Quantum computational complexity, phase transitions and glassiness

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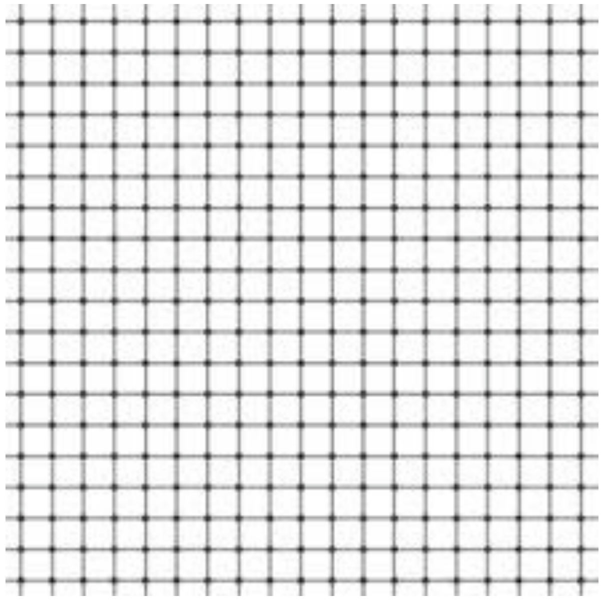
Plan

- Complexity theory for physics
- Physics for complexity theory

Complexity theory: the lightning intro

Complexity theory classifies problems according to the scaling of the resources a computer requires to solve large instances



Complexity theory: the lightning intro

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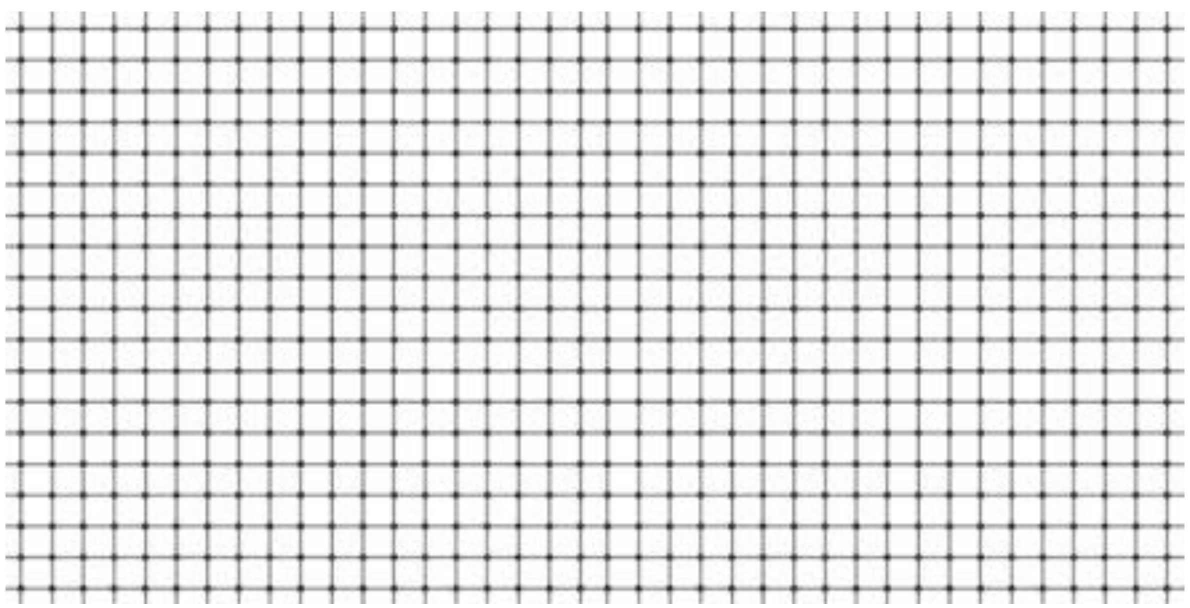
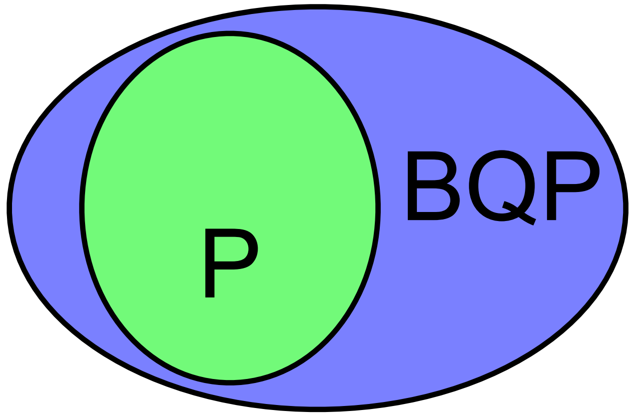


Complexity theory: the lightning intro

Complexity theory classifies problems according to the scaling of the resources a computer requires to solve large instances

- P -- solvable in polynomial time by a classical computer
- BQP -- solvable in polynomial time by a quantum computer



P and BQP

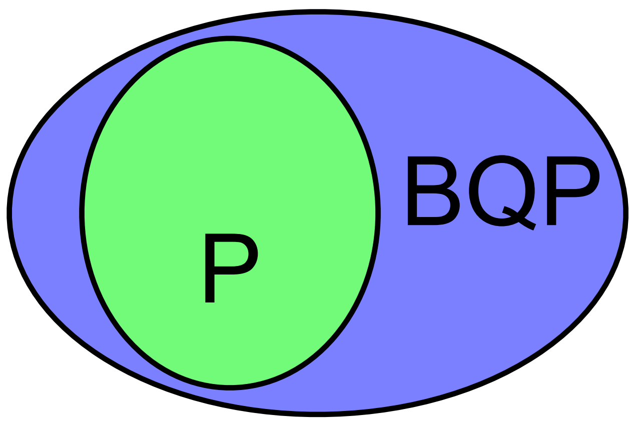
Can identify efficiently solvable (easy!) problems directly:
find polynomial time algorithms

- Classical (P):
  - What is energy of given configuration in 3D Ising model?

Arithmetic ~500AD

- Quantum (BQP):
  - Is there a factor $1 < p < m$ of the integer N?

Shor 1994

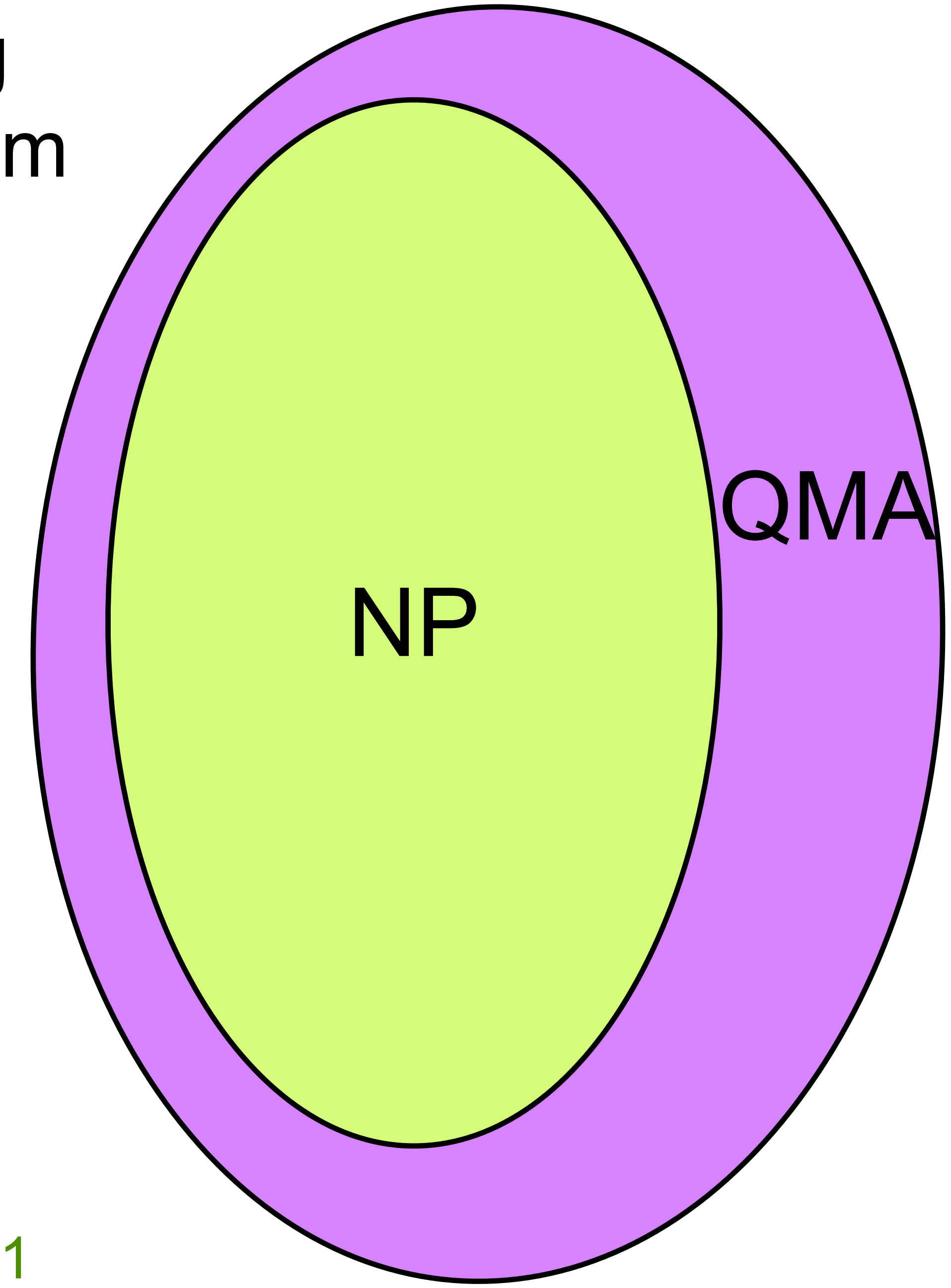


NP and QMA

Identify reasonable problems by finding polynomial times algorithms to check them

