Kavli Institute of Theoretical Physics, “Earthquake, Friction and Fracture”, Santa Barbara, 8/16/05

Energetic-Probabilistic Size Effects in Cohesive Fracture and Asymptotic Matching

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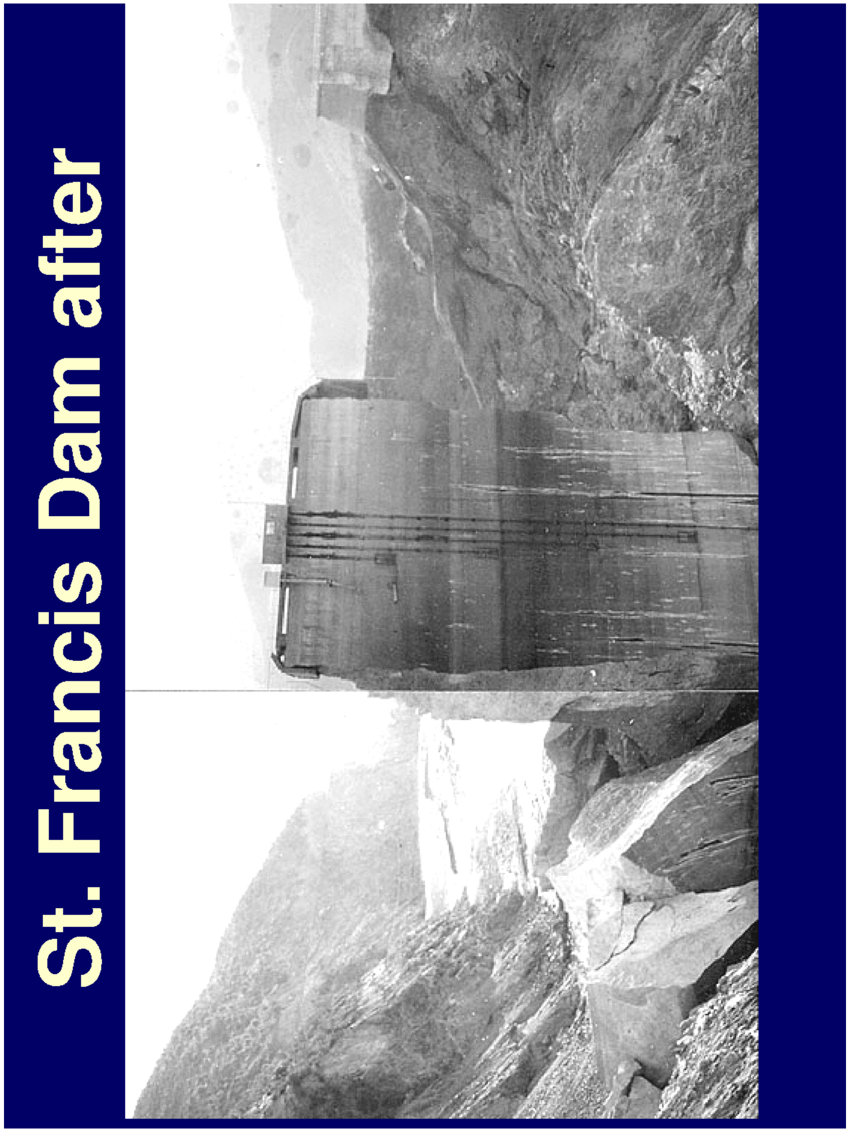
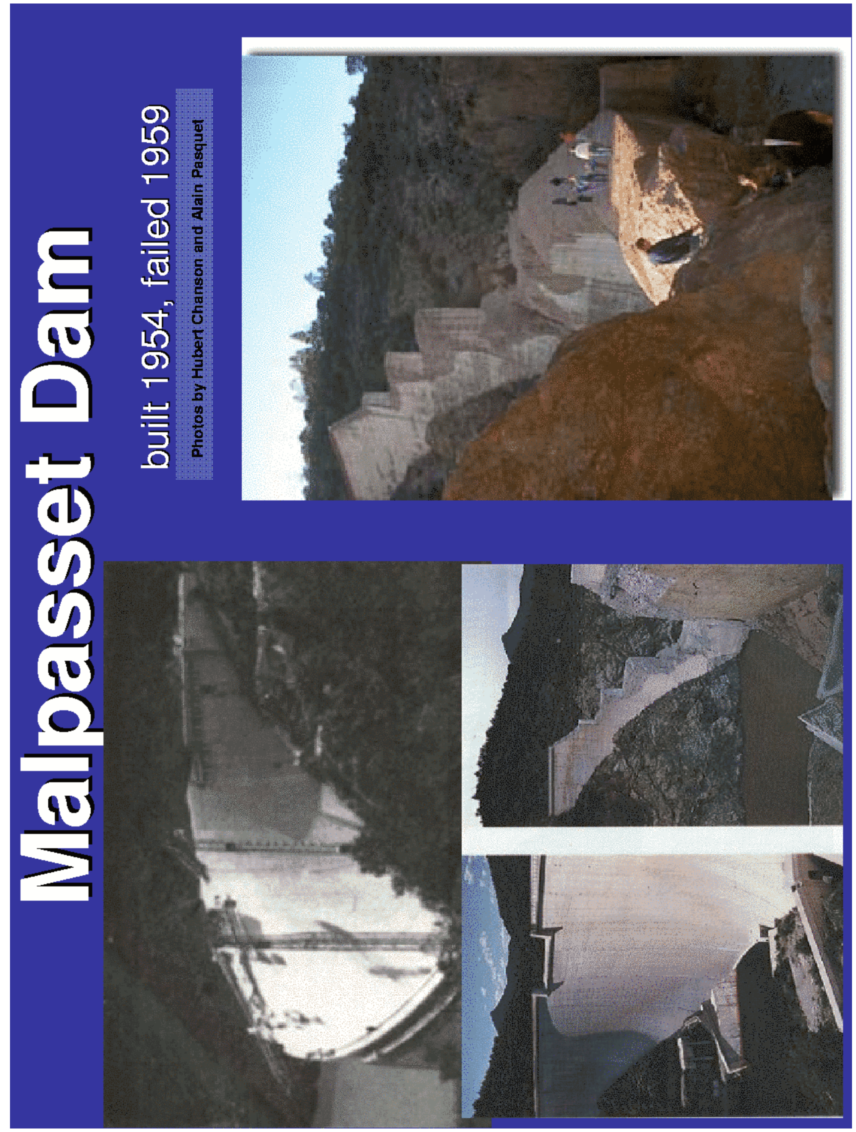
Sponsors: ONR, NSF, FAA, Infrastructure Technology Institute

