Magnitude-dependent Omori law

Multifractal Scaling of Thermally Activated Rupture Processes

The earthquake deformation flow as a multifractal measure / conditional Poisson process

G. Ouillon and D. Sornette
Institute of Geophysics and Planetary Physics
and Department of Earth and Space Science
University of California, Los Angeles, CA, USA
and
LPMC, CNRS UMR6622 and Univ. Nice, France

J.-F. Muzy
CNRS, University of Corte, Corsica, France

E. Ribeiro (under-graduate)
LPMC, CNRS UMR6622 and Univ. Nice, France

A. Saichev
Mathematical department, Nishny Noygorod, State University, Russia

M. Werner (graduate)
Department of Earth and Space Science
University of California, Los Angeles, CA, USA

Organization of seismicity

- Gutenberg-Richter law: \( \sim 1/E^{1+\beta} \) (with \( \beta \approx 2/3 \))
- Omori law: \( \sim 1/t^p \) (with \( p \approx 1 \) for large earthquakes)
- Productivity law: \( \sim E^a \) (with \( a \approx 2/3 \))
- Power law PDF of fault lengths: \( \sim 1/L^2 \)
- Fractal/multifractal structure of fault networks: \( \zeta(q), f(\alpha) \)
- Power law PDF of seismic stress sources: \( \sim 1/s^{2+\delta} \) (with \( \delta \geq 0 \))
Hierarchical geometry of faulting
Ouillon, Castaing, Sornette (JGR 1996)

Map A: linear size=10 m, orig. scale=1:1
Map B: linear size=60 m, orig. scale=1:220
Map C: linear size=11 km, orig. scale=1:62,500
Map D: linear size=45 km, orig. scale=1:125,000
Map E: linear size=150 km, orig. scale=1:250,000
Map F: linear size=400 km, orig. scale=1:1,000,000

Spatial and temporal organization of seismicity in California

Temporal decay of the rate $N(t)$ of aftershocks after a mainshock at $t=0$

$N(t) = \frac{K}{(t+c)^p}$

$p$ is in the range $[0.3, 2]$, often close to 1

[Omori, 1894; Utsu, 1960]
rate of seismic events of magnitude $M > m$ occurring in a cell of size $L \times L$

**Monofractal view:**
\[
\lambda(m,L,T) = a \cdot 10^{-bm} L^d T^{-p}
\]

Unified Scaling Law for Earthquakes (Bak et al, PRL 2002; Corral, 2003; Baiesi and Paczuski, 2004)

**Multifractal view (“metric”):**
\[
\lambda_i(m,L,T) = a_i \cdot 10^{-b_i m} L^{d_i} T^{-p_i}
\]

exponents are inter-related

Earthquakes as thermally activated processes

- Thermal activation controls creep rupture [Scholz, 2002]

- Eyring rheology and other thermal-dependent friction laws describe creep failure in many compounds and material interfaces [Liu and Ross, 1996; Vulliet, 2000]

- Stress corrosion with pre-existing cracks in rocks [Atkinson, 1984] and hydrolytic weakening [Griggs et al, 1957]

- Ruina-Dieterich state-and-velocity dependent friction law [Dieterich, 1979; Ruina, 1983; Scholz, 1998]
thermal rupture activation process

Poisson Intensity (average conditional seismicity rate)
At position \( \vec{r} \) and time \( t \)

\[
\lambda(\vec{r}, t) \sim \exp \left[ -\beta E(\vec{r}, t) \right]
\]

\[
E(\vec{r}, t) = E_0(\vec{r}) - V \Sigma(\vec{r}, t) \quad \text{(Zhurkov, 1965)}
\]

stress corrosion, damage, state-and-velocity dependent friction
and mechano-chemical effects

\[
\Sigma(\vec{r}, t) = \Sigma_{\text{far field}}(\vec{r}, t) + \int_{-\infty}^{t} \int d\vec{r} \Delta \sigma(\vec{r}', \tau) g(\vec{r} - \vec{r}', t - \tau)
\]

\[
\lambda_i(t) = \lambda_{\text{tec}}(t) \exp \left[ \beta \sum_j \int_{-\infty}^{t} d\tau \Delta \sigma_j(\tau) g_{ij}(t - \tau) \right]
\]

Generalization of stress release models [Vere-Jones et al.]