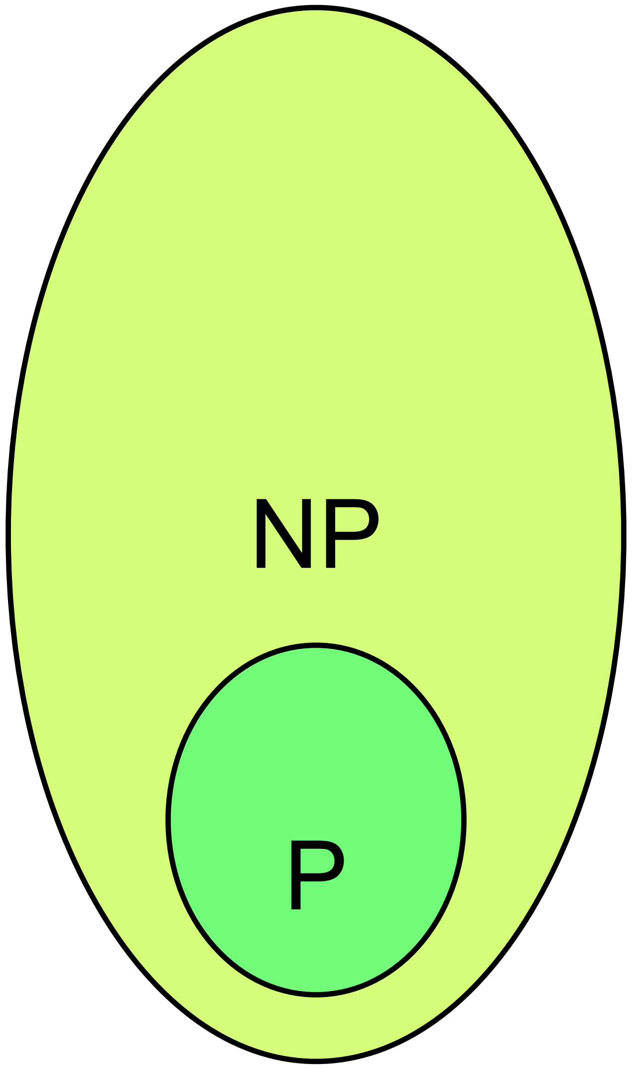
- NP -- checkable in poly time by a classical computer
  - Is the ground state energy of Ising model below E?
- QMA -- checkable in poly time by a quantum computer
  - Is the ground state energy of a local Hamiltonian below E?

Kitaev 2001



P$\neq$NP

There are hard problems.*

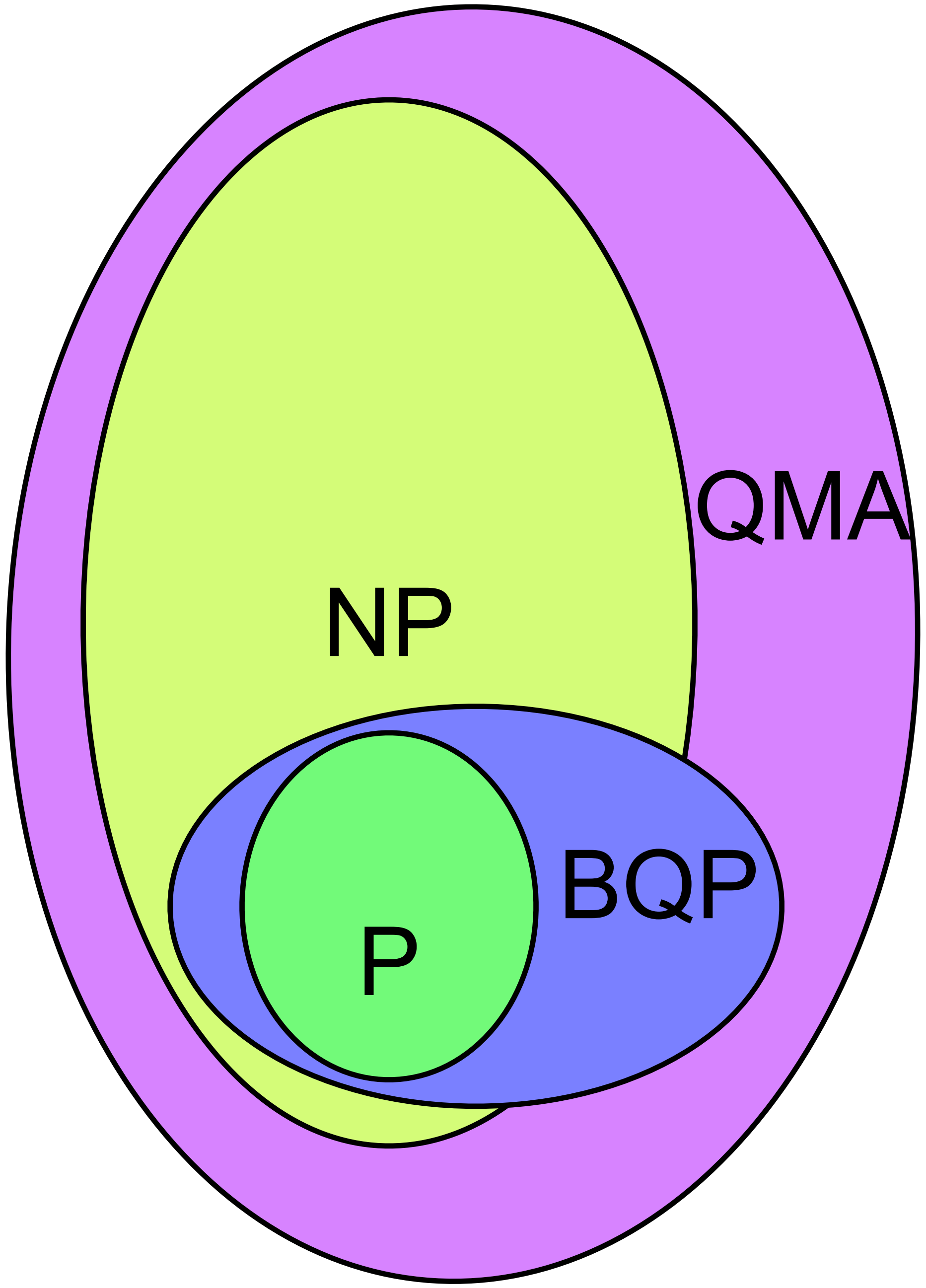


* Conjecture

P≠NP

There are hard problems.*

(even with quantum computers)



* Conjecture

Strong Church-Turing hypothesis

A computer can *efficiently* simulate any physical model of computation.

- All physical models of computation are equivalent
- Any physical object undergoing natural dynamics can be viewed as a computer.
- If $P \neq NP$ there must be glassy physical systems

Guilt by association: -completeness

- Identify hard problems by circumstantial evidence

All classical verifiable problems (NP)

reduce to

3D Ising Glass

Is the ground state energy of

$$H = \sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j$$

< E?

Barahona 1982

All quantum verifiable problems (QMA)

reduce to

2D Pauli Glass

Is ground state energy of

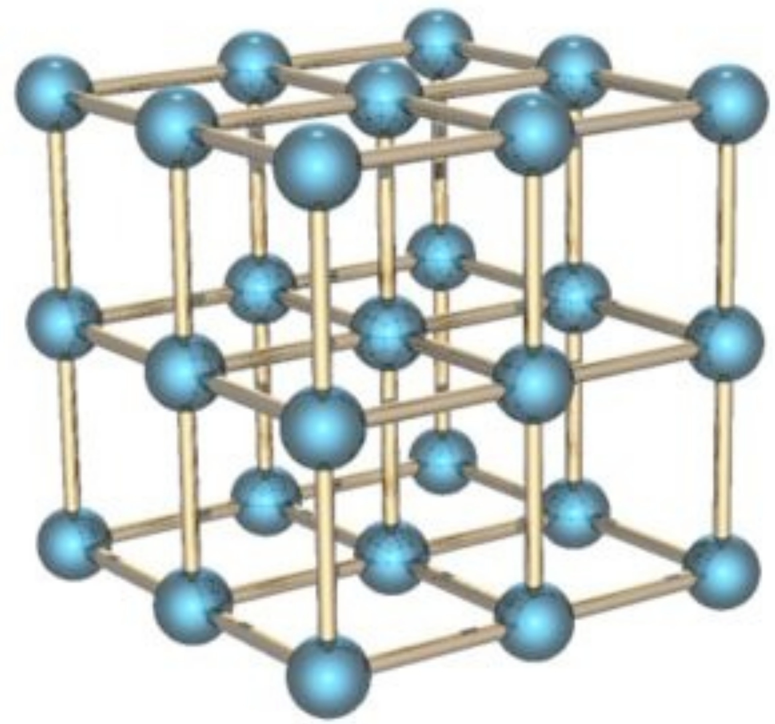
$$H = \sum_{\langle ij \rangle} J_{ij} \sigma_i^{\alpha_{ij}} \sigma_j^{\alpha_{ji}}$$

< E?

Oliveira, Terhal 2005

Extended hardness

- No physical process can find ground state of H efficiently.



