r}-\mathbf{r}') + G_0(\mathbf{r}-\mathbf{r}'^*)$



- 2 short-range divergencies for $\langle \Phi^2_{\mathrm{st}}(\mathbf{r}) \rangle |_{\mathbf{r}\in S}$
  - "bulk" $\Rightarrow (G_P - G_0)(\mathbf{r},\mathbf{r})$
  - "wall": integrated along S... compensation?



Regularized force

$$\mathbf{F}_S = F_0 \kappa \lim_{\epsilon\to 0} \int_S \mathrm{d}\sigma\, \mathbf{n}(\mathbf{r}) \left[G - G_0\right] (\mathbf{r} - \epsilon \mathbf{n}(\mathbf{r}), \mathbf{r} - \epsilon \mathbf{n}(\mathbf{r}))$$

$F_0 = \frac{p_0'' \Gamma}{4 D \kappa}$, $\Gamma = \Gamma_{\mathrm{nc}} - \kappa^2 \Gamma_{\mathrm{c}} \gtrless 0$: sign control

Self-force/torque on a deformed circle

- General solution for bounded obstacle:

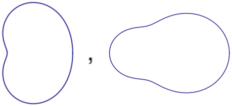
$$G(\mathbf{r},\mathbf{r}') = G_0(\mathbf{r}-\mathbf{r}') + \sum_{m,n\in\mathbb{Z}} \frac{\mathrm{e}^{im\theta+in\theta'}}{2\pi} a_{mn}\, \mathrm{K}_m(\kappa\rho)\mathrm{K}_n(\kappa\rho')$$

- Deformed circle: $\rho(\theta) = R + \eta s(\theta), \quad \eta \ll R, \kappa^{-1}$

- a_{mn} and then **F** determined perturbatively in $\eta \ll R, \kappa^{-1}$

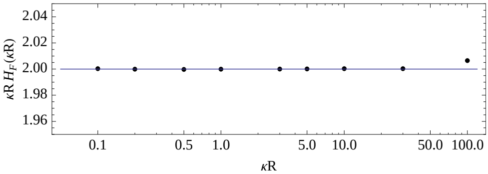
- Self-force: $s(\theta) = 2s_1 \cos(\theta) + 2s_2 \cos(2\theta)$

$$\mathbf{F} \sim -F_0\, s_1\, s_2\, (\kappa\eta)^2 H(\kappa R)\, \hat{\mathbf{x}}$$



$$H(\kappa R) = \lim_{\epsilon\to 0} \sum_{n\in\mathbb{Z}} \left\{ \sigma_{n,1-n}\big(\kappa R, \kappa(R+\epsilon)\big) + \sigma_{n,-2-n}\big(\kappa R, \kappa(R+\epsilon)\big) \right\}$$

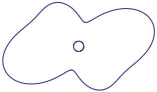
$$= \frac{2}{\kappa R} \;?$$

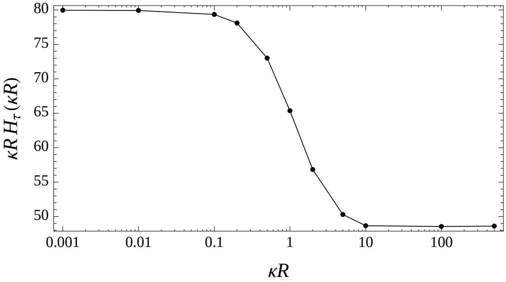


$$\sigma_{n,q}(x,y) = \frac{I'_n(x)}{K'_n(x)} \left\{ K_n(y)K'_n(y) + x\left[K'^2_n(y) + K_n(y)K''_n(y) \right] \right\}$$
$$+ \frac{K_n(y)K_{n+1}(y)}{K'_n(x)K'_{n+1}(x)} \left\{ \left[1-\frac{(n+1)q}{x^2}\right] \left[1-\frac{nq}{x^2}\right] \frac{K_q(x)}{K'_q(x)} - \frac{1}{2x}\left[1-\frac{n(n+1)}{x^2}\right] \right\}$$
$$+ \frac{1}{K'_n(x)K'_q(x)} \left[1-\frac{nq}{x^2}\right] \left\{ \frac{n+q}{x} K_n(y)K_q(y) - K'_n(y)K_q(y) - K_n(y)K'_q(y) \right\}$$

- Self-torque: $s(\theta) = 2s_2 \cos(2\theta) + 2s_4 \sin(4\theta)$

$$\mathbf{T} \sim -\frac{F_0}{\kappa}\, s_2^{\,2}\, s_4\, (\kappa\eta)^3\, H_\tau(\kappa R)\, \hat{\mathbf{z}}$$