St. Francis Dam

Located north of LA, record tall
Built 1926, ***Collapsed 1928***


St. Francis Dam after


Malpasset Dam
built 1954, failed 1959
Photos by Hubert Chanson and Alain Pasquel

Ruins of Malpasset Dam

Failed 1959, at Fréjus
French Maritime Alps

Photos by Hubert Chanson and Alain Pasquet

QUASIBRITTLE MATERIALS

(FPZ not << structure size *D*)

concrete (archetypical)	*nanocomposites*
fiber composites	*metallic thin films*
sea ice	*biological shells*
rocks, coal	*mortar, masonry*
toughened ceramics	*fiber-reinforced concrete*
rigid foams	*stiff clays*
wood	*silts, cemented sands*
consolidated snow	*grouted soils*
particle board	*particle board*
paper, carton	*refractories*
	bone, cartilage
	cast iron
	modern tough alloys,…

Two Causes of Size Effect:

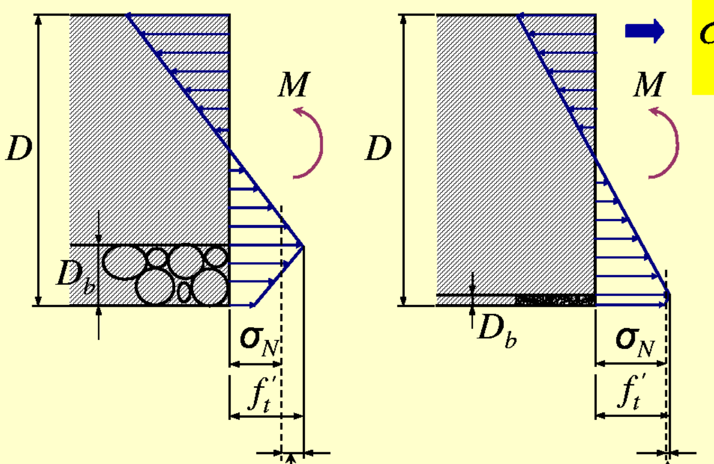
- **statistical**
  - random material strength

    (Mariotte 1650, Weibull 1939)
- **energetic**
  - energy release due to stress redistribution

    (Northwestern 1983)

1. Energetic Size Effect

Intuitive Explanation Type 1 Size Effect (Failure at Crack Initiation)



$$\sigma_N = \sigma_\infty \left(1 + \frac{rD_b}{D}\right)^{1/r}$$

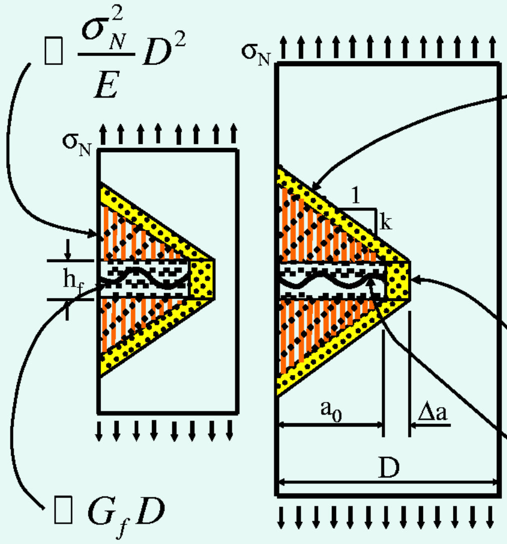
(Alternative derivation --by fracture mechanics for a ✹ 0)

Cause of size effect
(Can this be denied and replaced by fractals?)

Vanishing size effect, but strength randomness becomes important

Intuitive Explanation of Type 2 Energetic Size Effect for Fracture of Mode I or II

(geometric similarity assumed)



□ $\frac{\sigma_N^2}{E} D^2$

□ $G_f D$

Energy released

$= (\sigma_N^2 / 2E) 2ka_0 \Delta a$

□ $\sigma_N^2 (\frac{a_0}{D}) D \Delta a$

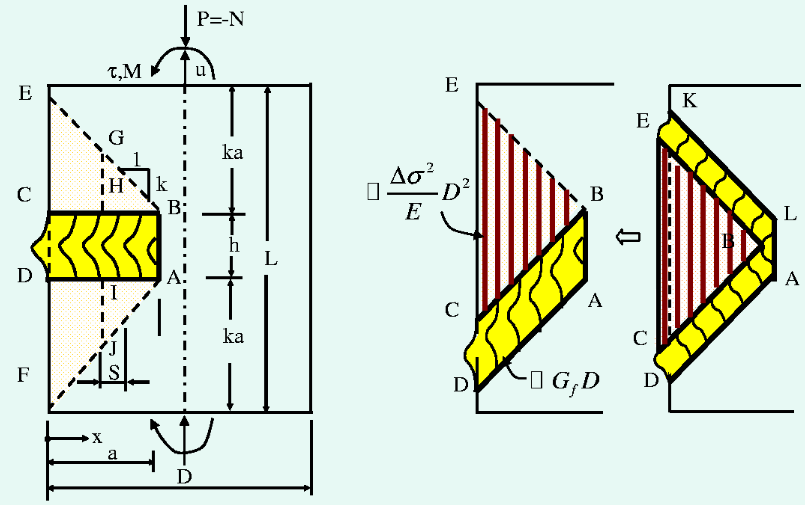
const.

$(\sigma_N^2 / 2E) h_f \Delta a$

Energy dissipated

$= G_f \Delta a$

Intuitive Explanation of Size Effect in COMPRESSIVE Fracture



Asymptotic Properties of Size Effect of Cohesive Crack Model

For $D \to 0$:

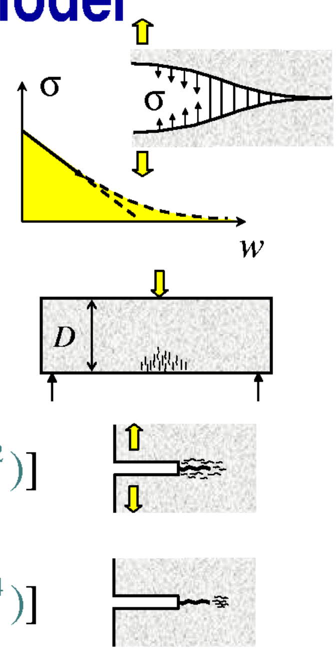
$$\sigma_N \propto 1 - c_0 D + O(D^2)$$

For $D \to \infty$:

Type 1 : $\sigma_N \propto 1 + c_1 D^{-1} + O(D^{-2})$

Type 2 : $\sigma_N \propto D^{-1/2}[1 - c_2 D^{-1} + O(D^{-2})]$

Type 3 : $\sigma_N \propto D^{-1/2}[1 - c_3 D^{-2} + O(D^{-4})]$



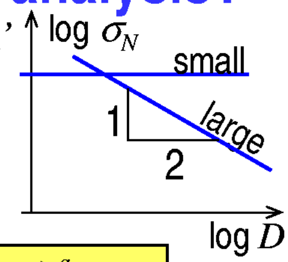
Size Effect of Large Cracks (Type 2)

What follows from dimensional analysis?

