1

Algebra vs. Analysis in loop & web models for TQFTs

Michael Freedman
Microsoft Research

2

Algebra: ideals and irreps of certain algebras, categories, and tensor cat.

Analysis: spectra gap (?) for certain local Hamiltonians

determining the relevant perturbations of certain critical phases

We know much more about the algebra than the analysis.

(2+1) TQFT

dim	object	invariant	example
3	link	scalar	$\text{Jones}_L(e^{2\pi i/5}) = \frac{1}{8\pi}\int d\mathfrak{a}\, e^{2\pi i\cdot 3\, cs(a)}\, \text{tr hol}_p(L)$
2	surface Σ	finite dimensional Hilbert space	$V(\Sigma)$
1	circle (boundary of a surface)	representation category	$\hat{\hat{U}}_q(sl,2)$
0	point	2-category	No name (too complicated to think about)

My focus is on the Hilbert space $V(\Sigma)$.

A large class of $V(\Sigma)$ are of the form:

"admissible pictures / local relation"

↑ local

More physically we would begin with an extensive Hilbert space H of pictures and introduce local projectors A_i which "check" admissibility and another set of projectors B_j (the "dynamics") which enforce the desired local relations:

$$H = \sum_i A_i + \sum_j B_j$$

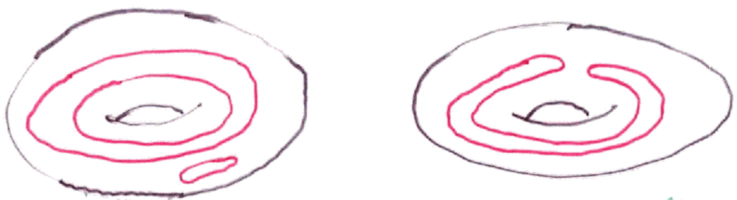
5

Examples

I Kitaev's "toric code"

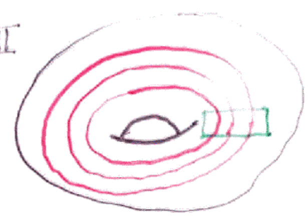
$A_i = -\prod_{+} \sigma^z \qquad B_j = -\prod_{\square} \sigma^x$

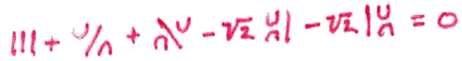
II (d=1) - isotopy + Jones-Wenzl)(= ∪∩



I and II both yield $\mathbb{Z}_2$-gauge theory (top. phase)

III





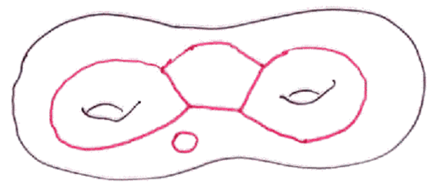
and $d = \sqrt{2}$

III is a doubled Pfaffian phase.

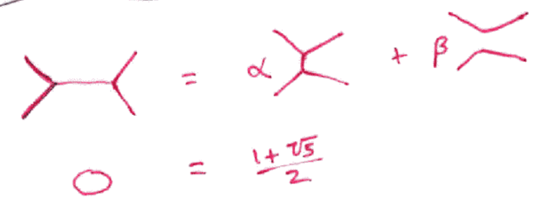
6

Examples (cont.)

IV simple webs

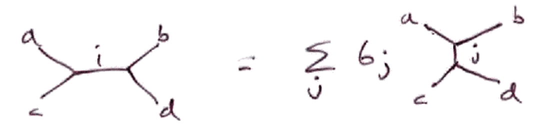


Modulo



V general webs: colors, arrows, flags, ☺

obeying



orthogonality + Eliot-Biedenhiem

Can realize: all ~~Reshetikhin~~ Turaev-Viro models

- SS Lie group, level
- finite gauge theories
- Haargrap TQFT

These V are "globalizations" of the representation category of known algebraic strutune (nearly, usually)

Easiest example

Thm 1954 (Rumor, Teller, Weyl)

$\mathrm{Inv}_{SU(2)}(\underbrace{2\otimes 2\otimes\cdots\otimes 2}_{k}) \cong \mathbb{C}$ (imbedded arc diagrams bounding k points $\subset \partial D^2$)

$\uparrow\downarrow - \downarrow\uparrow \longleftrightarrow$ 

$\mathrm{Inv}(\otimes_4 2) \longleftrightarrow$ span 

$\cong$ compatable with: 1. ($\otimes$; sum)

2. contraction

3. cyclic rotation

careful of -1's

Examples II + III globalize rep cat of $U_2(sl_2)$ and $U_3(sl_2)$ resp.

(see Kuperberg "spiders" on math. archive)

Simplest admissible pictures are unoriented multi loops ($\approx$ reps $U(sl_2)$)

I know 3 ways that microscopics can specify a multiloop:

① The obvious way

operator A checks for two* "up" at each site

* or zero

② As the surround loops of edge configurations

③ From a dimer covering of a non-bipartite lattice (use a fixed "background" dimer covering to build the multi loop

9

The meaning of the degree of freedom in each of the three methodes is <u>also</u> a variable at your disposal.

Drawing an edge might mean:

"bond occupied",

"a spin = 1/2 particle living on bond is ↑",

or in case ③ the edge might represent a singlet state between two spin = 1/2 particles,

etc...

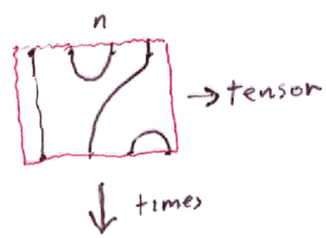
10

How does one study multiloops $\subset \Sigma$?