Extended hardness

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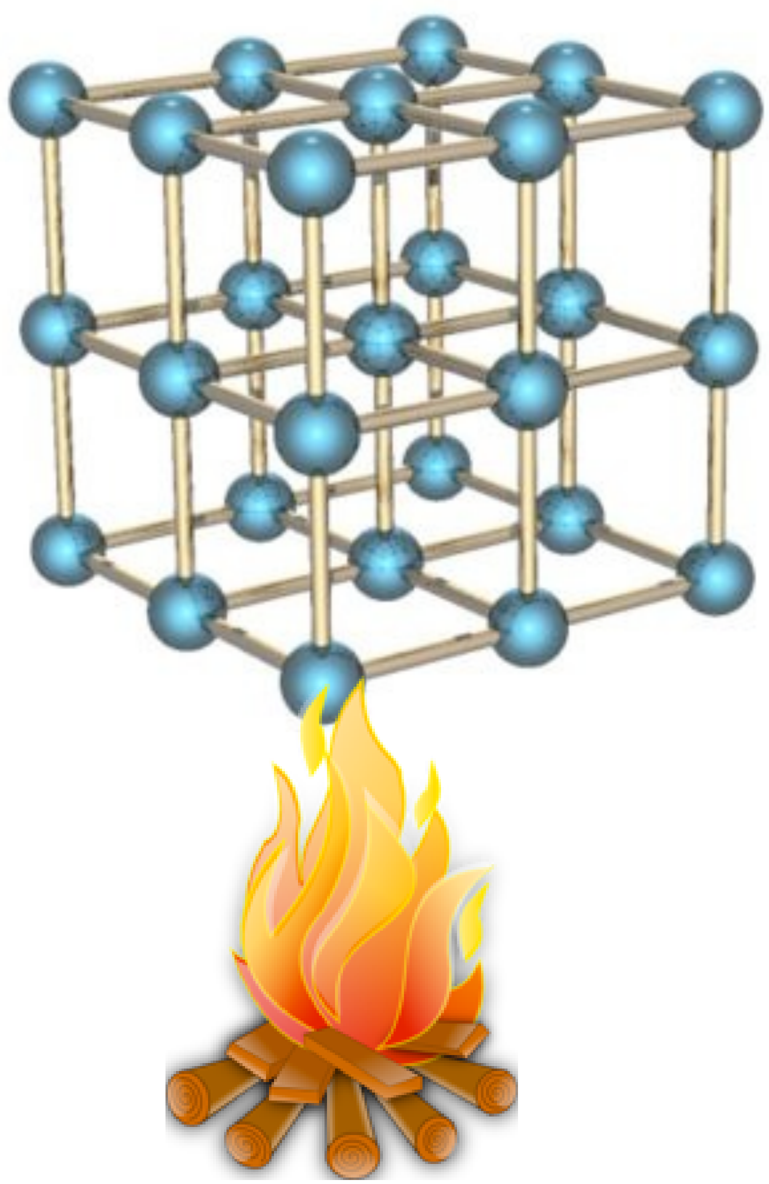
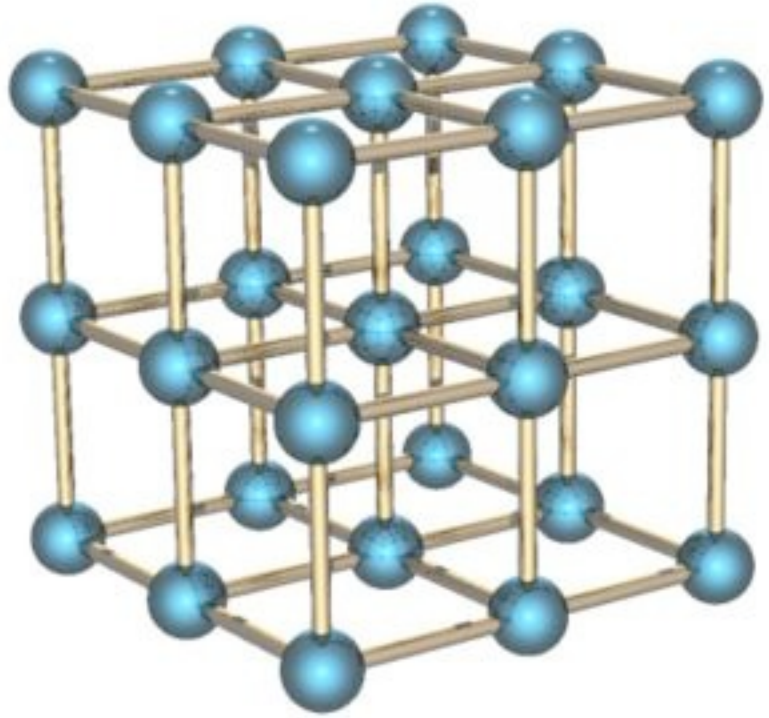


Thermal annealing

Extended hardness

- No physical process can find ground state of H efficiently.



Thermal annealing

Adiabatic annealing

Some -complete problems for physicists

- NP-complete ground state energy problems:

k-SAT	q-state Potts
2D translation-invariant tiling	3D Ising glass
Planar Ising glass in a field	

- QMA-complete ground state energy problems:

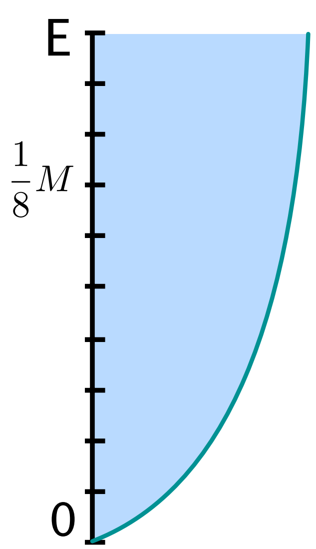
k-Local Hamiltonian	k-QSAT*
2D Pauli Glass	1D translation-invariant Hamiltonian

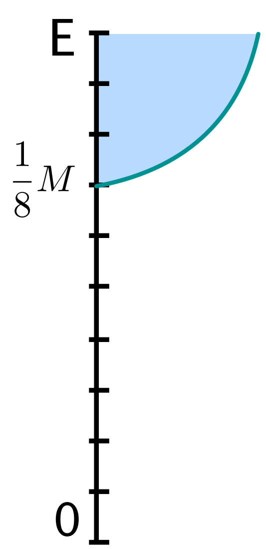
Cook 1971; Levin 1973
Garey, Johnson 1979
Barahona 1982
Kitaev 2001
Bravyi 2006
Oliveira, Terhal 2005
Gottesman 2009

*QMA_1-complete

Finite temperature slowness and PCP

- The PCP theorem asserts that distinguishing certain optimization problems with zero ground state energy from those with extensive ground state energy is NP-complete.
- Annealing to a finite temperature must be slow for these.
- Best current result: 3-SAT cannot be cooled to *any* finite temp.



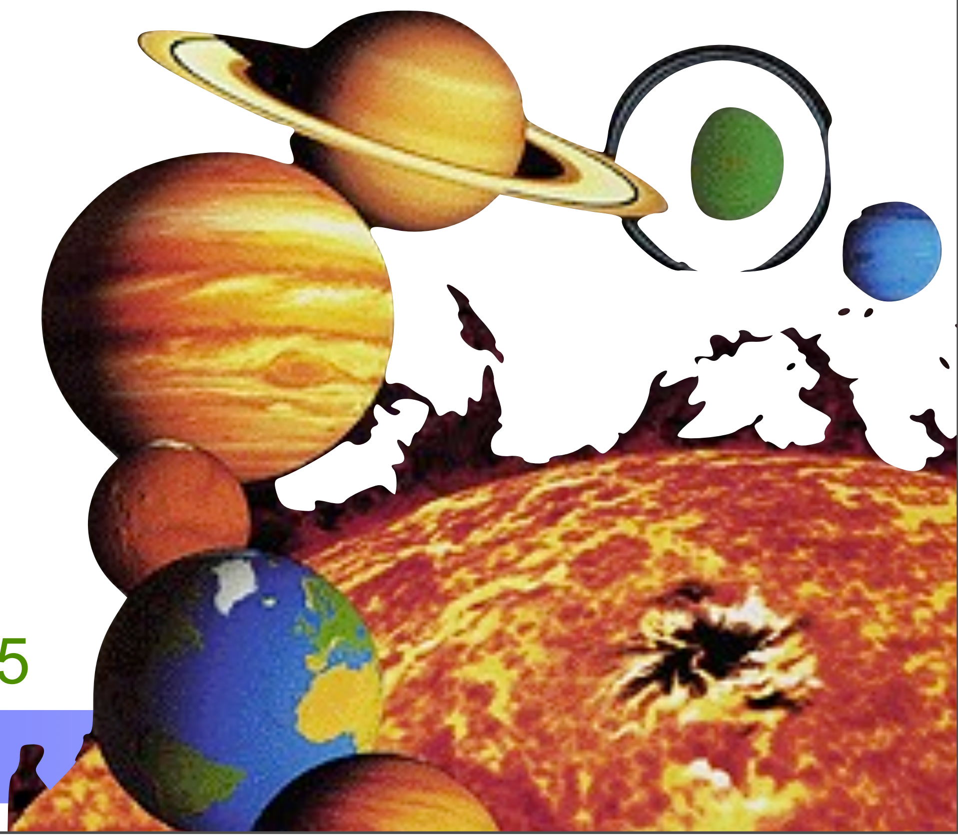


Arora, et al. 1998
Hastad 2001

P≠NP as physical principle

- No physical process can solve NP-complete problems in polynomial time.
  - Quantum mechanics must be linear
  - Closed time-like curves are forbidden
  - No hidden variables (almost).
  - etc