4 variables (geometry fixed):

$f_t',\ K_c = \sqrt{E'G_f},\ D,\ \sigma_N$

If G_f or f_t' is absent:



Π–Theorem of dimensional analysis:

-only 4 – 2 dimensions = 2 dimensionless variables:

$$F(\Pi_1, \Pi_2) = 0$$

Choose: $\Pi_1 = \left(\dfrac{\sigma_N}{f_t'}\right)^p \left(\dfrac{D}{l}\right)^u,\ \Pi_2 = \left(\dfrac{\sigma_N}{f_t'}\right)^q \left(\dfrac{D}{l}\right)^v$

$l = E'G_f / f_t'^2$ = Irwin's char. length ≈ FPZ size

Choose $\Pi_1 = 0$ if $D \ll l$, i.e. if G_f is irrelevant (FPZ > D)

$\sigma_N{}^q D^v = \text{const} \Rightarrow v = 0$ ... small-size asymptote: $\sigma_N = \text{const.}$

$\Pi_2 = 0$ if $D \gg l$, i.e. if f_t' is irrelevant (FPZ = point)

$\sigma_N{}^u D^p = \text{const} \Rightarrow u = p/2$ .. large-size asymptote: $\sigma_N \propto D^{-1/2}$

Special Variant of Asymptotic Matching

Basic idea: In each asymptotic case,
all $\Pi_1, \ldots \Pi_n$ vanish except one.

To match the **second-order terms** of asymptotic expansions, take **inverse** asymptotic expansion: $D \propto \sigma_N^{-2}(1 - C_2\sigma_N^2 + \ldots)$

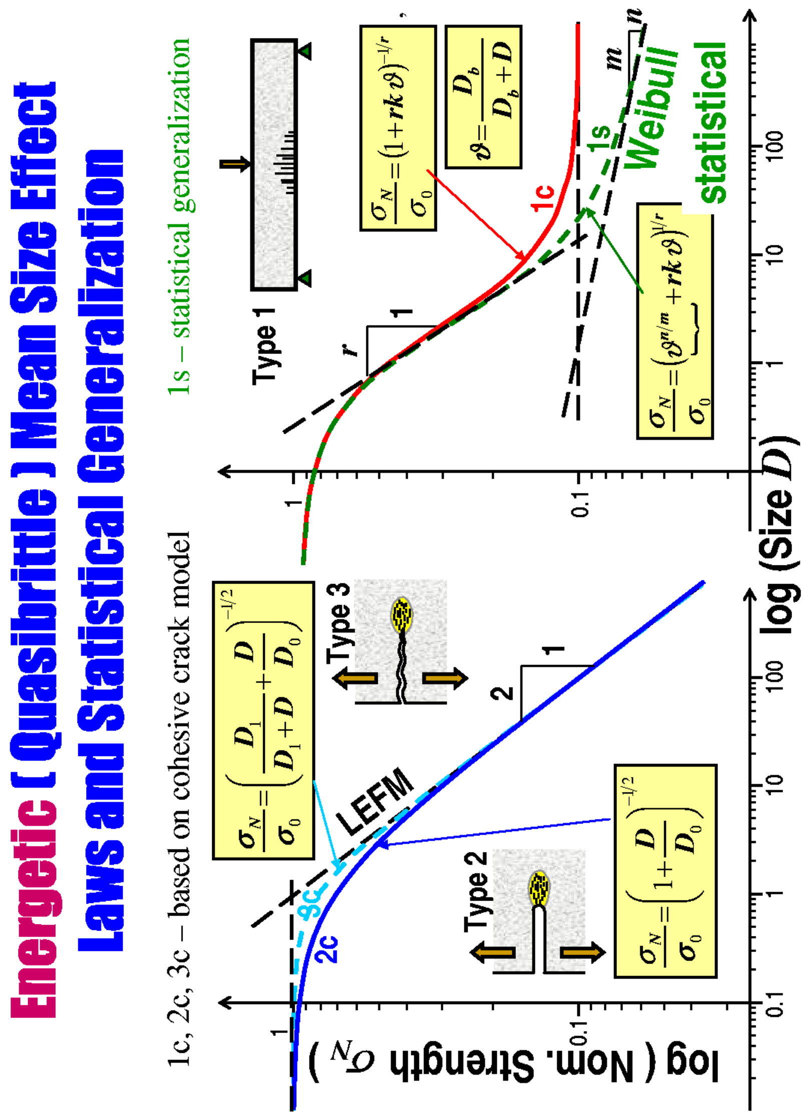
$$\Rightarrow p = q = 2 \quad \Rightarrow \quad \Pi_1 = \left(\frac{\sigma_N}{f_t'}\right)^2 \frac{D}{l}, \quad \Pi_2 = \left(\frac{\sigma_N}{f_t'}\right)^2$$

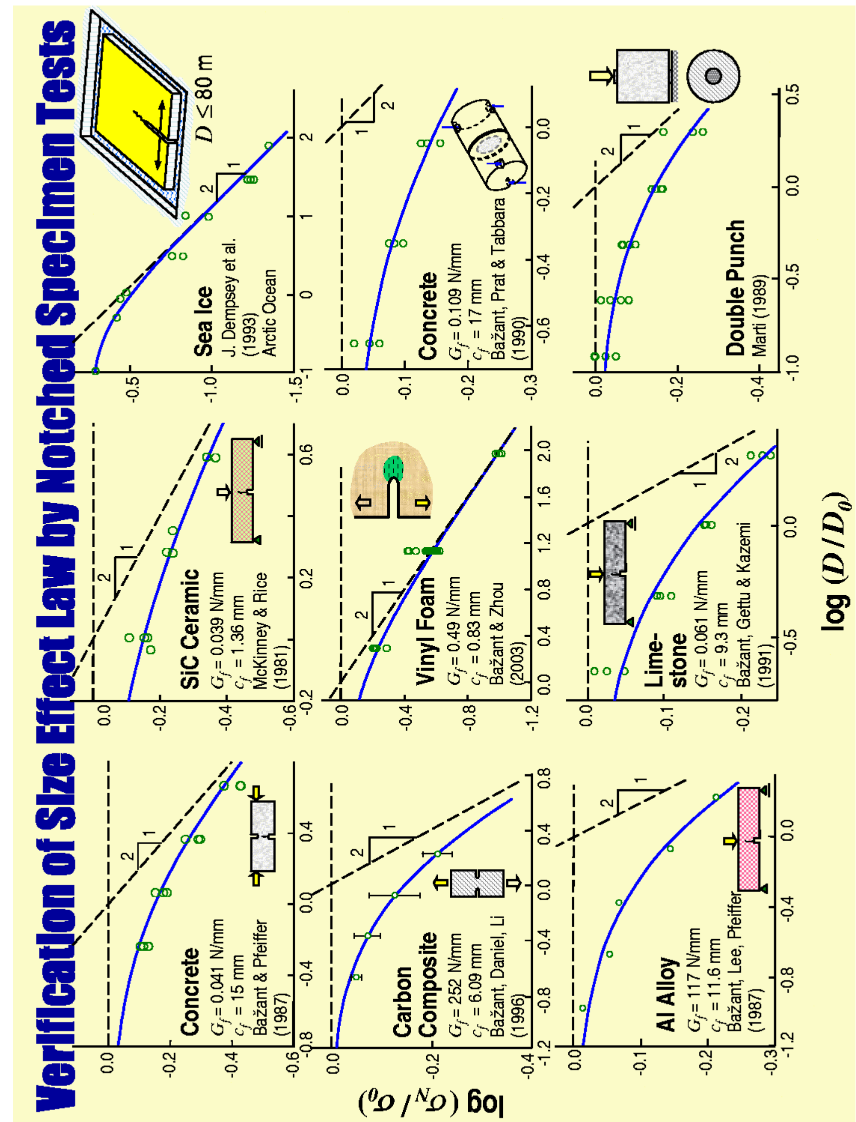
$F(\Pi_1, \Pi_2) \approx F_0 + F_1\Pi_1 + F_2\Pi_2 = 0$ (first–order approximation)

