Quantum Loop Gas Approach to Topological States of Matter

An elastic medium is a simple caricature describing many states of matter:

![](image)

The ground state breaks a symmetry, and the low-lying excitations can be thought of as ripples in the medium which tend to restore the symmetry. They are gapless because in the long-wavelength limit, such a ripple is a symmetry operation.
Quantum Loop Gases

In this talk, I will describe states of matter for which the appropriate caricature is a sea of fluctuating loops.
• The loops may arise as domain walls, dimers, chains of up-spins or occupied sites, etc.
Kitaev:

\[ H = -J_1 \sum_i A_v - J_2 \sum_p F_p \]

\[ A_v \equiv \prod_{\alpha \in N(v)} \sigma^z_\alpha \]

\[ F_p \equiv \prod_{\alpha \in p} \sigma^x_\alpha \]

- The loops obey a certain quantum dynamics; depending on the topological rules it imposes, the state may be a stable, gapped topological state or a gapless critical point.
• Excitations are violations of these rules, e.g. broken loops:

• The rules obeyed by loops determine the braiding properties of the quasiparticles, ground state degeneracy, etc.
Basic Structure of a Class of Theories

- Wavefunctions $\Psi[\alpha]$ on multi-loops $\alpha$ which are invariant under smooth deformations of the loops.

\[
\Psi \left[ \begin{array}{c}
\text{loops 1}
\end{array} \right] = \Psi \left[ \begin{array}{c}
\text{loops 2}
\end{array} \right]
\]

We would expect this for any topological phase.
• A ‘fugacity’ $d$ for small, contractible loops.

\[
\Psi \left[ \begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{loop1}}
\end{array} \right] = d \cdot \Psi \left[ \begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{loop2}}
\end{array} \right]
\]

In Kitaev’s model, $d = 1$.

Without such a relation, the ground state would be degenerate even on the sphere.
• Invariance of the wavefunction under a ‘surgery relation’ which cuts and rejoins loops,

e.g.

\[ \Psi \left[ \begin{array}{c}
\ldots
\end{array} \right] = \Psi \left[ \begin{array}{c}
\ldots
\end{array} \right] \]

Without such a relation, the ground state would be infinitely degenerate on the torus.
By generalizing the latter two conditions, we will construct a family of topological states of matter, all of which can be described as quantum loop gases.
Consistency Conditions for Quantum Loop Gases

If $d \neq 1$, then the surgery relation must be modified or else there is a contradiction:

$$\Psi[\text{---}] = \Psi[\text{---} \circlearrowleft] = \Psi[\text{---} \circlearrowright] = d \cdot \Psi[\text{---}]$$

Hence, we must look at surgery relations involving $3, 4, \ldots$ curves.
Important Mathematical Result: For almost all $d$, there is no consistent surgery relation.

Consistent surgery relations can be found only for $d = 2 \cos \left( \frac{\pi}{k+2} \right)$ (Jones-Wenzl projectors)

e.g. for $d = \sqrt{2}$,

\[
\Psi[ ] - \sqrt{2} \Psi[ ] - \sqrt{2} \Psi[ ] + \Psi[ ] + \Psi[ ] = 0
\]
Surgery and Physical Properties

For a given $k$, the value $d$ assigned to a contractible loop and the associated $k + 1$-curve surgery relation defines a **topological state**.
e.g. for $k=2$, 

$$\Psi[\big|\big|] - \sqrt{2} \Psi[\big|\big|\big]\bigg]\big|\bigg]- \sqrt{2} \Psi[\big\big\big|\big\big\big|] + \Psi[\big\big\big|\big\big\big|\bigg]\bigg]\big|\bigg] = 0$$

$$\Rightarrow$$

$\Psi[\big|\big|] - \sqrt{2} \Psi[\big|\big|\big]\bigg]\big|\bigg]- \sqrt{2} \Psi[\big\big\big|\big\big\big|] + \Psi[\big\big\big|\big\big\big|\bigg]\bigg]\big|\bigg] = 0$

9 Ground States on $T^2$:
Field Theoretic Description

- The associated field theories are gauge theories.
- Braiding statistics from the generalized Aharonov-Bohm effect
- Wilson loop operators act in a simple ‘pictorial’ manner on the argument of wavefunctions.
- Unoriented loops are a feature of SU(2).
‘Doubled’ $SU(2)_k$ Chern-Simons theories.