Aaronson 2005

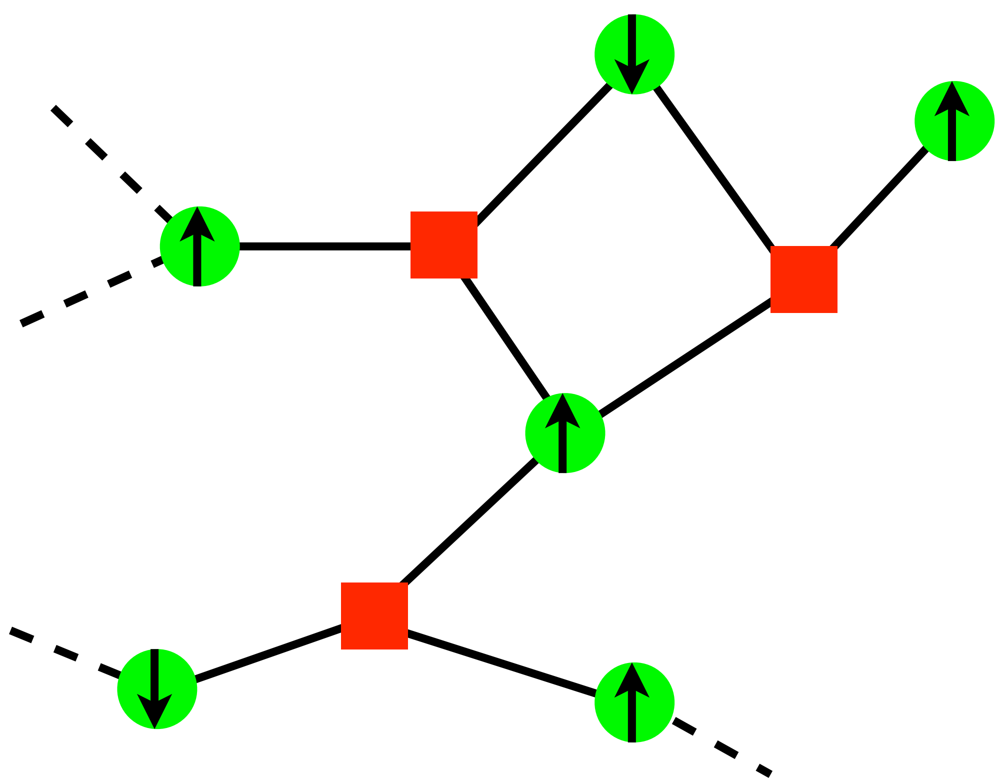


Statistical mechanics of hard problems

- Complexity theory shows hardness of -complete problems
- Suggests optimization problems which should exhibit glassiness
- Does not reveal mechanisms or features underlying slowness

- Studying ensembles of -complete problems not only shows glassiness but also glass transitions

Classical 3-SAT: An ‘Ising’ model



N bits $\vec{\sigma} \in \{\pm 1\}^N$

M constraints $E^m = \delta_{\sigma_{m_1},\phi_1^m}\delta_{\sigma_{m_2},\phi_2^m}\delta_{\sigma_{m_3},\phi_3^m}$

$$H = \sum_{m \in G} E^m(\sigma_{m_1}, \sigma_{m_2}, \sigma_{m_3})$$

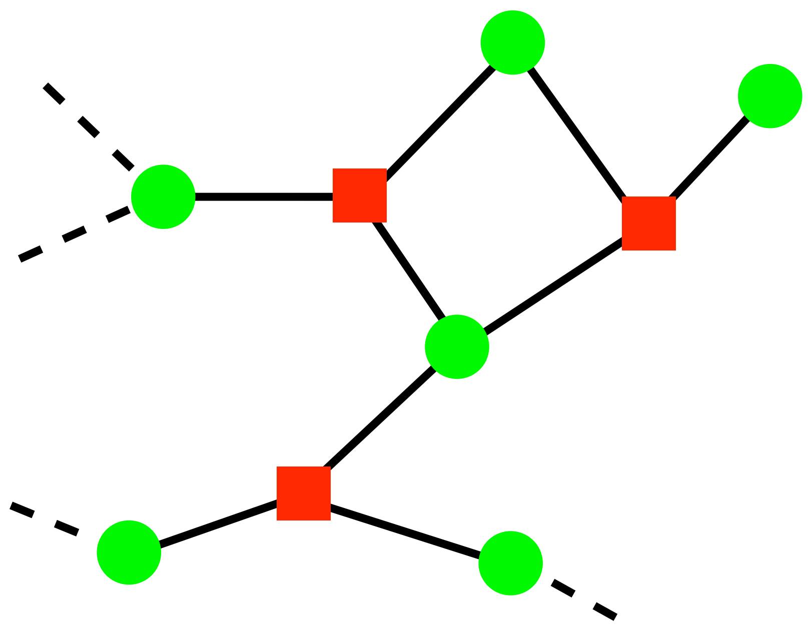
E^m is 0 for satisfying states

Is the ground state energy zero?

$$\exists \vec{\sigma} \text{ s.t. } E^m(\sigma_{m_1}, \sigma_{m_2}, \sigma_{m_3}) = 0 \ \forall m \in G?$$

Ensemble of 3-SAT: Average complexity

Random 3-SAT



Clause density $\alpha = M/N$

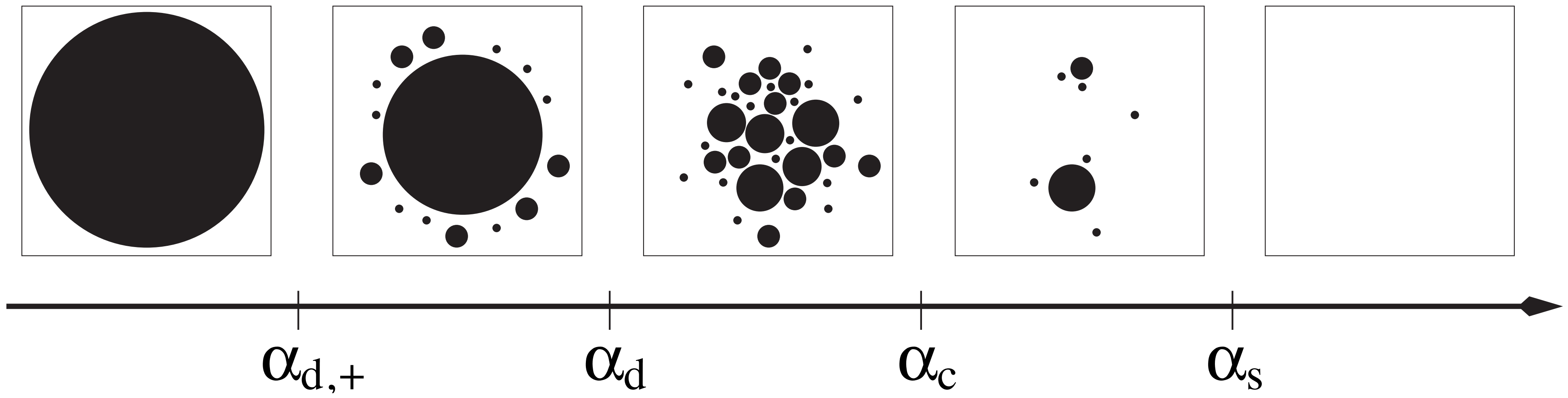
Random graph

Disordered couplings

- Which instances are hard?
- Ensembles of ‘typical’ instances
- Control parameters
- Spin glass physics

Fu, Anderson 1985; Levin 1986

Classical glass theorist's phase diagram

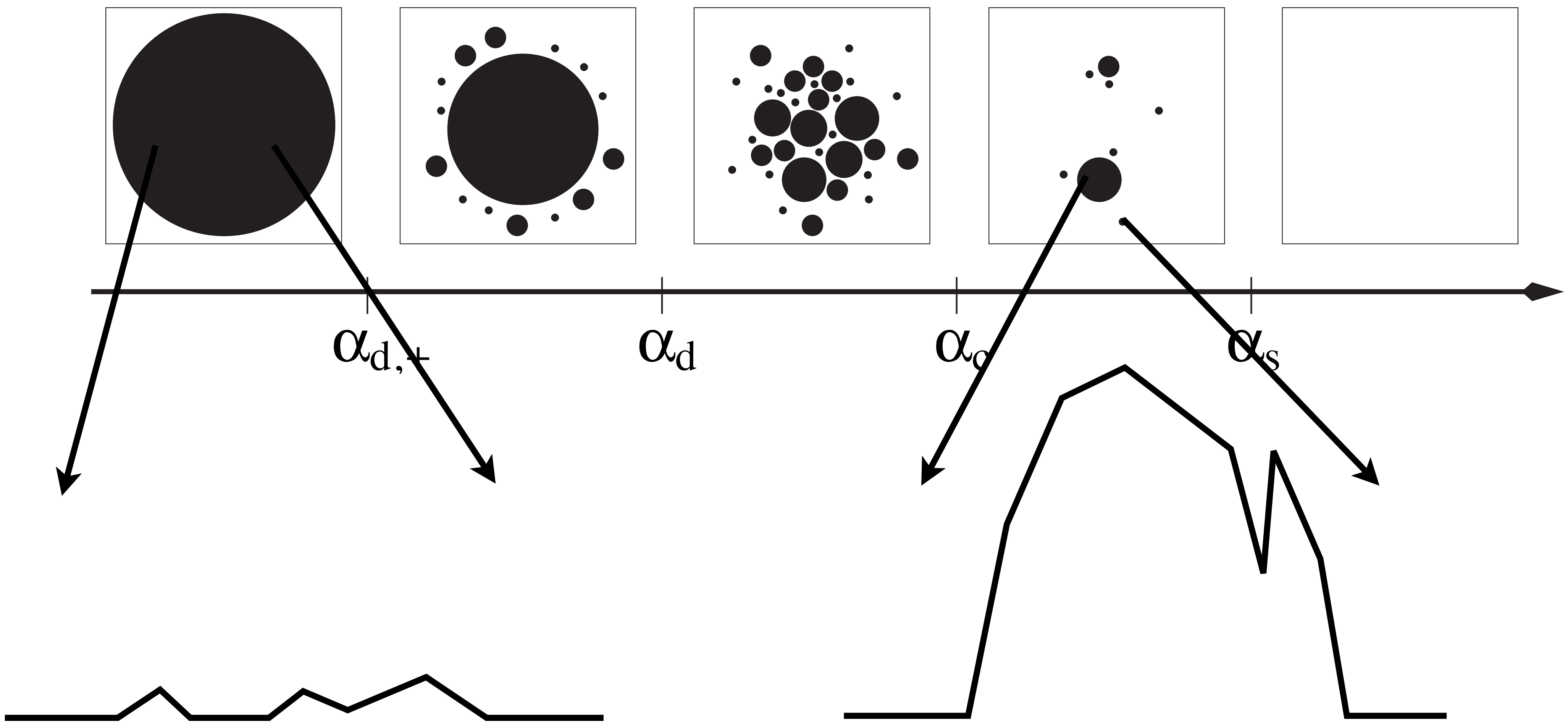


Krzakala, et al., PNAS 2007

- Qualitative phase diagram of random constraint satisfaction problems
- Phase transitions: clustering of satisfying assignments
- Based on RSB cavity methods — cf. Mezard, et al., Science, 2002
- Quantum cavity methods?