Solution:

$$\sigma_N = \sigma_0 \left(1 + \frac{D}{D_0}\right)^{-1/2}$$

where $\sigma_N = f_t'(-F_0F_2)^{-1/2}$, $D_0 = lF_2/F_1$


Energetic (Quasibrittle) Mean Size Effect
Laws and Statistical Generalization
1c, 2c, 3c – based on cohesive crack model
1s – statistical generalization
log (Nom. Strength σ_N)
log (Size D)
Type 1
Type 2
Type 3
LEFM
Weibull
statistical
$\frac{\sigma_N}{\sigma_0} = \left(1+\frac{D}{D_0}\right)^{-1/2}$
$\frac{\sigma_N}{\sigma_0} = \left(\frac{D_1}{D_1+D}+\frac{D}{D_0}\right)^{-1/2}$
$\frac{\sigma_N}{\sigma_0} = (1+rk\vartheta)^{-1/r}$
$\vartheta = \frac{D_b}{D_b+D}$
$\frac{\sigma_N}{\sigma_0} = \left(\vartheta^{n/m}+rk\vartheta\right)^{1/r}$

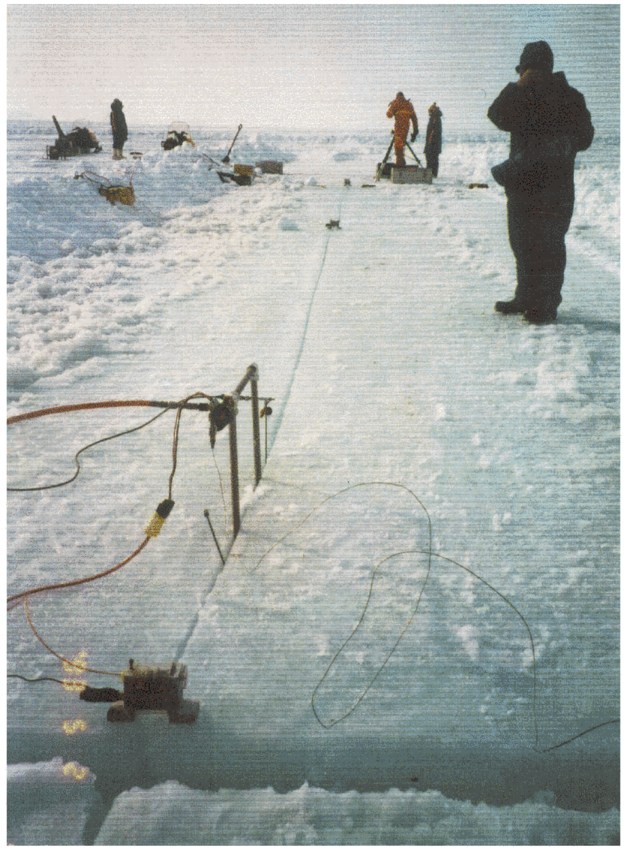

Verification of Size Effect Law by Notched Specimen Tests
Sea Ice
J. Dempsey et al. (1993) Arctic Ocean
$D \le 80$ m
SiC Ceramic
McKinney & Rice (1981)
Concrete
Bažant, Prat & Tabbara (1990)
Concrete
Bažant & Pfeiffer (1987)
Vinyl Foam
Bažant & Zhou (2003)
Carbon Composite
Bažant, Daniel, Li (1996)
Lime-stone
Bažant, Gettu & Kazemi (1991)
Double Punch
Marti (1989)
Al Alloy
Bažant, Lee, Pfeiffer (1987)
log (σ_N / σ_0)
log (D/D_0)

Scaling of Sea Ice Fracture

Arctic Ocean
1993, 1994
West of Resolute
Ice 1.8m thick

ONR Project,
Director &
Organizer:
John Dempsey

Notched Specimen :
<u>80×80 ×1.8 m</u>



2. Combination with Extreme-Value Statistics

(Type 1 Size Effect Only)

Three (and only three) possible extreme value distributions:

N statistically independent random variables X_i ($I=1,2,\ldots N$)

$P_N(y) = 1 - e^{-NP_1(y)}$ where $P_N(y) = \text{Prob}(\min_i^N X_i \leq y)$

where $P_N(y) = P_f$ =failure probability of structure.

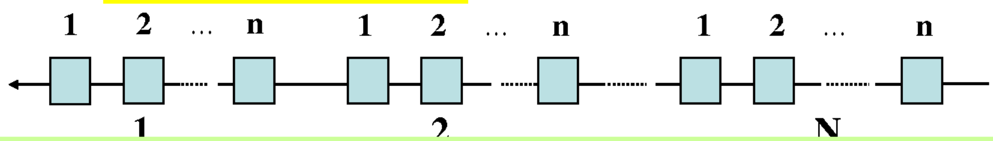
1) **Weibull distribution:** $P_N(y) = 1 - e^{-y^m}$

