Convection

- Driving forces and resistance
- Governing equations
- Convection modeling
- Phase changes in convection
- Applications:
  - Subduction zones
  - plumes

Driving forces and resistance

- Buoyancy force due to horizontal density gradients

$$\mathbf{f} = \rho(T, C, \Gamma)\mathbf{g}$$

- Thermal expansion

$$\rho(T) = \rho_0[1 - \alpha(T - T_0)]$$

- Resisting motion: viscous forces (and diffusivity)

$$\sigma_{ij} = 2\eta\dot{\varepsilon}_{ij}$$

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right]$$

Ra: a most dangerous number

- Non-dimensional ratio of driving vs braking:

$$Ra = \frac{\rho_0 \alpha g \Delta T h^3}{\kappa\eta}$$

Basic equations (simplified)

Conservation of mass leads to incompressibility constraint

$$\nabla \cdot \mathbf{u} = 0$$

Conservation of momentum for very viscous fluid leads to Stokes equations

$$\nabla P = \nabla \cdot \sigma + \rho \mathbf{g}$$

Conservation of energy leads to the heat advection-diffusion equation

$$\rho c_p \left[\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T \right] = \nabla \cdot (k \nabla T) + Q$$

- Non-dimensionalization

$$x' = x/h \; ; \; t' = t \frac{\kappa}{h^2} \; ; \; etc.$$

- And assumption of constant properties leads to

$$\nabla P = \nabla^2 \mathbf{u} - Ra T \hat{\mathbf{z}}$$

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \nabla^2 T + Q$$

Non-dimensional heat flow

$$\mathrm{Nu} = \frac{q_{conv} + q_{cond}}{q_{cond}}$$

Characteristic non-dimensional values $Ra = 10^7$, $Nu = 20$, $V_{surf} = 5000$.

- Linear stability analysis
  - Derive whether harmonic perturbations grow or decay
  - Find critical Rayleigh number and growth rates

- Boundary layer analysis

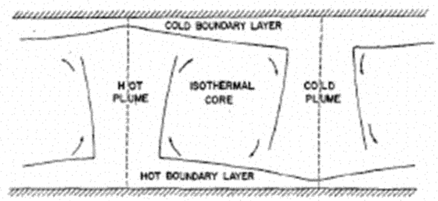


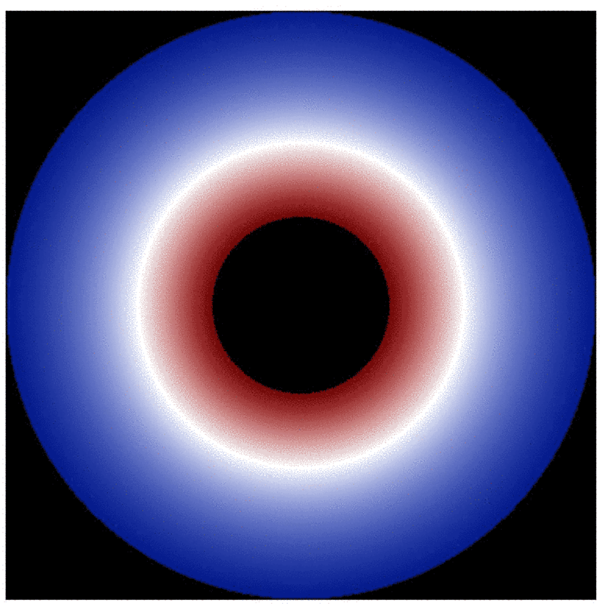
FIGURE 7. Illustration of the boundary-layer model for two-dimensional thermal convection.