approximation

\[
\int dN [d\vec{r}' \times d\tau] \Delta \sigma(\vec{r}', \tau) g(\vec{r} - \vec{r}', t - \tau) \approx dN[\tau] s(\tau) h(t - \tau)
\]

\[
\lambda(\vec{r}, t) = \lambda_{\text{tec}}(\vec{r}, t) \exp \left[ \beta \int_{-\infty}^{t} d\tau s(\vec{r}, \tau) h(t - \tau) \right]
\]

\[
s(\vec{r}, \tau) = \int d\vec{r}' \Delta \sigma(\vec{r}', \tau) f(\vec{r} - \vec{r}')
\]

Effective source at time \( \tau \) at point \( \vec{r} \) resulting from all events occurring in the spatial domain at that time \( \tau \)
Physical model

- Rupture of triggered events is a thermally activated processes (creep rupture, subcritical crack growth, state and rate friction...), depending exponentially on stress.
- Bulk rheology displays a slow relaxation of stress, with a long relaxation time $\tau$ (much larger than $T=1$ year). This relaxation takes the form:
  
  $$ h(t) = \frac{h_0}{(t+c_0)^{1+\theta}} , 0 < t < \tau $$

- At any place, stress fluctuations due to past events obey a power-law distribution:
  
  $$ P(s) \propto \frac{C}{s^{1+\mu}} $$

  (Kagan, 1994; Marsan, 2004)

- In continuous form, the seismicity rate can thus be written:
  
  $$ \lambda(t) = \lambda_{\text{tec}}(t) \exp \left[ \beta V \int_{-\infty}^t dt' s(t') h(t-t') \right] $$

  where $\lambda_{\text{tec}}(t)$ is the average long-term seismicity rate imposed by tectonic loading and $\beta$ is the inverse temperature. $V$ is the activation volume.

Theoretical predictions using tail covariance concept (Ide-Sornette, 2001)

$$ \Pr[\lambda(t) > \lambda_M | \lambda_{\text{tec}}] = \Pr[e^{\beta \omega(t)} > \frac{\lambda_{\text{tec}}}{\lambda_M} | \omega_M] = \Pr[\omega(t) > (1/\beta) \ln \left( \frac{\lambda_{\text{tec}}}{\lambda_M} \right) | \omega_M] $$
\[ \lambda_q(t) = A_q \lambda_{tec} e^{\beta \gamma(t) \omega_M} \]

\[ \gamma(t) = \frac{h_0^2}{\Delta t^{2/\mu}} \left( \frac{1}{t^{2m-1}} \int_0^{t^{2m-1}} dy \left( \frac{1}{y+1} \right)^m \right)^{2/\mu} \]

\[ m = (1 + \theta) \mu / 2. \]

Since \( \gamma(t) \sim \ln(t) \) and \( \omega_m \sim m \), we obtain \( p(m) = a + b \).

\[ A(t) = \int_{-\infty}^{t} d\tau \eta(\tau) K(t - \tau) \]

**Endogeneous shock**

\[ E[X(t)|Y = A_0] - E[X(t)] = (A_0 - E[Y]) \frac{\text{Cov}(X(t), Y)}{E[Y^2]} \]

\[ \text{Cov}(A(t), A(0)) = \int_{-\infty}^{0} d\tau K(t - \tau) K(-\tau) \]

\[ E_{\text{endo}}[A(t)|A(0) = A_0] \propto A_0 \int_{0}^{+\infty} du K(t + u) K(u) \]
Data used for analysis

- We use the SCEC catalog (32° to 37°N, -122° to -128°W)
- We define 4 subcatalogs, according to their completeness
  1932-2003 for events with M > 3.0
  1975-2003 for events with M > 2.5
  1992-2003 for events with M > 2.0
  1994-2003 for events with M > 1.5
- Each subcatalog will be analyzed separately
Data processing

- An event is considered as triggered by another event of magnitude $M$ if it falls within a spatial window of size $d$ or $L$ or $L_{\text{previous}}$ around that event within $T=1$ year after its occurrence.
- Size $L$ is taken either equal to the estimated main rupture length ($L=10^{-2.57+0.6M}$), or twice that length.
- We bin mainshock magnitudes in consecutive intervals $[1.5;2.0]$, $[2.0;2.5],...$ up to $[7.0;7.5]$
- In each main event magnitude interval $[M_1;M_2]$, we translate each triggered sequence to a common origin time $t=0$, and stack all sequences.
- We fit composite sequences by $N(t) = B + a/(t+c)^p$ using linear least-squares or use Maximum Likelihood.
- We can then obtain the average value of $p$ as a function of main event magnitude.
- Use of different definitions of mainshocks and robustness of the results.
Figure 10. Average p-values and error bars obtained from Figure 9 as described in the text. The straight line is the linear fit with $p(M) = 0.12M_L + 0.28$. (first declustering method)