2) Gumbel (min.) distribution: $P_N(y) = 1 - e^{-e^y}$

3) Fréchet (min.) distribution: $P_N(y) = 1 - e^{|y|^{-m}}$

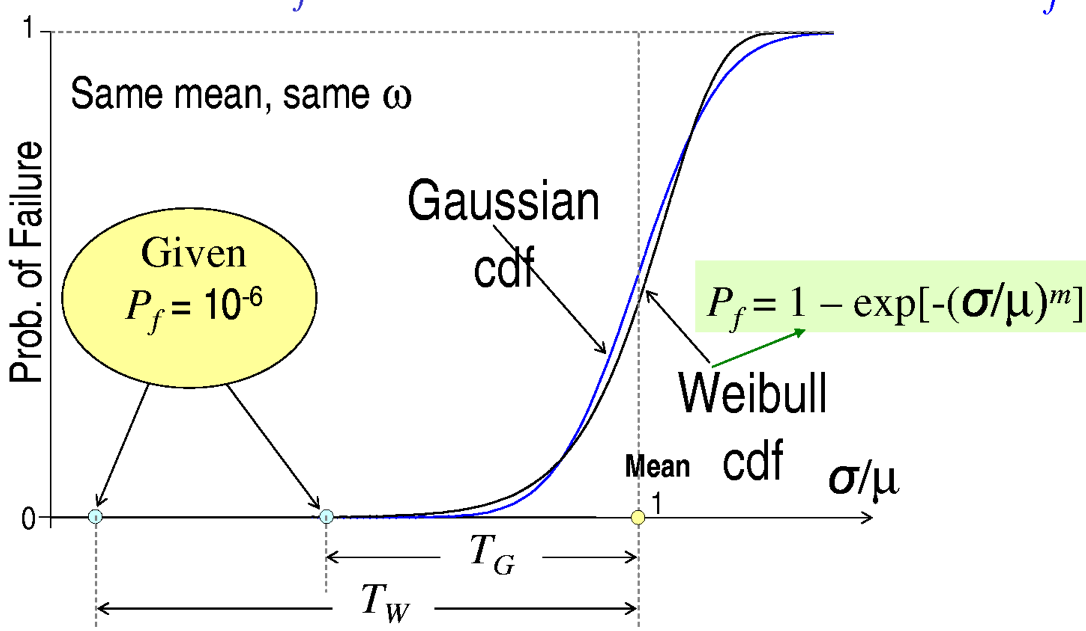
Stability Postulate:

$F^N(\sigma) = F(a_N \sigma + b_N)$ $F(\sigma) = 1 - P_f$ = survival probability



But what are the elemental pdf tails? — control convergence rate!

Tail Offset σ_f of Point of Given Failure Prob. P_f



Ratio of Tail Offsets $\theta_f = \dfrac{T_W}{T_G}$ = function of P_f and ω_N

Basis of Probabilistic Model for Quasibrittle RVE

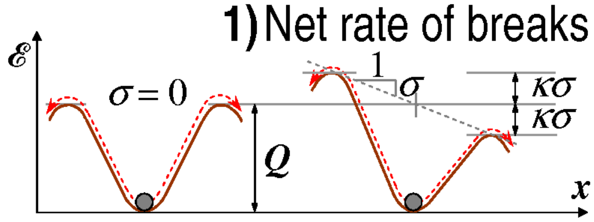
Maxwell-Boltzmann distribution

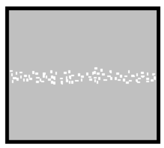
⇒ frequency of exceeding *activation energy* Q

$$f = e^{-Q/kT}$$

1) Net rate of breaks $= \underbrace{e^{-(Q-\kappa\sigma)/kT}}_{\text{breaks}} - \underbrace{e^{-(Q+\kappa\sigma)/kT}}_{\text{restorations}}$

$$= 2e^{-Q/kT}\sinh\frac{\kappa\sigma}{kT}$$



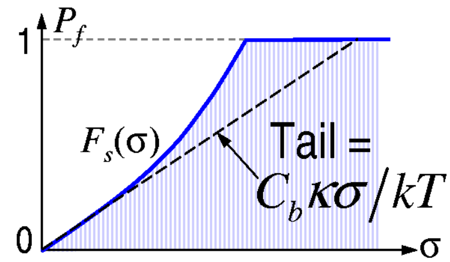


2) Critical fraction of broken bonds $\varphi_b(\tau)$ reached within stress duration τ

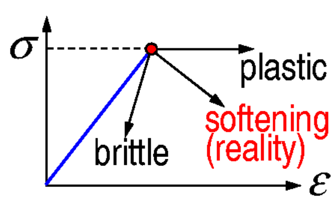
3) cdf of break surface:

$$F_s(\sigma) = \min\left(C_b \sinh\frac{\kappa\sigma_i}{kT},\ 1\right)$$

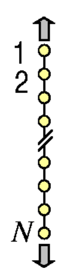
$$C_b = 2\varphi_b(\tau)e^{-Q/kT}$$



Transmission of Power Tail Exponents in Chains & Bundles – brittle or plastic

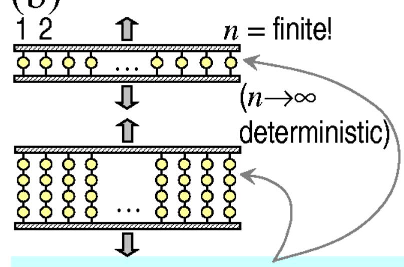


(a)



- If each link has tail x^m, the chain has the same tail x^m

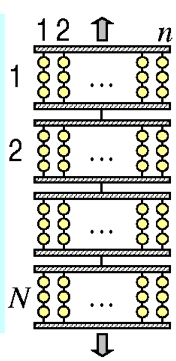
(b)



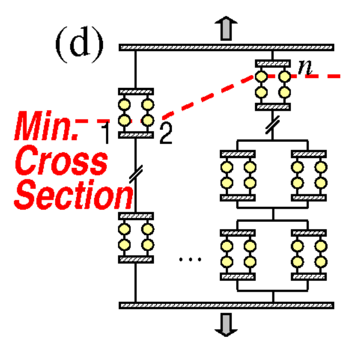
- If each fiber has tail x^p, the bundle has tail x^{np}.
(but power tail too short)

(c)

- If each link of each fiber has tail x^p, the model has tail x^{np}.



(d)



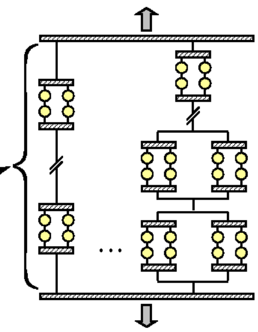
- If each link in the min. cross-section has tail x^p, the model has tail x^{np}.

Power Tail Length for Bundles & Chains

1) Brittle bundle (Daniels' model, 1945) with 24 fibers having
   Weibull cdf with $m = 1$ – Gaussian core down to 0.3
   Power tail up to 10^{-18} irrelevant (D=1000 l.y.)
2) Plastic bundle (Central Limit Theorem) with $n = 24$, fibers having:
   Weibull cdf with $m = 1$ – Gaussian core down to 0.01
   Power tail up to 10^{-15} – irrelevant
3) Brittle bundle with $n = 2$, fibers having:
   Weibull cdf with $m = 12$ – Gaussian core down to 0.3
   Power tail up to 10^{-5} – longer but not enough
4) Plastic bundle with $n = 2$, fibers having:
   Weibull cdf with $m = 12$ – Gaussian core down to 0.3
   Power tail up to 10^{-3}

$\Rightarrow$ Plastic fibers extend Weibull tail to 10^{-3} (need 10^{-2})