$$\mathrm{Nu} = c_1 Ra^{1/3} \quad ; \qquad V_{surf} = c_2 Ra^{2/3}$$

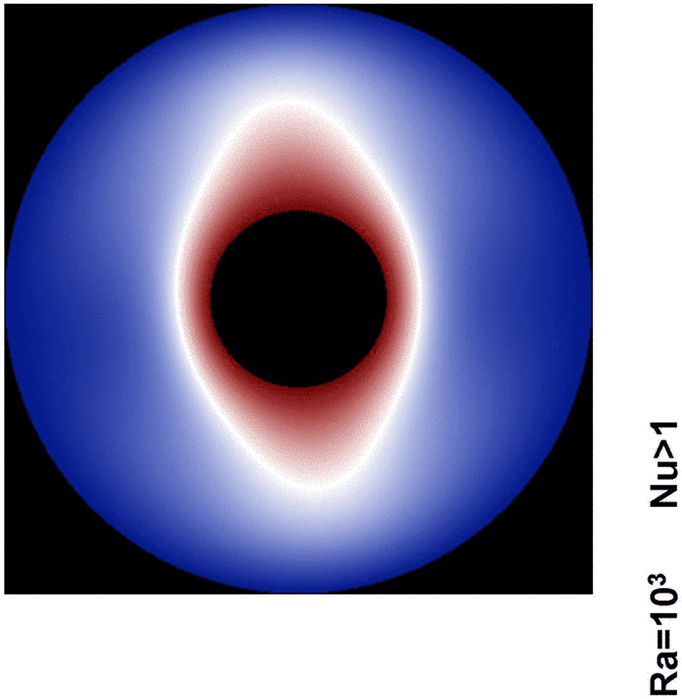
See, e.g., Turcotte and Schubert's Geodynamics

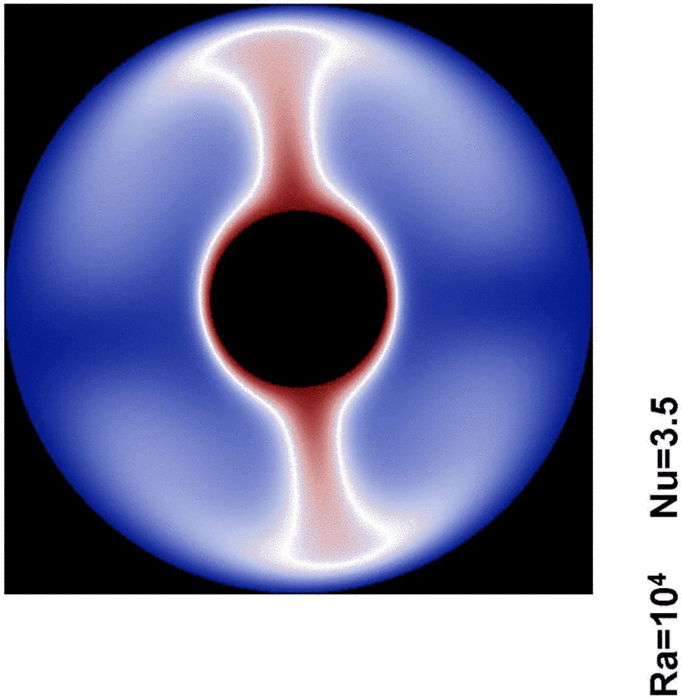
- Numerical solution

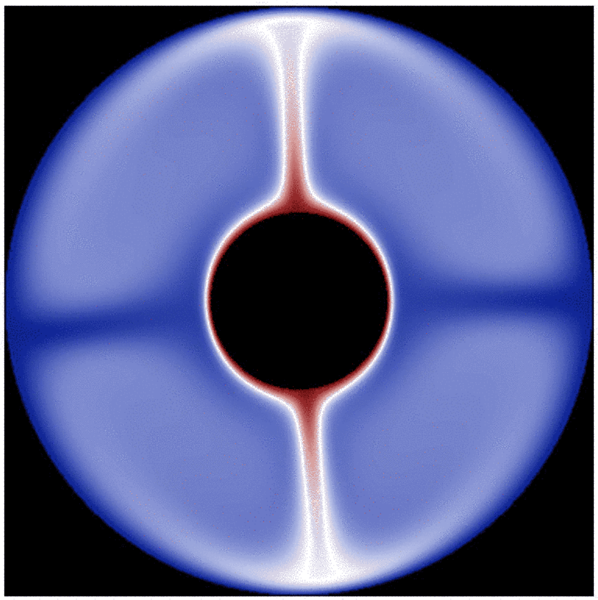
Example: Boussinesq convection, isoviscous, no internal heating