CRL, et al., PRB 2008
Hastings, PRB 2007
Leifer, Poulin, Ann Phys 2008

Clustering



- Barrier to local search dynamics and relaxation classically
- Many-body localization in adiabatic algorithm

Altshuler, et al., 2009

Statistical physics of constraint satisfaction

	Classical
Worst-case complexity	NP-completeness
Statistical physics	Satisfiability transitions Dynamical transitions Clustering transitions
Heuristic algorithms	Simulated annealing Belief propagation Survey propagation

Statistical physics of constraint satisfaction

	Classical	Quantum
Worst-case complexity	NP-completeness	QMA-completeness
Statistical physics	Satisfiability transitions Dynamical transitions Clustering transitions	?
Heuristic algorithms	Simulated annealing Belief propagation Survey propagation	??

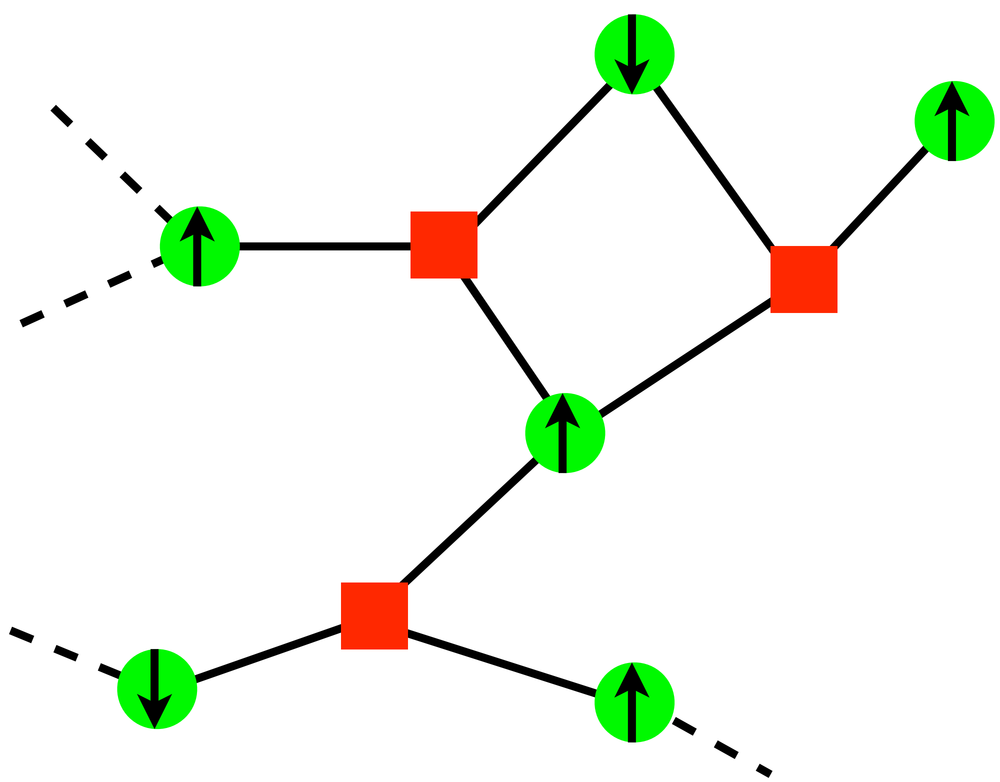
Statistical physics of constraint satisfaction

	Classical	Quantum
Worst-case complexity	NP-completeness	QMA-completeness
Statistical physics	Satisfiability transitions Dynamical transitions Clustering transitions	Satisfiability transitions Entanglement transitions
Heuristic algorithms	Simulated annealing Belief propagation Survey propagation	Adiabatic algorithm? Quantum Metropolis?

Quantum Satisfiability

- Natural quantum generalization of classical satisfiability (k-SAT)
- Quantum hard worst-case complexity: QMA_1-complete
- Are typical instances hard?
- Motivated by classical story, but has its own features...

Quantum k-QSAT: A k-local qubit model



N qubits $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$

M constraints $\Pi^m = |\phi^m\rangle\langle\phi^m|$

$$H = \sum_{m \in G} \Pi^m$$

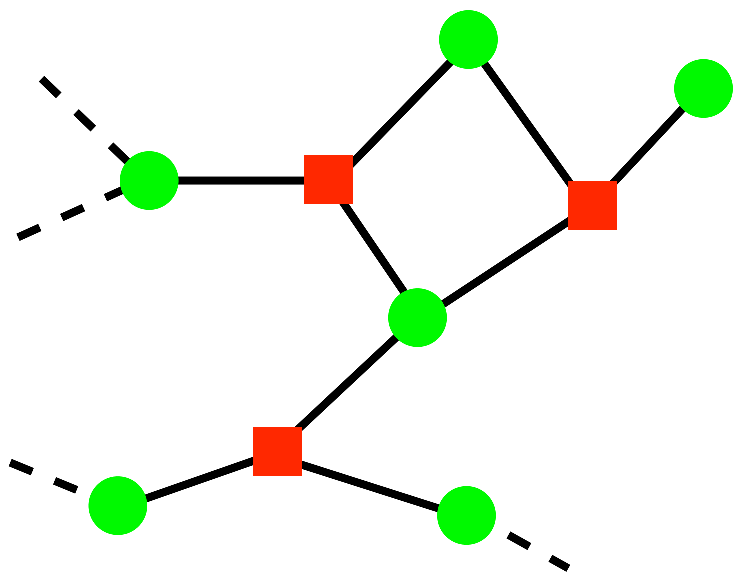
Π^m penalizes 1 out 2^k states

Is the ground state energy zero?

$$\exists|\psi\rangle \in \mathcal{H} \text{ s.t. } \Pi^m|\psi\rangle = 0 \ \forall m \in G?$$