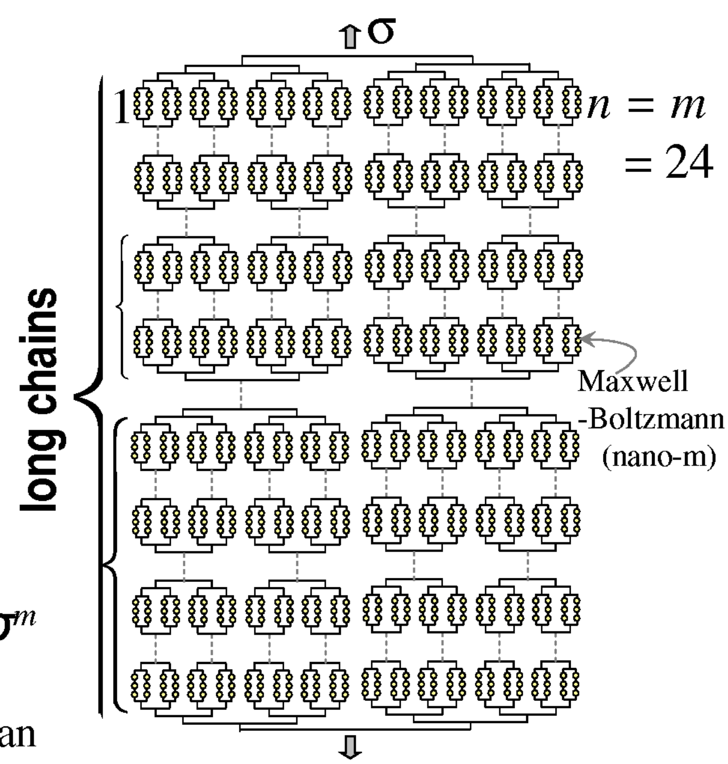
Chains tend to extend the power tail!



Conclusion: Need softening fibers and a hierarchy of parallel–series couplings.

An Acceptable Complete Model for **Extreme-Value Statitistics of a RVE** (10^{-9}—10^{-1} m)

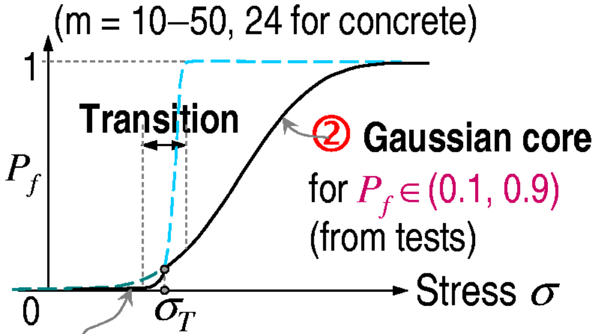
-achieves long enough power tail σ^m and short enough transition to Gaussian



3. Stochastic Analysis of Structures

Practical RVE Model: Composite Weibull Gaussian cdf

(m = 10–50, 24 for concrete)



① **Weibull tail** σ^m for $P_f \in (0, 0.01)$
or else a chain of 1000 RVE
would not be totally Weibull

Fiber bundles – plastic or brittle, with Weibull or combined Weibull–Gaussian fibers, can't meet requirements ① and ②

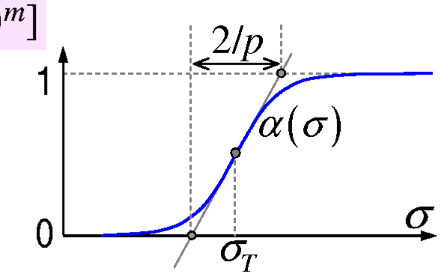
⇒ RVE model: **A hierarchy of parallel–series couplings**, with Weibull–Gaussian softening elements (between brittle and plastic).

$$P_f = 1 - (1 - P_{RVE})^N \Rightarrow 1 - \exp[-(\sigma/\mu)^m]$$

Smooth Transition:

$$\mathrm{cdf} = \left[1 - \alpha(\sigma)\right] F_{Weib} + \alpha(\sigma) F_{Gaus}$$

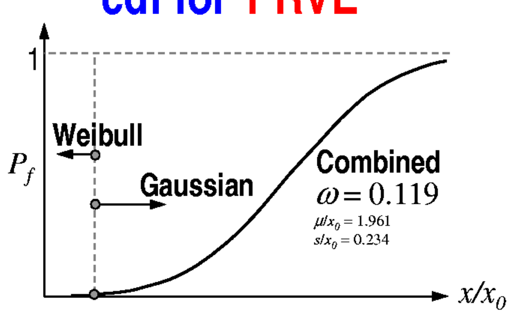
$$\alpha(\sigma) = \tfrac{1}{2}\left[1 + \tanh p(\sigma - \sigma_T)\right]$$



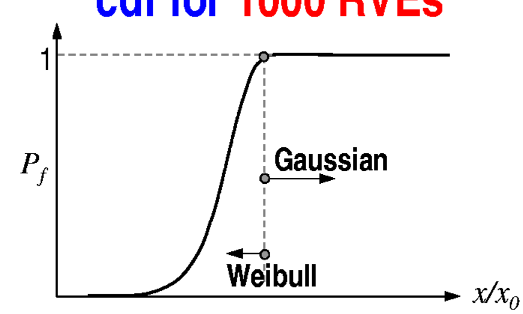
How an Apparently Gaussian Material Produces the Weibull cdf for a Structure

Assume $P_{trans} = 0.01$

cdf for 1 RVE



cdf for 1000 RVEs



Nonlocal Weibull Theory

Chain model ⇒ failure probability of structure P_f

$$-\ln\left(1-P_f\right)=\int_V \varphi\left(\sigma(\mathbf{x})\right)dV(\mathbf{x})/V, \qquad \text{where } \varphi(\sigma)=\left\langle\frac{\sigma}{s_0}\right\rangle^m$$

⇒ **Weibull Size effect:** $\sigma_N = k_v V^{-1/m} = k_0 D^{-n/m}$

Nonlocal generalization:

--- to capture stress redistribution approximately (1991):

$$-\ln\left(1-P_f\right)=k_1\int_V \varphi\left[E\bar{\varepsilon}(\mathbf{x})\right]dV(\mathbf{x})/V_r$$

$$\bar{\varepsilon}(\mathbf{x})=\varepsilon^{el}(\mathbf{x})+\overline{\varepsilon''(\mathbf{x})}$$

(local: ε^{el}; averaged: $\overline{\varepsilon''}$)

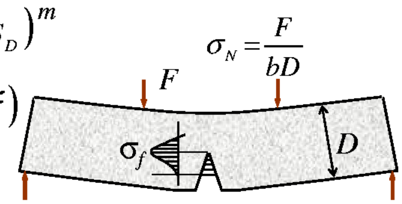
= **way to combine statistical & energetic size effects**

$\bar{\varepsilon}$ = weighted **volume average of strain** over a repres. mat. element.