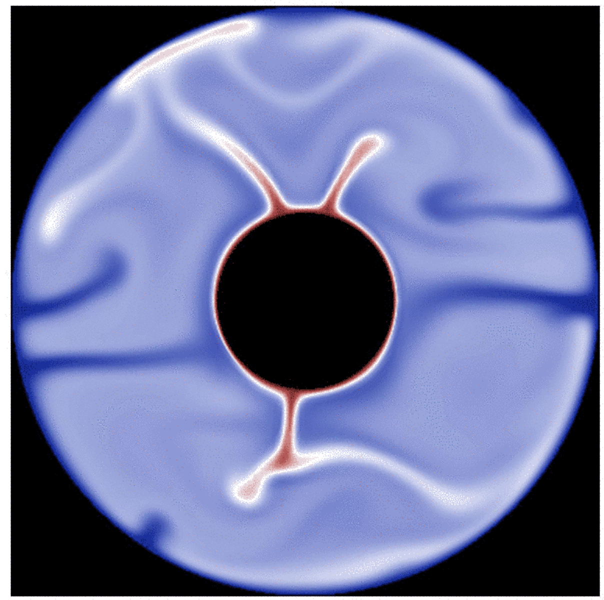
Ra=1 Nu=1


Ra=10³ Nu>1

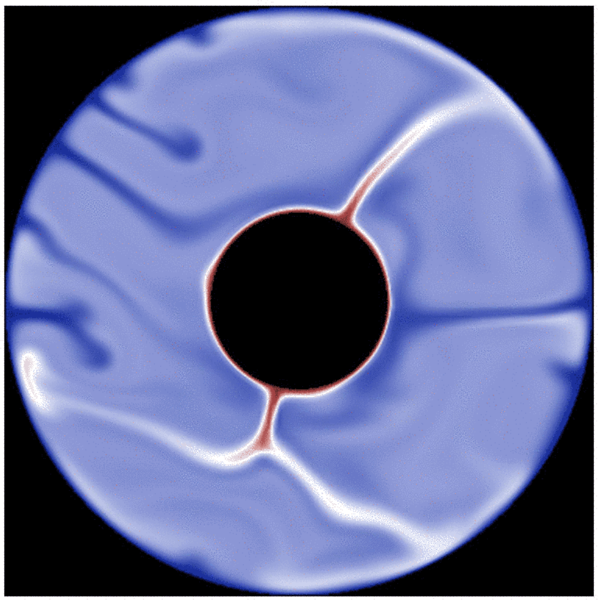

Ra=10⁴ Nu=3.5



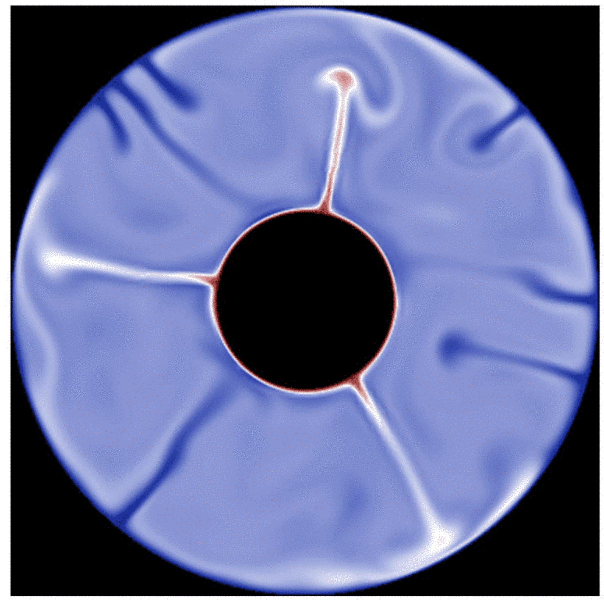
$Ra=10^5$ $Nu=8$