Ensemble of k-QSAT: Average complexity

Random graph



Discrete

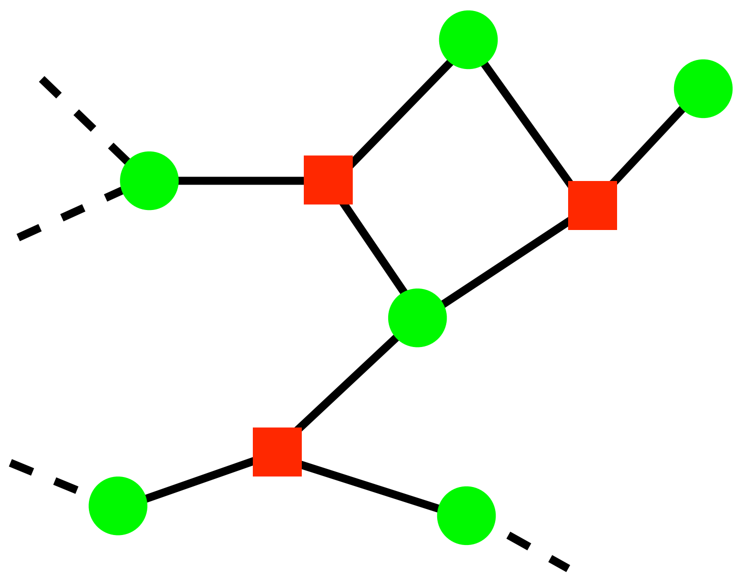
Clause density $\alpha = M/N$

Place edges w.p. $p = \alpha / \binom{N}{k}$

CRL, et al., 2009

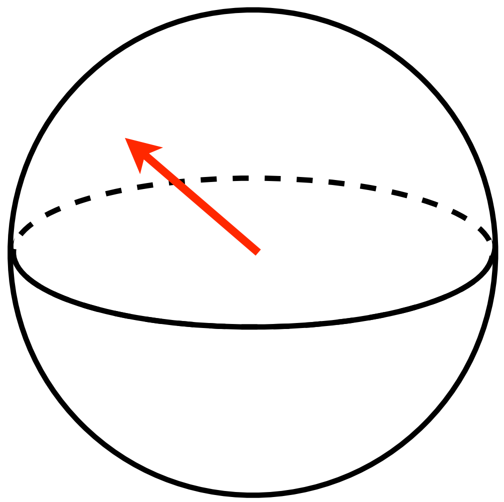
Ensemble of k-QSAT

Random graph



Discrete

Generic projectors



Continuous

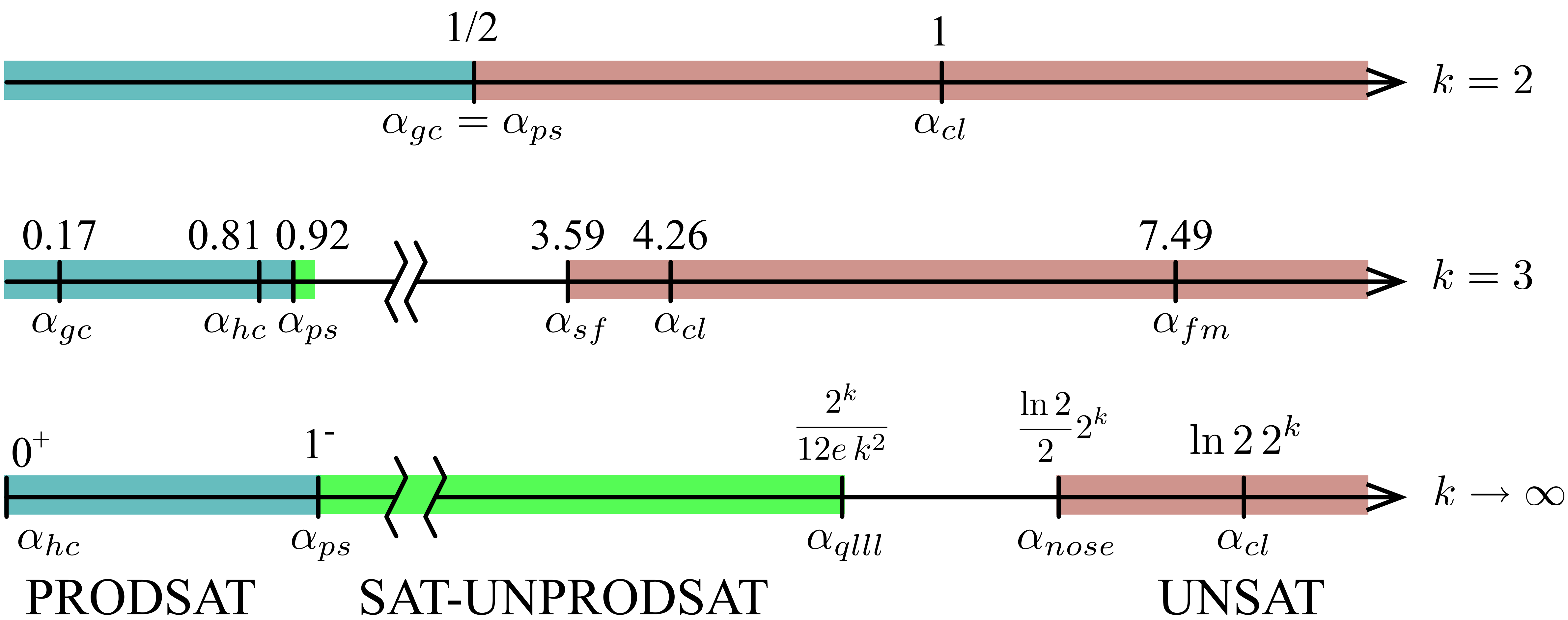
Clause density $\alpha = M/N$

Place edges w.p. $p = \alpha / \binom{N}{k}$

$\Pi^m \leftarrow \mathbb{CP}^{2^k - 1}$

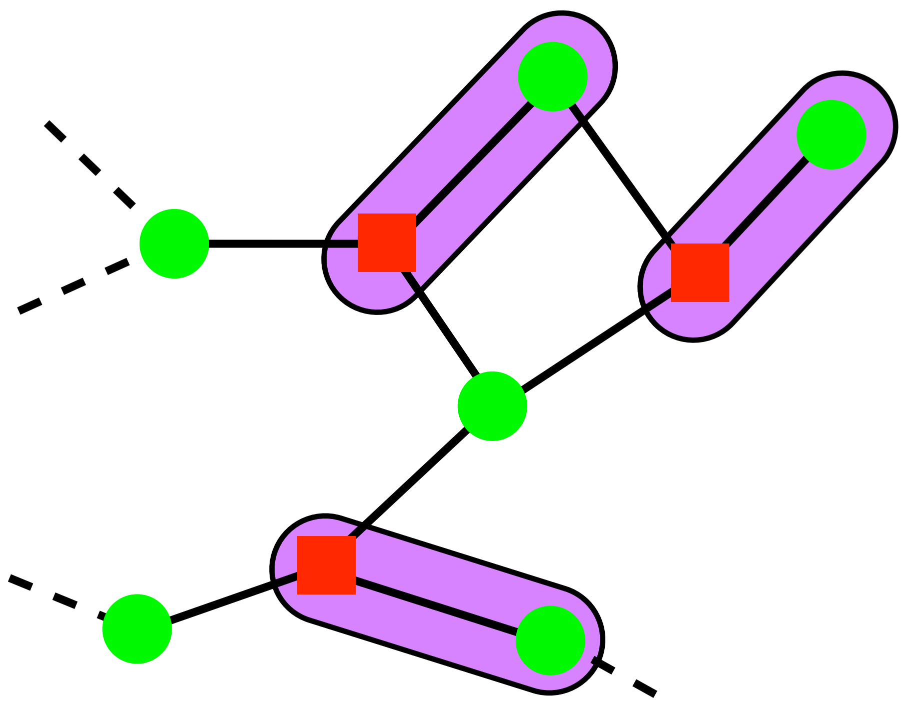
CRL, et al., 2009

Phase diagram of k-QSAT



- k=2 has direct PRODSAT-UNSAT transition
- Large k has entangled SAT phase, barrier to description of GS
- Numerics for k=3 (small sizes): $\alpha_c \approx 1 \pm 0.06$

PRODSAT: dimer covering states



$$|\phi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle$$

$$|\psi_{dc}\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_N\rangle$$

$$\langle\phi_1|\psi_1\rangle\langle\phi_2|\psi_2\rangle\langle\phi_3|\psi_3\rangle = 0$$

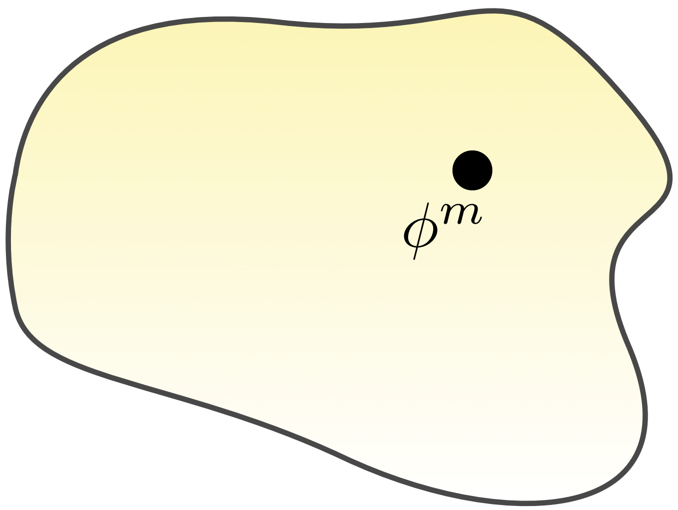
Dimer covering ⟺ Satisfying product state

- Simple argument for product projectors
- Product state perturbation theory for generic projectors

CRL, et al., 0910.2058, 2009

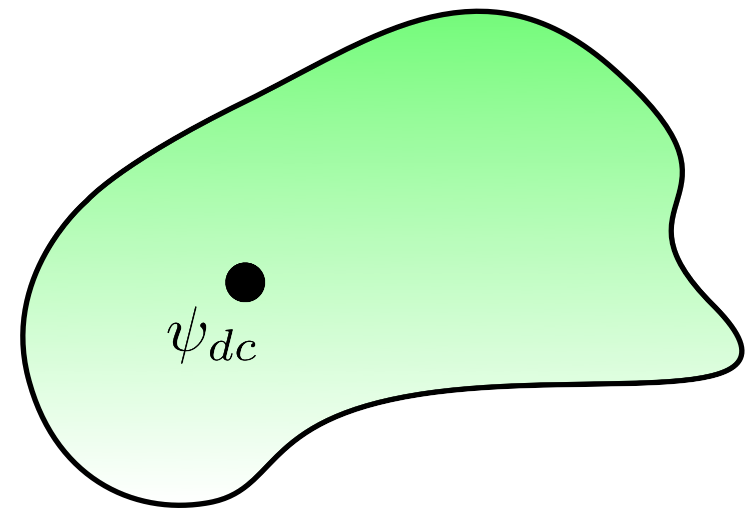
Product state perturbation theory

Projectors



$$(\mathbb{CP}^{2^k-1})^M$$

Product States



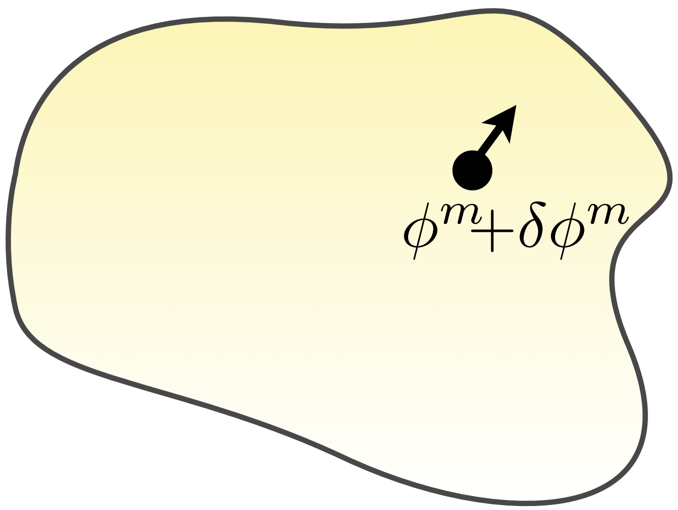
$$(\mathbb{CP}^1)^N$$

$$\phi^m_{i_1 i_2 \cdots i_k} z^{i_1}_{m_1} z^{i_2}_{m_2} \cdots z^{i_k}_{m_k} = 0$$

- Linearization of product state satisfiability condition
- Generically solvable if and only if G has dimer covers