Derivation of Size Effect on Entire pdf from Nonlocal Weibull Theory

$$-\ln(1-P_f)=\int_V\left\langle\hat{\sigma}(x)/s_0\right\rangle^m dV(x)/V_r=\left(\sigma_N/s_D\right)^m$$

$$s_D^m=\left(s_0/H\right)^m\left(l/d\right)^{n_d} \qquad H^m=\int_v\left\langle\hat{S}(\xi)\right\rangle^m dv(\xi)$$



$\hat{S}$ = nonlocal stress $\qquad \sigma_N\left(P_f\right)=\bar{\sigma}_N(D)\Phi$

Mean: $\bar{\sigma}_N(D)=\frac{s_\infty}{H_\infty}\vartheta^{n_d/m}\left(1+n\,r\,c_m\vartheta\right)^{1/r}$ $\qquad \vartheta=\frac{l}{\eta l+D}$

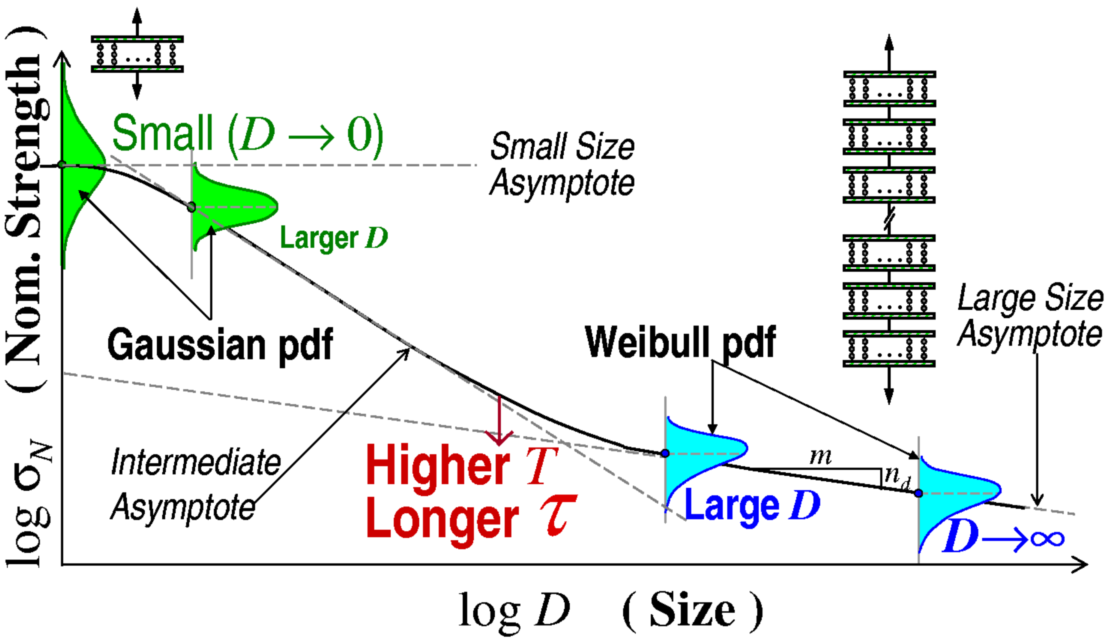
Large size: Weakest Link Model → Weibull size effect

Generally $H\approx H_0\left[1-c_m(l/D)\right]$ error ~ l^2/D^2 ...mid-point estimate

⇒inverse Weibull cdf (up to 2nd order in *l*/*D*): $\Phi=\Phi_{Weibull}\left(P_f\right)=\left[-\ln\left(1-P_f\right)\right]^{1/m}$

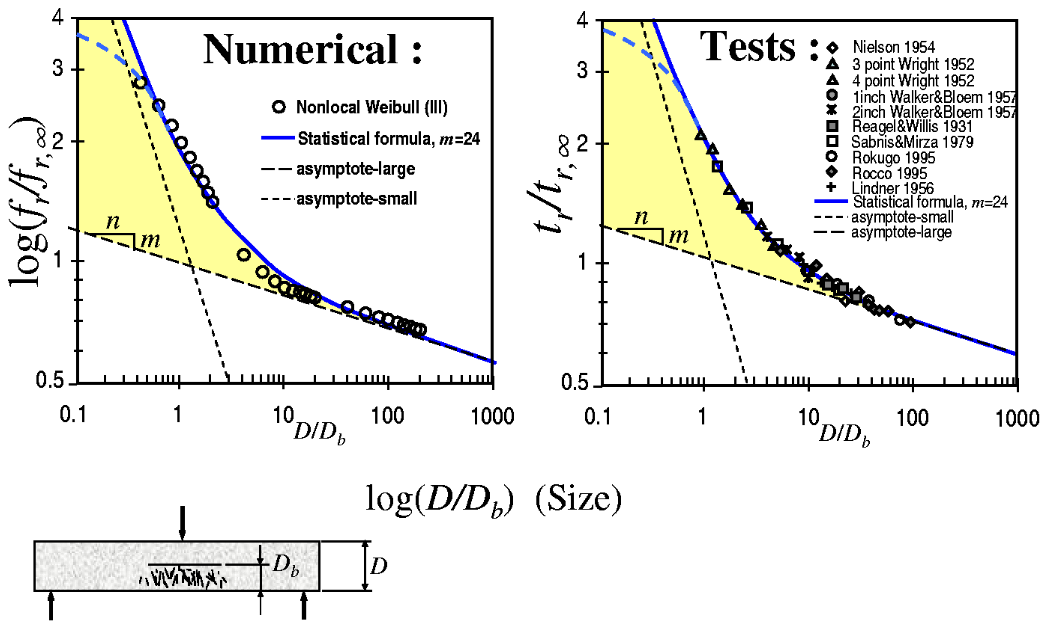
Mean $\bar{\sigma}_N = s_D\Gamma(1+1/m)$, C.o.V = $\sqrt{\frac{\Gamma(1+2/m)}{\Gamma^2(1+1/m)}-1}$