We obtained very similar results using slightly different declustering methods.

\[ p(M) = 0.10M + 0.37 \]

Predicts minimum earthquake magnitude for triggering \( m_g = 3 \) (Ben-Zion, 2005)

\[ p(M) = 0.3 + 0.11m \]
Multifractal stress activation (MSA) model:

\[
\lambda(\vec{r}, t) = \lambda_{\text{tec}}(\vec{r}, t) \exp \left[ \beta \int_{-\infty}^{t} d\tau \, s(\vec{r}, \tau) h(t - \tau) \right]
\]

\[
\lambda(t) = \lambda_{\text{tec}} \prod_{i \mid t_i < t} \exp \left[ \beta s(t_i) \, h(t - t_i) \right]
\]

\[
\beta s(t_i) h(t - t_i) = \beta s(t_i) h_0 \, e^{-t/T} \cdot c^\theta / (t + c)^{1+\theta}
\]

For \( \beta s(t_i) h_0 \) small, expand the exponential and get

**ETAS conditional Poisson intensity:**

\[
\lambda(t) = \lambda_{\text{tec}} + \sum_{i \mid t_i < t} \rho_i h(t - t_i)
\]

with \( \rho_i \equiv \beta s(t_i) \)

**ETAS = mono-fractal approximation of richer Multifractal model**

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**Epidemic Type Aftershock Sequence (ETAS)**


- each earthquake can be both a mainshock, an aftershock and a foreshock
- each earthquake triggers aftershocks according to the Omori law, that in turn trigger their own aftershocks
  \[
  \phi(t) = \frac{K}{(t + c)^{\alpha + \theta}}
  \]
- the number of aftershocks triggered by a mainshock depends on the mainshock magnitude:
  \[
  N(m) \sim 10^{am}
  \]
- aftershock magnitudes follow the Gutenberg-Richter distribution, independently of the time and of the mainshock magnitude
  \[
  P(m) \sim 10^{-bm}
  \]

\[
\phi_m(\phi, t) \, dr \, dt = K \, 10^{a(M - M_0)} \frac{\theta \, c^\theta \, dt}{(t + c)^{1+\theta}} \frac{\mu \, d\mu \, dr}{(r + d)^{1+\mu}}
\]
"We found that the rate of triggered events decays with time according to Omori's law $1/(t+c)^p$ with $p=0.9$ and $c<3$ minutes (after correcting for the increase in the magnitude of completeness after a large mainshock). This decay is independent of the mainshock magnitude $m$ for $2<m<7.5$.

Figure 2. Same as Figure 1 except that we have used $m_c = 2$ and we have corrected the seismicity rate for missing early aftershocks (assuming GR law with $b = 1$).

We fit the seismicity rate in the time interval $0.002 < t < 10$ days and for $\lambda(t, m_c) > 0.5$ day$^{-1}$. The fit of $K(m_c)$ give $K_0 = 0.008$ day$^{-1}$ and $\alpha = 1.01$."

Helmstetter, Kagan & Jackson, 2005
Arguments supporting our results

- The model predicts that $p=aM+b$ is independent of inverse temperature $\beta$. Until now, no clear empirical relationship between $p$ and temperature has ever been presented.
- Bohnenstiehl et al (2003) sum several triggered sequences whatever the magnitude of the mainshock, and note that raising the magnitude threshold of the mainshocks increases the inverted $p$-value.
- Marsan et al (2003), using all pairs of events in a mine, obtain a global $p$-value of 0.4 – using our empirical $p(M)$ relationship, this corresponds to a magnitude of 0.3, which is a rather reasonable estimate of the size of mining-induced events.
- Assuming that the mean modulus of stress variation in the area where aftershocks occur is $S_0$, then the ratio between the number of triggered events in regions of stress increase to the number of triggered events in regions of stress decrease is of order $R=\exp(2\beta S_0 V)$, where $V$ is the activation volume. Considering that $R$ varies from 1.5 (Parsons, 2002) to 10, that $S_0$ varies from 0.01 to 1MPa and that temperature at seismogenic depth is about 600K, then one can invert for $V$. We then obtain an activation scale=$V^{1/3}$ of about 1 nanometer, which is in agreement with the microscopic process that is thermal activation.