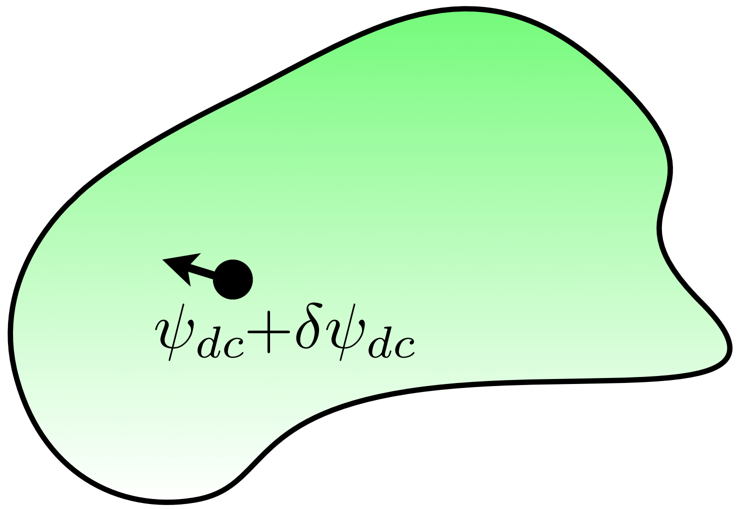
Product state perturbation theory

Projectors



$$(\mathbb{CP}^{2^k-1})^M$$

Product States



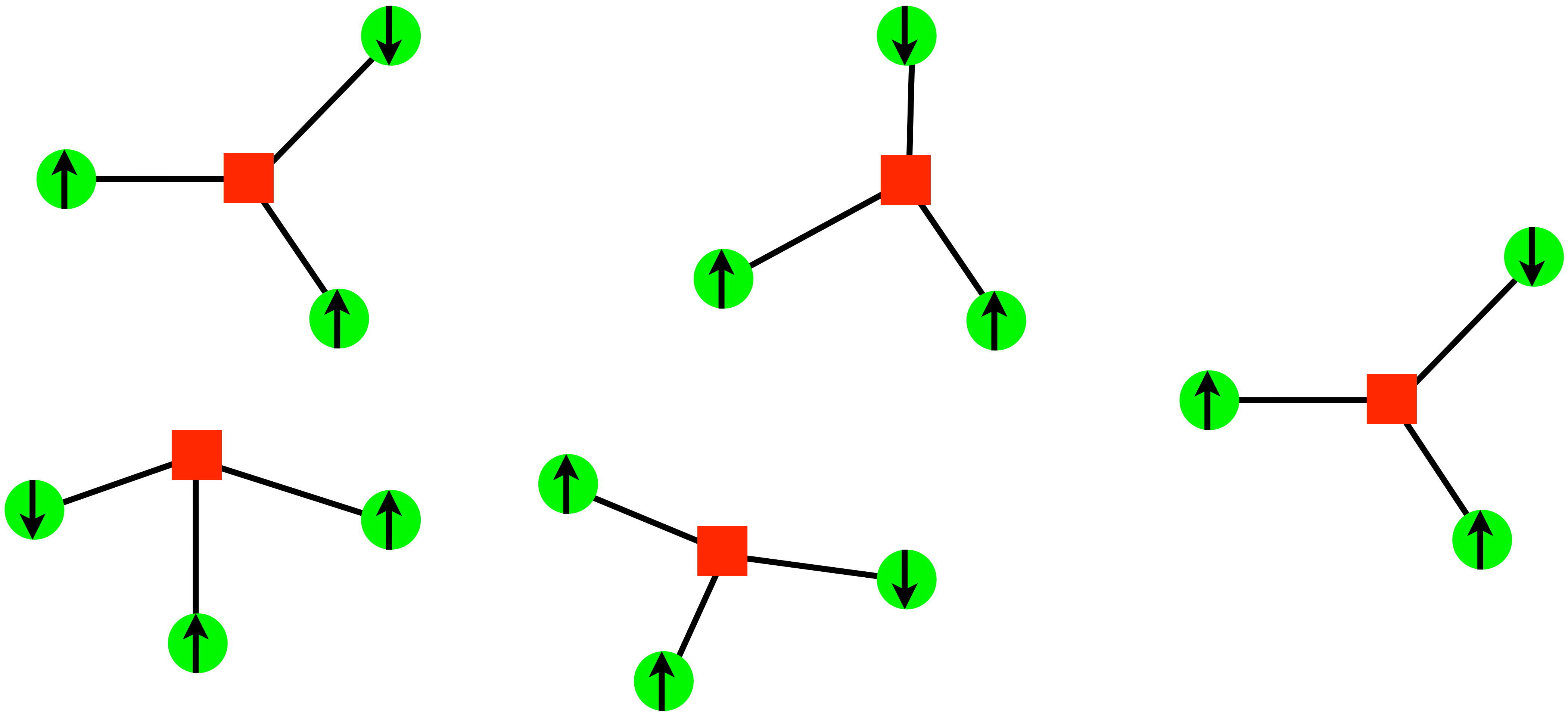
$$(\mathbb{CP}^1)^N$$

$$\phi^m_{i_1 i_2 \cdots i_k} z^{i_1}_{m_1} z^{i_2}_{m_2} \cdots z^{i_k}_{m_k} = 0$$

$$\phi^m_{10\cdots 0} \delta\psi_{m_1} + \phi^m_{01\cdots 0} \delta\psi_{m_2} + \cdots + \phi^m_{00\cdots 1} \delta z_{m_k} = -\delta\phi^m_{00\cdots 0}$$

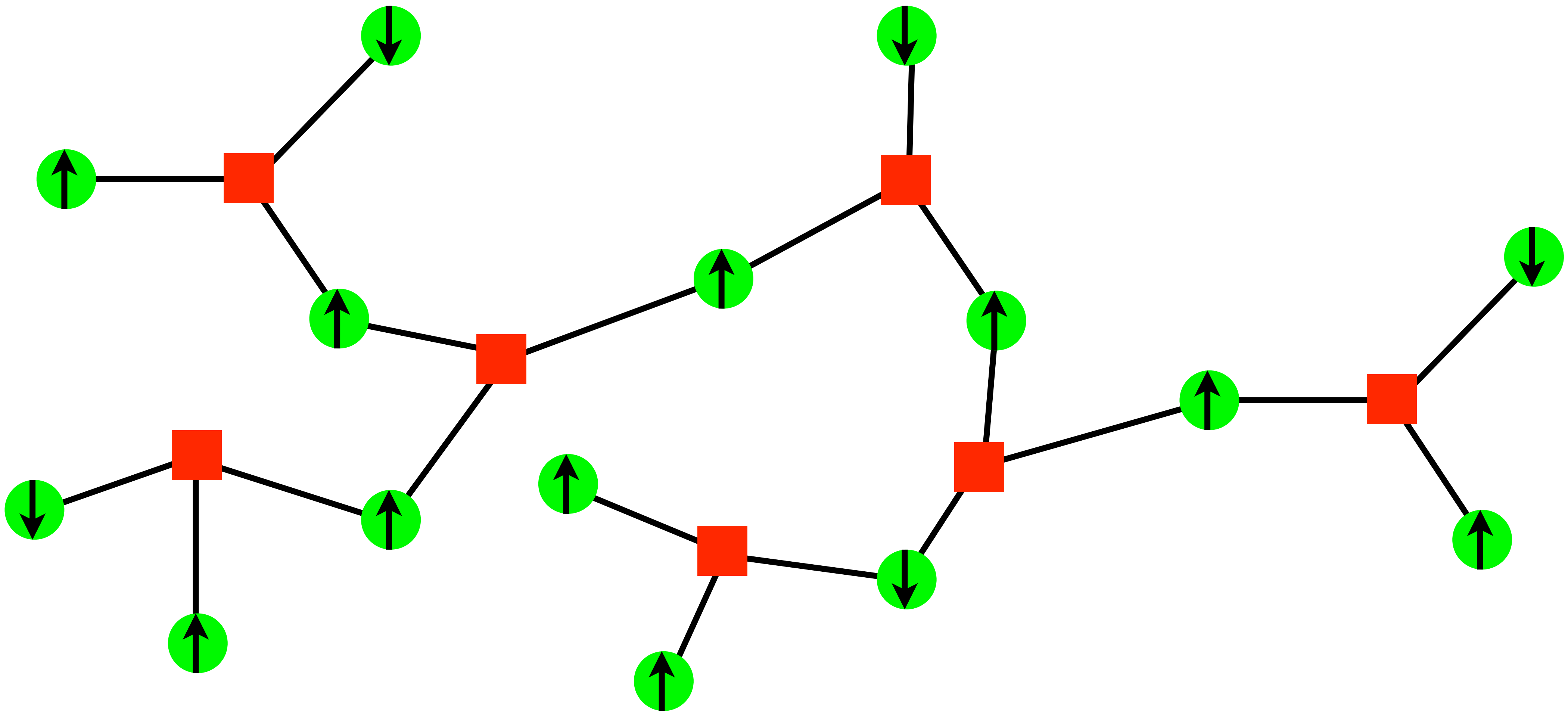
- Linearization of product state satisfiability condition
- Generically solvable if and only if G has dimer covers

Quantum Lovász Local Lemma



- Satisfying subspace exists for nonintersecting projectors

Quantum Lovász Local Lemma



- Satisfying subspace exists for nonintersecting projectors
- Lovász: if $D \leq 2^k/e - 1$ then the subspace still exists

Ambainis, Kempe, Sattath, 2009

Random QSAT: Open questions

- Where's the SAT-UNSAT phase transition?
- What is the complexity of computing $R_g(G)$? Does generic QSAT have a classical test?
- Is the entanglement transition a thermodynamic transition in the entropy?
- What about higher rank projectors? Chains at higher rank.
- *Glass physics:* Quantum analogs of clustering/dynamical phase transitions from classical glass problems? Many-body localization?
- *Stat mech:* Cavity methods for QSAT problem? Finite energy/temperature behavior? Does it anneal?
- *Quantum algorithms:* Quantum generalizations of belief propagation, survey propagation? How does the adiabatic algorithm fare? Quantum metropolis?