Mean Size Effect Curve for Failure at Macroscopic Crack Initiation

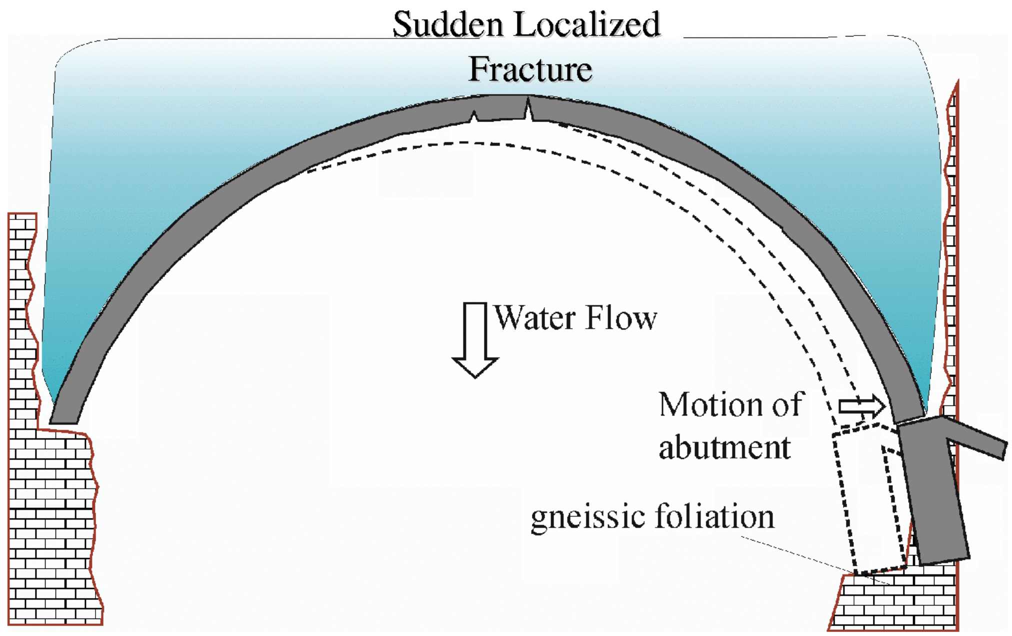


Corroboration by Nonlocal Weibull Theory

Optimum Fit of Existing Test Data

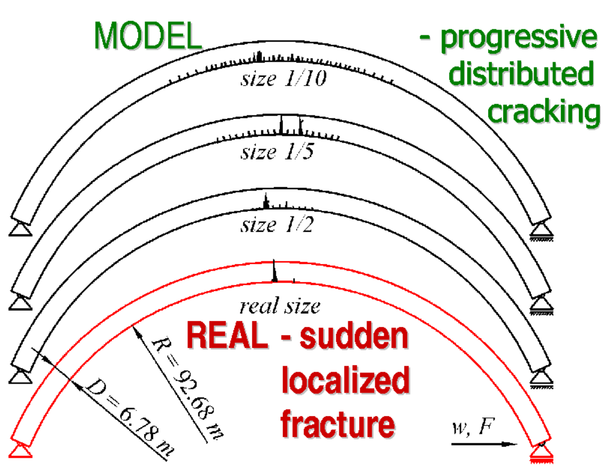


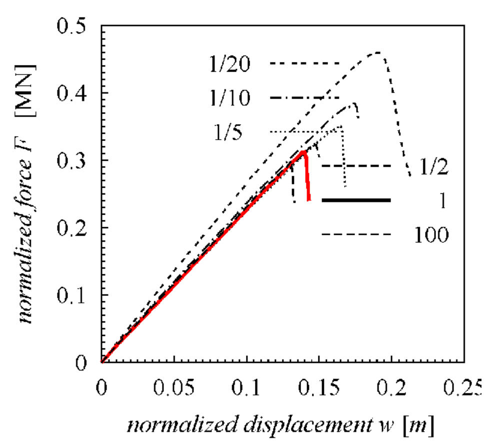
Cause of Failure of Malpasset Dam



Tolerable movement of abutment would today be **8 × smaller !**

Deterministic Computations by ATENA with Microplane Model for Scaled Dams of Various Sizes





Bažant, Vořechovský and Novák, 2004

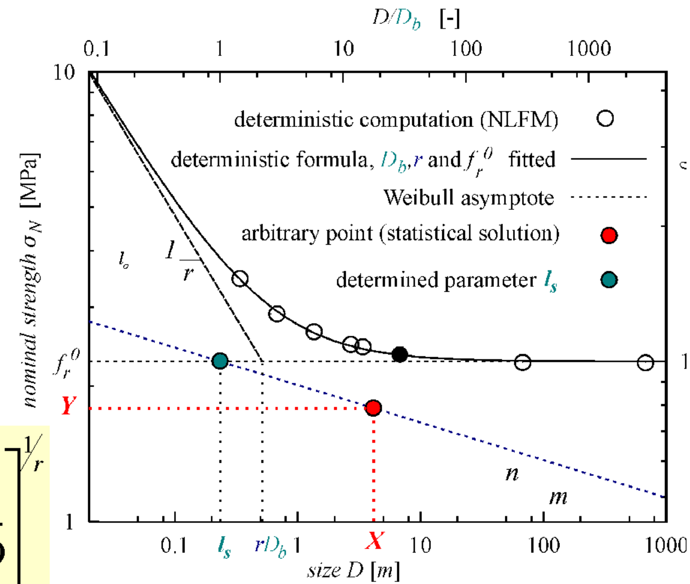
Predicting Energ.-Probabilistic Scaling without Nonlocal Stochastic Simulations

- Fit of deterministic computations for at least 3 sizes gives r, D_b, f_r^0
- One evaluation of Weibull probability integral gives l_s

… Asymptotic matching formula fixed:

$$\sigma_N = f_r^0 \left[\left(\frac{l_s}{l_s + D} \right)^{m/m} + \frac{rD_b}{l_o + D} \right]^{1/r}$$

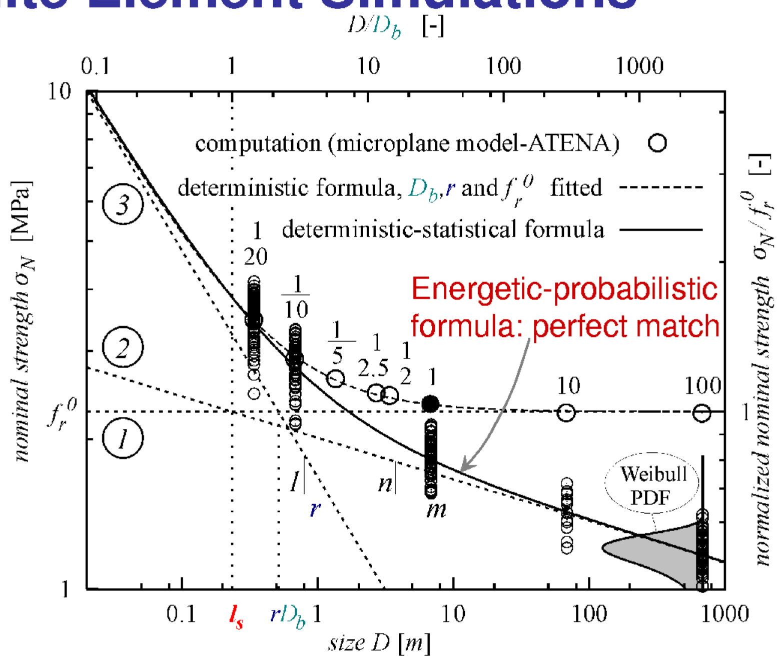
(l_0 neglected)



Bažant, Vořechovský and Novák, 2004

Verification by Energetic-Probabilistic Finite Element Simulations

$D_b = l_s$ assumed



Bažant, Vořechovský and Novák, 2004

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Closure

- **Asymptotic matching for both mean size effect and tails**
- **Power tails: Maxwell-Boltzmann**
- **Structural Analysis: Nonlocal Weibull**