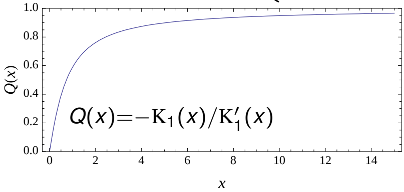


$$H_\tau(\kappa R) \approx \frac{80}{\kappa R}, \quad \kappa R \ll 1$$

$$H_\tau(\kappa R) \approx \frac{48.5}{\kappa R}, \quad \kappa R \gg 1$$

Fluctuations of the force

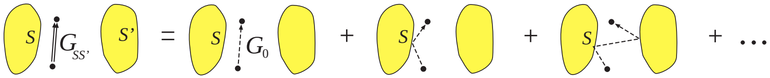
- $$\langle F_S^\mu F_S^\nu \rangle - \langle F_S^\mu \rangle \langle F_S^\nu \rangle = p_0'^2 \int_S d\sigma_1 \int_S d\sigma_2 \, n_1^\mu n_2^\nu \, \langle \Phi_{\mathrm{st}}(\mathbf{r}_1) \Phi_{\mathrm{st}}(\mathbf{r}_2) \rangle$$
  $$= \frac{p_0'^2}{2D} \left\{ \frac{\Gamma_c}{\epsilon} \int_S d\sigma \, n^\mu n^\nu + \Gamma \int_S d\sigma_1 \int_S d\sigma_2 \, n_1^\mu n_2^\nu \, G(\mathbf{r}_1, \mathbf{r}_2) \right\}$$
- Force fluctuations on the (undeformed) circle:
  $$\langle F_C^\mu F_C^\nu \rangle = \delta_{\mu\nu} \frac{p_0'^2 |C|}{4D} \left\{ \frac{\Gamma_c}{\epsilon} + \frac{\Gamma}{\kappa} Q(\kappa R) \right\}$$



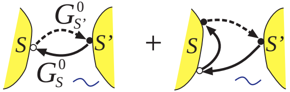
- $\Longrightarrow$ Force noise can be lowered by noneq. fluctuations

Two obstacles

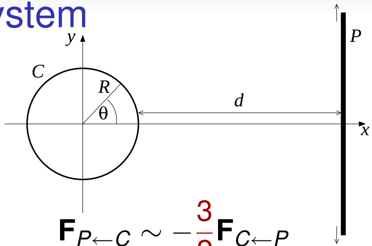
- 2 objects, S, S'. Total force $\mathbf{F}_S$ on S decomposed as
  - “self-force” $\mathbf{F}_S^0$
  - “two-body” force $\mathbf{F}_{S\leftarrow S'} \equiv \mathbf{F}_S - \mathbf{F}_S^0 = F_0\kappa \int_S \mathrm{d}\sigma\, \mathbf{n}\,(G_{SS'} - G_S^0)$
- Action–reaction? $\mathbf{F}_{S\leftarrow S'} + \mathbf{F}_{S'\leftarrow S} = \mathbf{F}_{SS'} - \mathbf{F}_S^0 - \mathbf{F}_{S'}^0$
- Self-forces $\rightsquigarrow \nexists$ action–reaction
- Multiple scattering



$(G_{SS'} - G_S^0)(\mathbf{r},\mathbf{r})|_{\mathbf{r}\in S} \overset{\kappa d\to\infty}{\sim}$



Circle–plate system



- Regime $R \ll \kappa^{-1} \ll d$:

$$\mathbf{F}_{C\leftarrow P} \sim -F_0 \frac{\sqrt{\pi}(\kappa R)^2 \mathrm{e}^{-2\kappa d}}{\sqrt{\kappa d}} \hat{\mathbf{x}}, \qquad \mathbf{F}_{P\leftarrow C} \sim -\frac{3}{2}\mathbf{F}_{C\leftarrow P}$$

Total force on the circle–plate assembly:

$$\mathbf{F}_{CP} \sim \frac{1}{2} F_0 \frac{\sqrt{\pi}(\kappa R)^2 \mathrm{e}^{-2\kappa d}}{\sqrt{\kappa d}} \hat{\mathbf{x}}$$

same order of magnitude!

- Regimes $\kappa^{-1} \ll R, d$: (same as Derjaguin)

$$\mathbf{F}_{C\leftarrow P} \sim -F_0 \frac{\sqrt{\kappa R}\, \mathrm{e}^{-2\kappa d}}{\sqrt{\kappa d}} \hat{\mathbf{x}}, \qquad \mathbf{F}_{P\leftarrow C} \sim -\mathbf{F}_{C\leftarrow P}$$

Conclusions

Nonequ. ($\nexists$ time-reversal) + Nonplanar ($\nexists$ space symm.)

$\Downarrow$

Self-forces

- Directed motion (ratchets)
- Tunable
- Stresses on asymmetrical structures
- Motor axles with external energy source, self-assembly

Violation of action–reaction

- Special care in experimental measurements
- No effective potential (nonconservative forces)

P. R. Buenzli & R. Soto, PRE **78**, 020102(R) (2008)