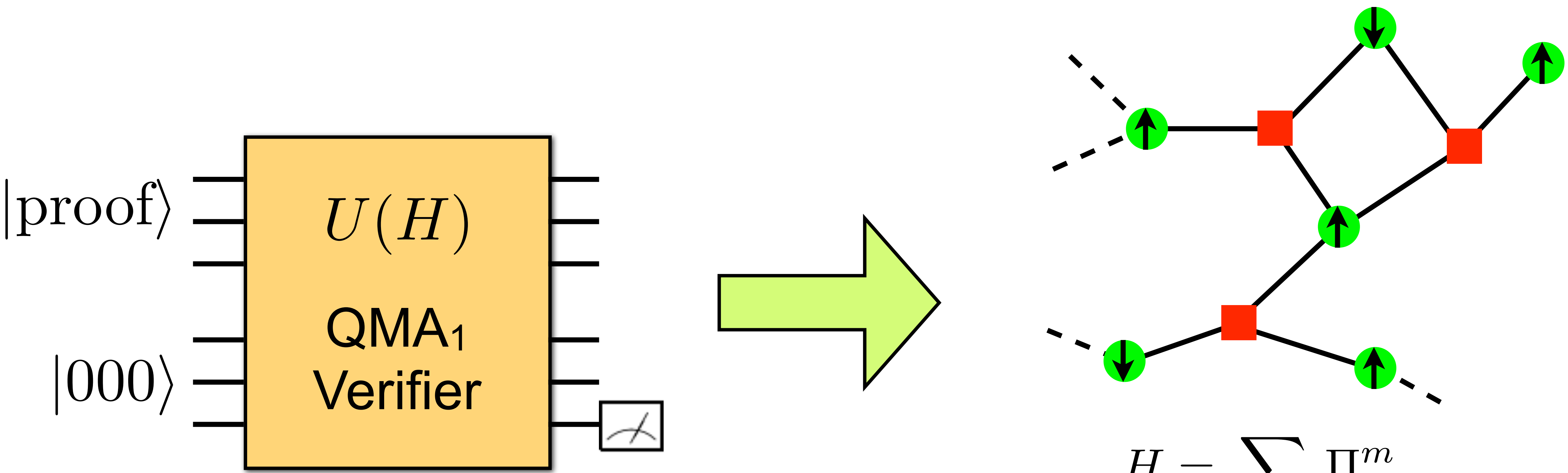
CRL, et al., QIC 10, 2010
CRL, et al., PRA 81, 2010

Conclusion

- Complexity theory attempts to identify problems which are hard.
- Hard problems in this sense are unsolvable by any physical process.
- Such hardness results provide indirect existence proofs for glassy dynamics.
- Random ensembles of -complete problems provide direct insight into those features which may lead to hardness and slow down.
- Phase transitions, clustering and replica symmetry breaking are characteristics of the classical statistical treatments of SAT.
- Quantum SAT is a quantum hard problem whose random ensemble exhibits a novel kind of ‘clustered’ phase: the entangled SAT phase.

Introductory Les Houches Lecture Notes to appear on arXiv soon with R. Moessner, A. Scardicchio and S. L. Sondhi.

k-QSAT: Worst-case complexity

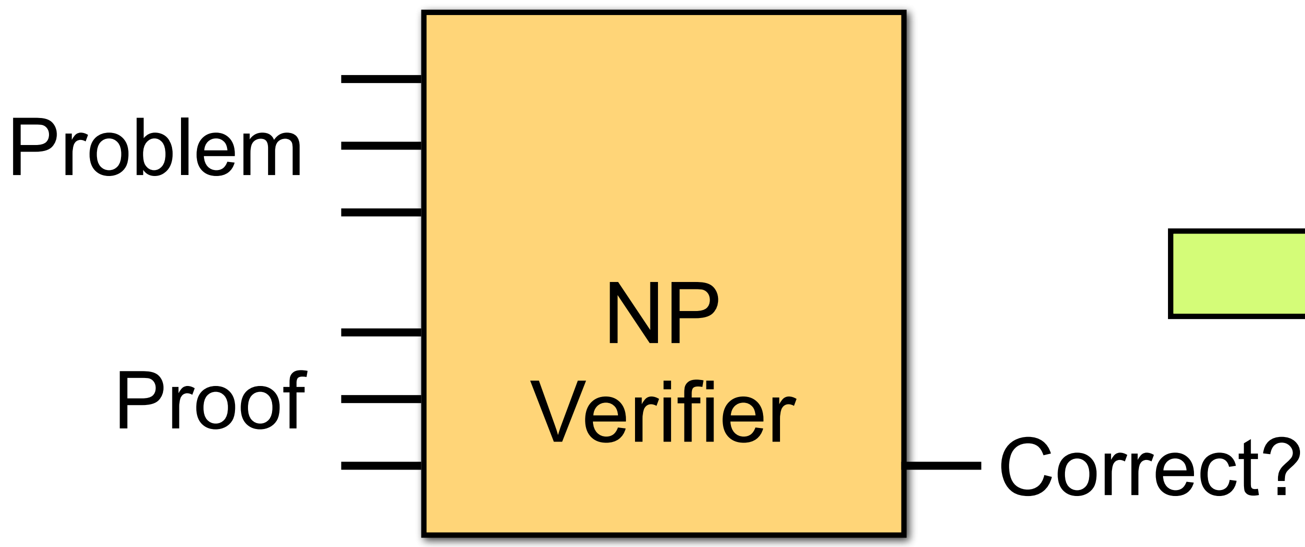


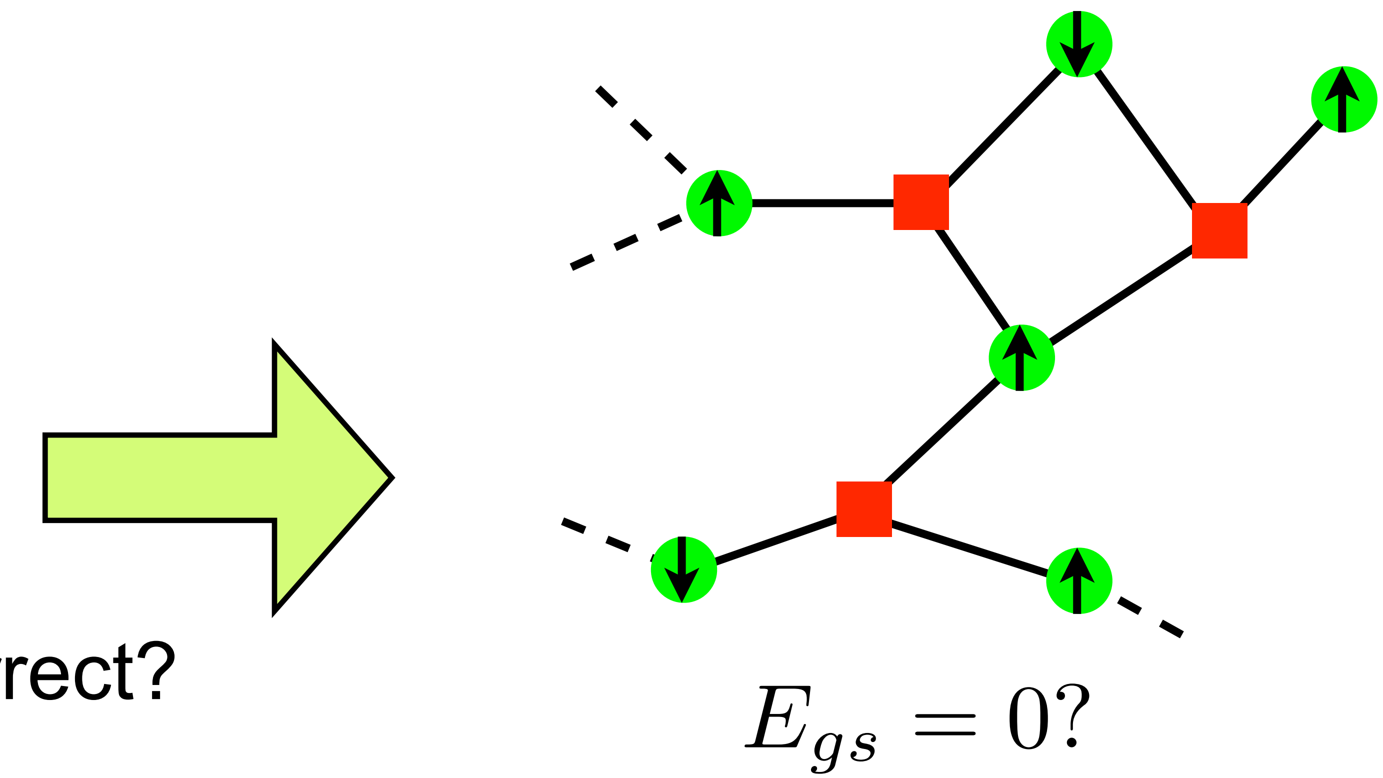
$$H = \sum_{m \in G} \Pi^m$$

- 2-QSAT in P -- classical algorithm to solve -- Easy!
- k-QSAT (k > 3) is QMA$_1$-complete -- Hard!

Bravyi, 2006

3-SAT: Worst-case complexity





- 3-SAT can encode the operation of the verification circuit.
- 3-SAT is NP-complete: solve 3-SAT efficiently and you could solve all of NP efficiently (P=NP)

Cook 1971; Levin 1973