$$S_{CS} = \frac{k}{4\pi} \int \text{tr} \left( a \wedge da + \frac{2}{3} a \wedge a \wedge a \right)$$

Important gauge-invariant operators:

$$W[\gamma] \equiv \text{tr} \left( \mathcal{P} e^{i \oint \gamma a_c T^c \cdot dl} \right)$$

Their commutator algebra:

$$[W[\gamma], W[\gamma']] = 2 \sin \left( \frac{\pi}{2(k + 2)} \right) \sum_i (W[\gamma \circ_i \gamma'] - W[\gamma' \circ_i \gamma])$$
Algebra of Wilson Loops

Can be represented on isotopy, $d$, surgery-invariant $\Psi[\alpha]$ if:

$$W_+[\gamma] \Psi[\beta] = \Psi[\beta \star \gamma]$$

where $\alpha \star \gamma = \alpha \cup \gamma$ with intersections resolved by:

$$\Psi[\alpha] = A \Psi[\alpha'] + A^{-1} \Psi[\alpha'']$$

$$A = i \exp \left( \frac{\pi i}{2(k + 2)} \right)$$
This guarantees that the desired commutation relations are obeyed. It also fixes $d$.

Suppose we deform $\gamma$ into $\gamma'$ which has two new intersections with $\alpha$,

Using the resolution of crossings, we see that $\Psi[\alpha \star \gamma] = \Psi[\alpha \star \gamma']$ iff

$$d = -A^2 - A^{-2} = 2 \cos \left( \frac{\pi}{k+2} \right)$$
Descriptions at Different Scales

Short scales: electrons/spins at points (0-D)

Intermed. scales: fluctuating curves/loops (1-D)

Long scales: degenerate ground states on genus-$g$ surfaces (2-D)
Intermediate Scales $\sim$ Nearby Critical Point

Intermed. length scale physics: ‘$d$-isotopy’.

Long-wavelength physics: Jones-Wenzl surgery relations restrict winding numbers and det. the energy gap.

A nearby critical point might determine intermediate length scale physics.
$d$-isotopy Hamiltonians

Spins on the links of the honeycomb lattice:

$$H_{d\text{-iso}} = \sum_v \left( 1 + \prod_{i \in \mathcal{N}(v)} \sigma_z^i \right) + \sum_p \left[ \frac{1}{d^2} (F_p^0) \dagger F_p^0 + (F_p^0) \dagger F_p^0 - \frac{1}{d} F_p^0 - \frac{1}{d} (F_p^0) \dagger \right]$$

$$+ \left( F_p^1 \right) \dagger F_p^1 + \left( F_p^1 \right) \dagger F_p^1 - F_p^1 - \left( F_p^1 \right) \dagger \left( F_p^2 \right) \dagger F_p^2 + \left( F_p^2 \right) \dagger F_p^2 - F_p^2 - \left( F_p^2 \right) \dagger$$

$$+ \left( F_p^3 \right) \dagger F_p^3 + \left( F_p^3 \right) \dagger F_p^3 - F_p^3 - \left( F_p^3 \right) \dagger \right]$$

$$F_p^0 = \sigma_1^- \sigma_2^- \sigma_3^- \sigma_4^- \sigma_5^- \sigma_6^- , \quad F_p^1 = \sigma_1^+ \sigma_2^- \sigma_3^- \sigma_4^- \sigma_5^- \sigma_6^- + \text{cyclic perm.}$$

$$F_p^2 = \sigma_1^+ \sigma_2^+ \sigma_3^- \sigma_4^- \sigma_5^- \sigma_6^- + \text{c. p.} , \quad F_p^3 = \sigma_1^+ \sigma_2^+ \sigma_3^+ \sigma_4^- \sigma_5^- \sigma_6^- + \text{c. p.}$$

Up-spins again form closed loops which satisfy $d$-isotopy, but without surgery.

\[ |\Psi_0\rangle = \sum_{\alpha} d^{n\alpha} |\alpha\rangle \]

Can be interpreted as a Loop Gas of fugacity \( d^2 \):

\[
\sum_{\alpha} |\Psi[\alpha]|^2 = Z_{O(n)}(x = n) \quad \text{where} \quad n = d^2
\]

\[
Z_{O(n)}(x) = \int \prod_i d\hat{S}_i \prod_{\langle i,j \rangle} (1 + x\hat{S}_i \cdot \hat{S}_j) = \sum_{\alpha} \left(\frac{x}{n}\right)^{\ell_\alpha} n^{n\alpha}
\]
Ground State Properties

For $x = n$, the $O(n < 2)$ loop model is in its critical low-temperature phase.