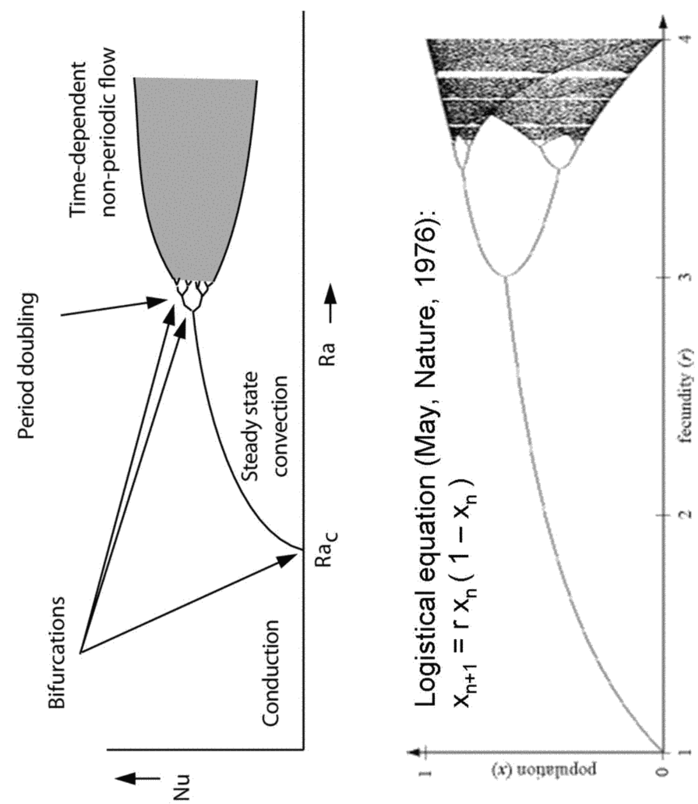
$Ra=10^6$ $<Nu>=16$



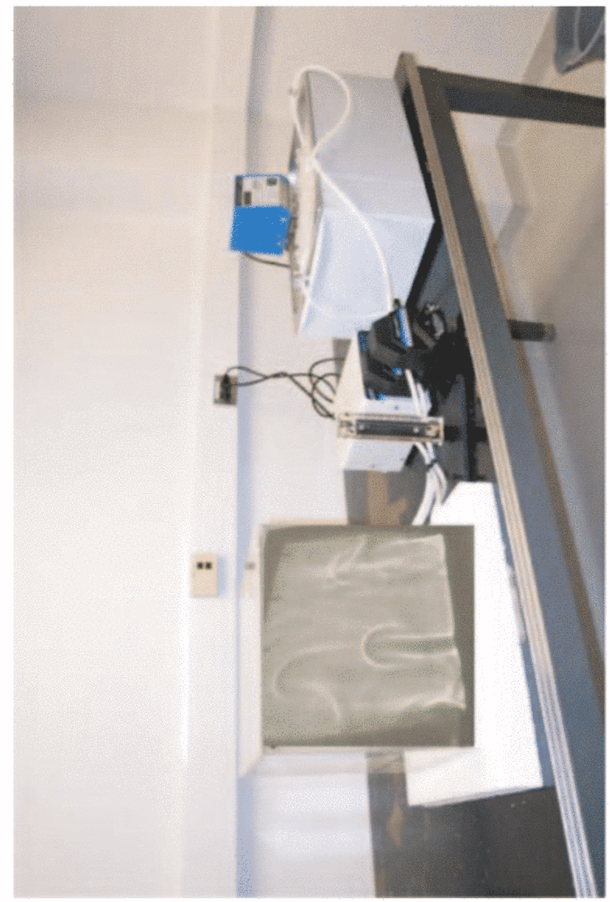
$Ra=2x10^6$ <Nu>=18



$Ra=5x10^6$ <Nu>=20



Laboratory modeling



Approximations ….