<u>Ans</u>: Temperley-Lieb Algebras:

- TL category
- TL Tensor category
- TL annular category
- TL planar algebra
- TL surface algebra

TL^n_d



 $= d$

Definition: Ideal : a linear space of morphisms <u>absorbtive</u> under times and tensor

11

Thm For all d the algebra TL_d^n has a unique projector P_n such that (turn back) $P_n = P_n$ (turn back) $= 0$.

$P_1 = 1$

$P_2 = || - \frac{1}{d}\, {}^{\cup}_{\cap}$

$P_3 = ||| + \frac{1}{d^2-1}\, {}_{\cap}\backslash^{\cup} + \frac{1}{d^2-1}\, {}^{\cup}/_{\cap} - \frac{d}{d^2-1}\, {}^{\cup}_{\cap}| - \frac{d}{d^2-1}\, |^{\cup}_{\cap}$

$\vdots$

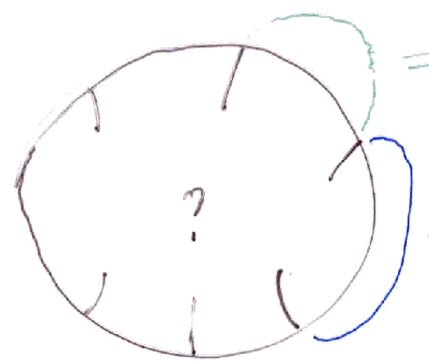
12

Thm The TL_d tensor category has an ideal $\mathcal{I}$ ($\neq TL_d$ or 0) iff $d = \alpha + \bar{\alpha}$ for some α a $2r^{th}$ root of 1.

If order$(\alpha) = 2r$ the $\mathcal{I} = \langle P_{r-1} \rangle$

$TL_d/\mathcal{I}$ is a $\mathbb{C}^*$-category iff $\alpha = e^{\pi i/r}$

proof via "minimal criminal"



$\Rightarrow \cup \Rightarrow ? = \alpha P_3$

$\Rightarrow 0 = \frac{\Delta_n}{\Delta_{n-1}} P_2 \Rightarrow d$ root Δ_n

(recall: $\Delta_n = tr(P_n) =$ 

$\Delta_{n+1} = d\Delta_n - \Delta_{n-1}$ Chebychev)

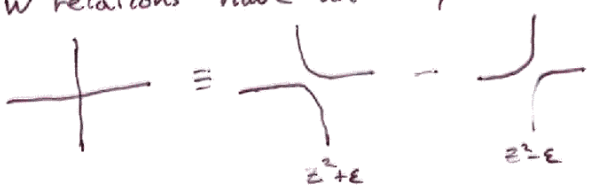
It is "just possible" to find / engineer a system H, near H_0 where

gsm H_0 = d-isotopy ($d = 2\cos \pi/r$)
or some other
"pretopological phase"

Beyond Z_2-gauge theory it is hopeless to engineer the final JW relation by hand but I believe that its <u>algebraic uniqueness</u> will show up analytically in the perturbation theory.

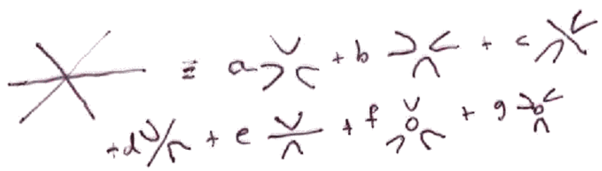
Q Do the JW relations have an analytic derivation?

$0 = \mathrm{re}(z^2)$



???

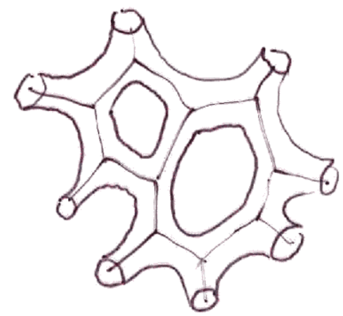
$0 = \mathrm{re}(z^3) =$

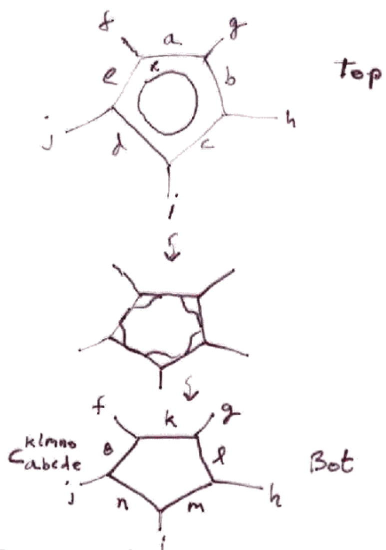


In particular we conjecture: (special d)

d-isotopy + JW is gapped.

<u>Evidence</u>: "true in web world" { Turaev – Viro, Kitaev – Kuperberg, Wen – Levine





$\sum c_{abcde}^{klmno}$

$H = \mathcal{C}(x^{\Sigma})$

1. vertex operators A_i check fusion rules
2. face operators B_j enforce d_x Top = Bot

B_j's are "disjoint" so commute

B_j + A_i commute Y ok ⇔ Y(ok

"passive observer"

A_i's are disjoint, so commute □

I would surprise a gapless H' could have same alg. structure of its gsm as a gapped H

15

	statements		
d-isotopy	$1 \le d \le \sqrt{2}$	is gapless	proved (from stat. mech. criticality)
d-isotopy	$d > \sqrt{2}$	gapped	very likely
d-isotopy + JW		always gapped	likely

The JW relations will be the "effective", "low energy" result of a large class of simple perturbations like $\sum_j -\sigma_j^x$?