- Loops meander over long distances with exponents $\eta_k = \frac{g}{4}k^2 - \frac{1}{g}(1 - g)^2$ where
  $n = -2 \cos(\pi g)$

- The $x \rightarrow \infty$ limit is the FK rep. of the critical $q = n^2$ state Potts model, which has the same exponents as the low-temp. $O(n)$ model.
• The ground state of $H_{d-\text{iso}}$ contains long loops characterized by exponents $\eta_k$ for $d \leq \sqrt{2}$, which arise in correlators of non-local operators – referring to the same loop.

• However, correlation functions of local ops. $\vec{\sigma}$ are short-ranged.

• A ‘Quasi-Topological Phase’.
Low-Energy Excitations

Trial wavefunction:

\[ |\Psi_1\rangle = \sum_{\alpha \in X} d^{n\alpha} |\alpha\rangle - \sum_{\alpha \in Y} d^{n\alpha} |\alpha\rangle \]

\( X = \text{configs. with long loops; } Y = \text{without.} \)

Since the O(n) model is critical for \( n \leq 2 \), we can define ‘long’ so that the prob. of a config. with a long loop is \( 1/2 \). Then \( \langle \Psi_1 | \Psi_1 \rangle = 0 \).
\[ \langle \Psi_1 | H_{d_{iso}} | \Psi_1 \rangle = 0 \] because the two sectors of configuration space are not directly connected by the Hamiltonian, i.e. there is a bottleneck.

This is a critical line parametrized by \( d \leq \sqrt{2} \).
Low-Energy Field Theory

An effective field theory would help us address stability, dynamics, etc.

- $\omega \sim k^2$

- SU(2) gauge theory

- Local operators equal-time correlations are short-ranged, but non-local operators have power-laws $\eta_k = \frac{g}{4} k^2 - \frac{1}{g} (1 - g)^2$. 
The first two requirements motivate the guess:

\[ S = \frac{1}{g^2} \int d^2 x \ d\tau \left( E_i^a \partial_\tau A_i^a + A_0^a D_i E_i^a + \frac{1}{2} E_i^a D^2 E_i^a + \frac{1}{2} B^a B^a \right) \]

But is this interacting theory actually critical?
The first two requirements motivate the guess:

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S = \frac{1}{g^2} \int d^2 x \, d\tau \left( E_i^a \partial_\tau A_i^a + A_0^a D_i E_i^a + \frac{1}{2} E_i^a D^2 E_i^a + \frac{1}{2} B^a B^a \right)
\]

But is this interacting theory actually critical?

At one-loop,

\[
\frac{dg}{d\ell} = 0
\]

This theory is also on a critical line.
For $g$ small, the perturbations $\lambda_1 (E_i^a E_i^a)^2 + \lambda_2 (E_i^a E_j^a)^2$ have runaway flows. This presumably corresponds to $d \geq 2$. *The classical limit is massive, as in the q-state Potts/O(n) models.*

If the conjecture is correct, then for $g$ sufficiently large, these become irrelevant, and in this regime we expect

$$\langle W[\gamma] \rangle = d$$

$$\langle \text{tr} \left( E_i(x) \mathcal{P} e^{i \int_0^x a} E_j(0) \mathcal{P} e^{i \int_0^x a} \right) \rangle \sim \frac{1}{|x| \eta_2} \delta_{ij}$$
Future Directions

With the pictorial-combinatorial description of topological phases in hand, there are many open questions which one hopes to address.

- Electrons, topology, statistical mechanics, gauge theories ... computer science.

- The pictorial representation motivates certain types of microscopic models.
Perturbing away from soluble models, towards more realistic ones.

- Imposing the Jones-Wenzl relations.

- *Stability of* $d$-*isotopy critical line.*

References
Foundations


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Quantum Loop Gases – Effective Field Theories to Microscopic Models


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Statistical Mechanics of Classical Loop Models


Related Quantum Critical Points


E. Fradkin et al., cond-mat/0311353.