- Classical Boussinesq approximation
  - Constant density, except in buoyancy term
- Lots of missing physics
  - Compressibility (adiabatic heating/cooling)
  - Viscous dissipation
  - Phase transitions
  - Variable thermodynamic properties
- Compressible convection
  - Navier-Stokes equations for very viscous fluids
  - Jarvis and Peltier, in Peltier's orange mantle convection book
  - Also Schubert, Turcotte and Olson, Mantle convection in the …; Jarvis&McKenzie, JFM, 1980, or www.geo.lsa.umich.edu/~keken/benchmarks/compr

New and improved physics

- Viscous dissipation

$$\phi = \tfrac{1}{2}\,\tau : \dot{\varepsilon} = \tau_{ij}\frac{\partial u_i}{\partial x_j}$$

- Reference state and deviations thereof

$$\rho = \bar{\rho}(\bar{T}, \bar{p}) + \rho'$$
$$T = \bar{T} + T'$$
$$P = \bar{p} + p' \qquad \nabla \bar{p} = \bar{\rho}\bar{\mathbf{g}}$$

- Dissipation number

$$Di = \frac{\alpha_r g_r h}{c_{p_r}}$$

The anelastic liquid formulation

$$\nabla \cdot \bar{\rho}\mathbf{u} = 0$$

$$0 = -\nabla p' + Di\,\frac{\bar{\rho} c_{p_r}}{\bar{K}_S \gamma_r c_{v_r}}\,\mathbf{g} p' - \mathbf{g}\bar{\rho}\bar{\alpha} Ra T' + \nabla \cdot \tau$$

$$\bar{\rho}\bar{c}_p \frac{D}{Dt}(\bar{T} + T') - Di\bar{\alpha}(\bar{T} + T' + T_0)\bar{\rho}\mathbf{u} \cdot \mathbf{g} =$$

$$\nabla \cdot (k\nabla(\bar{T} + T')) + \phi \frac{Di}{Ra} + \bar{\rho} H$$

Truncated Anelastic Liquid Approximation (TALA)

- Ignore pressure effect in buoyancy (probably ok for Di<1)
- Jarvis & McKenzie, 1980:
  - assumption of constant properties except density and 2D Cartesian

$$\bar{\rho} = \exp \frac{Diz}{\gamma_r}$$

$$\nabla \cdot \bar{\rho}\mathbf{u} = 0$$

$$0 = -\nabla p' - \bar{\rho} Ra T \hat{z} + \nabla \cdot \tau$$

$$\bar{\rho}\frac{DT}{Dt} + Di\bar{\rho} w (T + T_0) = \nabla^2 T + \frac{Di}{Ra}\phi + \nabla^2 \bar{T}$$

Extended Boussinesq Approximation

- Assume reference density = 1

$$\nabla \cdot \mathbf{u} = 0$$

$$0 = -\nabla p - RaT\hat{z} + \nabla \cdot \tau$$

$$\frac{DT}{Dt} + Diw(T + T_0) = \nabla \cdot (k\nabla T) + \frac{Di}{Ra}\phi + H$$

Boussinesq Approximation

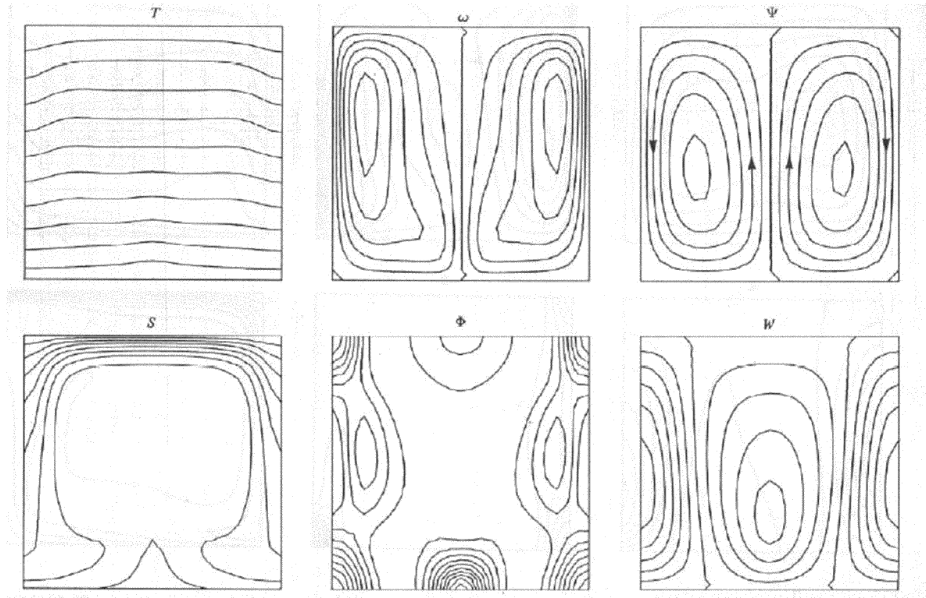
- Back to square one: Assume Di=0.