CONCLUSIONS

- Quantitative generic mechanism for multifractality in geophysics

- Implications for forecasts

- Spatio-temporal version

- Multifractal ETAS model

- Multifractal conditional Poisson model

- Log-gamma multifractal measure: continuous “deformation flow” (deriving GR, Omori and productivity law from multifractality flow)

Towards fulfilling Yan Kagan’s dream:

“IS AN EARTHQUAKE A PHYSICAL ENTITY?”
The Multifractal Random Walk (MRW) model

\[ r_{\Delta t}(t) = \epsilon(t) \cdot \sigma_{\Delta t}(t) = \epsilon(t) \cdot e^{\omega_{\Delta t}(t)} \]

\[ \mu_{\Delta t} = \frac{1}{2} \ln(\sigma_{\Delta t}^2) - C_{\Delta t}(0) \]

\[ C_{\Delta t}(\tau) = \text{Cov}[\omega_{\Delta t}(t), \omega_{\Delta t}(t + \tau)] = \lambda^2 \ln \left( \frac{T}{|\tau| + e^{-\frac{3}{2} \Delta t}} \right) \]

\[ \omega_{\Delta t}(t) = \mu_{\Delta t} + \int_{-\infty}^{t} d\tau \eta(\tau) K_{\Delta t}(t - \tau) \]

\[ \omega_{\Delta t}(t) \text{ is Gaussian with mean } \mu_{\Delta t} \text{ and variance } V_{\Delta t} = \int_{0}^{\infty} d\tau \text{ } K_{\Delta t}^2(\tau) = \lambda^2 \ln \left( \frac{T e^{\frac{3}{2} \Delta t}}{\Delta t} \right) \]

\[ C_{\Delta t}(\tau) = \int_{0}^{\infty} dt \text{ } K_{\Delta t}(t) K_{\Delta t}(t + |\tau|) \]

\[ \hat{K}_{\Delta t}(f)^2 = \hat{C}_{\Delta t}(f) = 2 \lambda^2 f^{-1} \left[ \int_{0}^{Tf} \frac{\sin(t)}{t} dt + O(f \Delta t \ln(f \Delta t)) \right] \]

\[ K_{\Delta t}(\tau) \sim K_0 \sqrt{\frac{\lambda^2 T}{\tau}} \text{ for } \Delta t << \tau << T. \]

"Conditional response" to an endogeneous shock

\[ E_{\text{endo}}[\sigma^2(t) \mid \omega_0] = \overline{\sigma^2(t)} \exp \left[ 2(\omega_0 - \mu) \cdot \frac{C(t)}{C(0)} - 2 \frac{C^2(t)}{C(0)} \right] \]

\[ = \overline{\sigma^2(t)} \left( \frac{T}{t} \right)^{\alpha(s)+\beta(t)} \]

where

\[ \alpha(s) = \frac{2s}{\ln \left( \frac{T^3}{e^{3/2} \Delta^2} \right)} \]

\[ \beta(t) = 2\lambda^2 \frac{\ln (t/\Delta t)}{\ln \left( T e^{3/2} / \Delta t \right)} \]

Within the range \( \Delta t < t << \Delta t e^{\lambda^2} \), \( \beta(t) << \alpha(s) \)

\[ E_{\text{endo}}[\sigma^2(t) \mid \omega_0] \sim t^{-\alpha(s)} \]
Real Data and Multifractal Random Walk model

Monofractal view:
\[ \lambda(m, L, T) = a \cdot 10^{-bm} \cdot L^c \cdot T^p \]

Unified Scaling Law for Earthquakes
Bak et al, PRL 2002)

Multifractal view ("metric"):
\[ \lambda_i(m, L, T) = a_i \cdot 10^{-b_i m} \cdot L^{c_i} \cdot T^{-p_i} \]

exponents are inter-related

Molchan and Kronrod (2004) have shown that
\( c_i \) is multifractal
Determination of the sources of endogeneous shocks

\[ W(t) = \int_{-\infty}^{t} d\tau \eta(\tau), \] where \( \eta(t) \) is a standardized Gaussian white noise

\[ E_{\text{endo}}[W(t) \mid \omega_0] = \frac{\text{Cov}[W(t), \omega_0]}{\text{Var}[\omega_0]} \cdot (\omega_0 - E[\omega_0]) \propto (\omega_0 - E[\omega_0]) \int_{-\infty}^{t} d\tau K(-\tau) \]

the expected path of the continuous information flow prior to the endogeneous shock (i.e., for \( t < 0 \)) grows like \( \Delta W(t) = \eta(t)\Delta t \sim K(-t)\Delta t \sim \Delta t/\sqrt{-t} \)

Similar to the expectation of random walk increments conditioned on the knowledge of the fixed values of the two end points