$$\nabla \cdot \mathbf{u} = 0$$

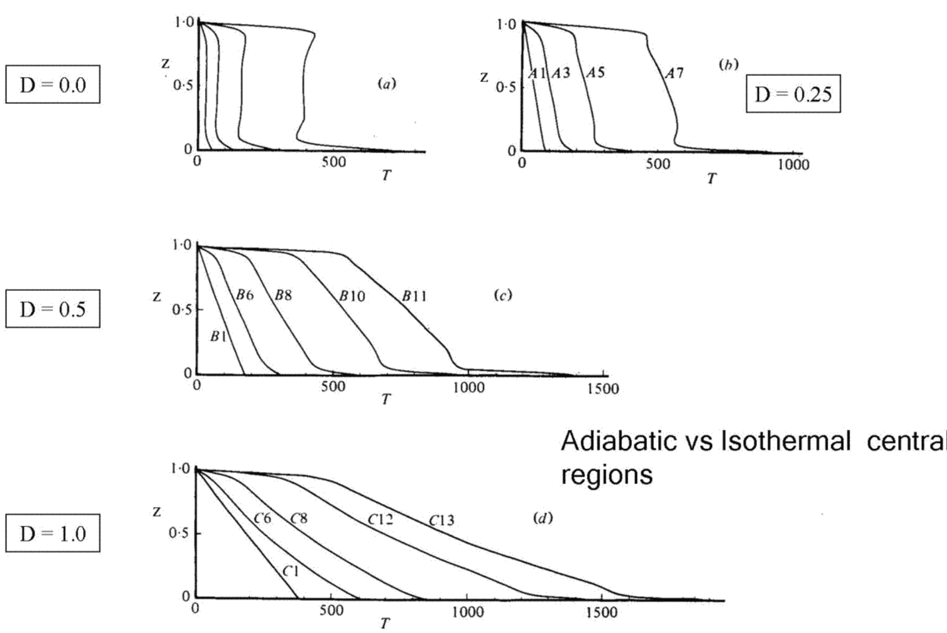
$$0 = -\nabla p - RaT\hat{z} + \nabla \cdot \tau$$

$$\frac{DT}{Dt} = \nabla \cdot (k\nabla T) + H$$

Example of TALA (JM80)

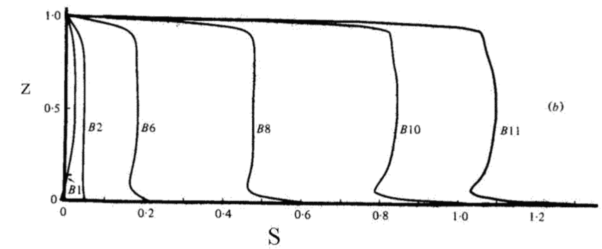


Temperature Profiles

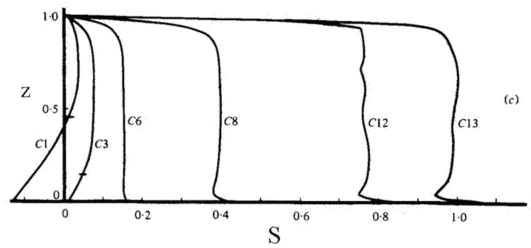


Entropy Profiles

D = 0.5

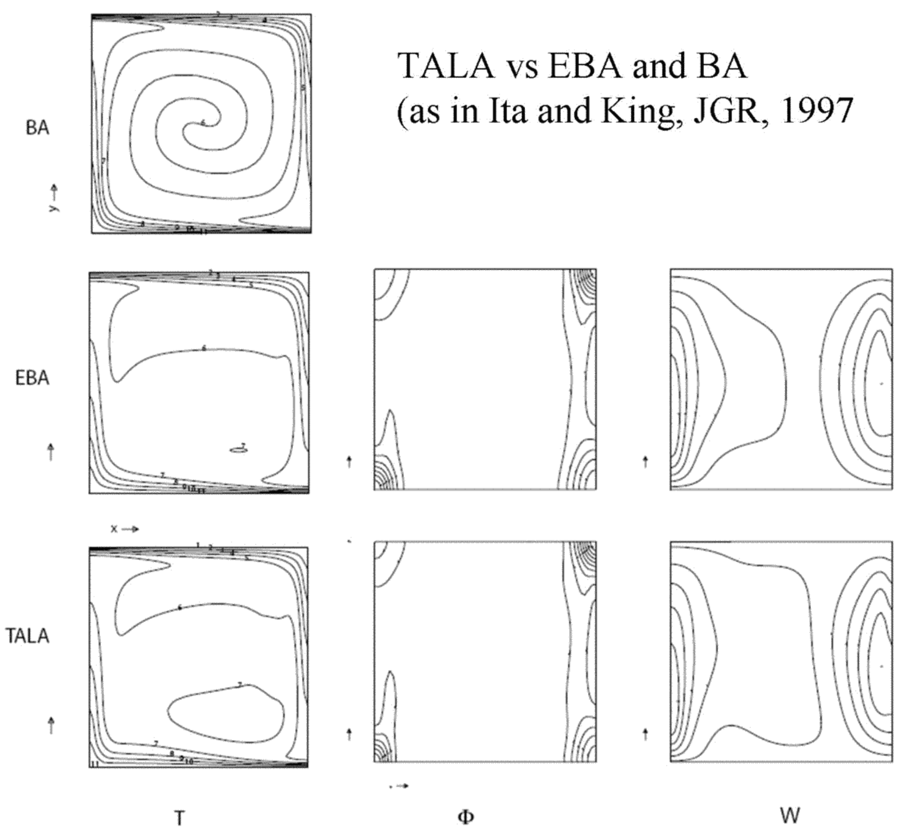


D = 1.0



Note: Isentropic central regions

TALA vs EBA and BA
(as in Ita and King, JGR, 1997



Some thoughts on numerical methods: FD

- Finite difference discretization

$$\frac{du}{dx} \approx \frac{u_i - u_{i-1}}{\Delta x} + O((\Delta x)^2)$$

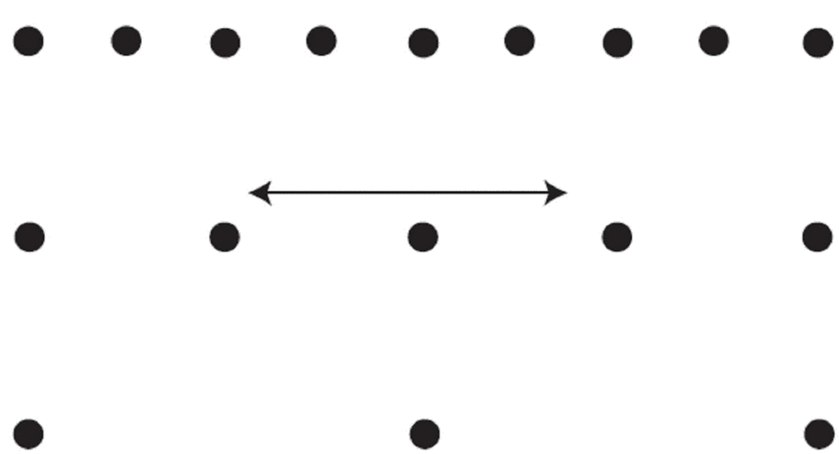
$$\frac{d^2u}{dx^2} \approx \frac{u_{i-1} - 2u_i + u_{i+1}}{(\Delta x)^2} + O((\Delta x)^3)$$

Differential form

$$-\frac{d^2u}{dx^2} = 1$$

Difference form

$$u_i = \frac{1}{2}\left[u_{i-1} + u_{i+1} + (\Delta x)^2\right]$$



Jacobi/Gauss-Seidel/successive overrelaxation:
successive sweeps to convergence
Computational cost $O(N^2)$ to $O(N^3)$

Solution: multigrid

- Relax long wavelength errors/residual on coarse grid

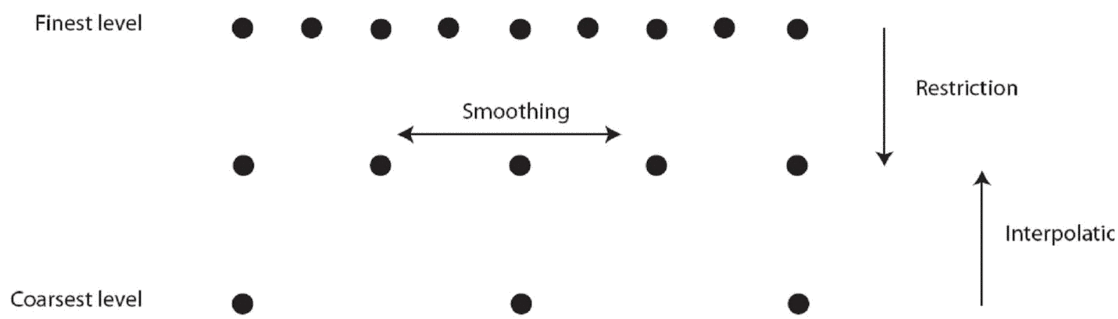


Figure 1. Geometry for 1D multigrid

- Computational work: O(N)
- Suited for FE, FD, FV discretization
- Caveat: requires fairly regular grids

Finite element & Galerkin approach

General 1D PDE

$$Lu = f \quad on\ \Omega$$

Pose approximate solution

$$\tilde{u}(x) = \sum_{j=1}^{N} u_j \phi_j(x)$$

Minimize residual in space spanned by shape functions

$$R(\tilde{u}) = L\tilde{u} - f \quad ; \quad (R, \phi_i) = 0 \quad i = 1 \ldots N$$

Example:

$$\frac{d^2u}{dx^2} = 1 \quad on\ \Omega = [0,1]$$

$$\int_0^1 \left(\frac{d^2\tilde{u}}{dx^2} - 1 \right) \phi_i dx = 0 \qquad i = 1, 2, \cdots, N$$

$$\frac{d\tilde{u}}{dx}(0)\phi_i(0) - \frac{d\tilde{u}}{dx}(1)\phi_i(1) - \int_0^1 \frac{d\tilde{u}}{dx}\frac{d\phi_i}{dx}dx = \int_0^1 \phi_i dx \quad i = 1, 2, \cdots, N$$

$$\sum_{j=2}^{N} u_j \int_0^1 \frac{d\phi_j}{dx}\frac{d\phi_i}{dx}dx = -\int_0^1 \phi_i dx \quad i = 2, \cdots, N$$

Benchmarks

- Blankenbach et al., GJI, 1989
  - 2D Cartesian thermal; internal heating, (T,z)-dependent viscosity
- Busse et al., GAFD, 1994
  - Low Ra 3D box convection
- Van Keken et al., JGR, 1997
  - 2D thermochemical convection
- Subduction zone benchmark (nearly ready for submission)
- Compressible convection (just started: CIG)

Computational Infrastructure for Geodynamics

- CIG: www.geodynamics.org
- Guest lecture by Eh Tan on Monday
- convection & seismological modeling
  - citcom
  - specfem

Phase change implementation

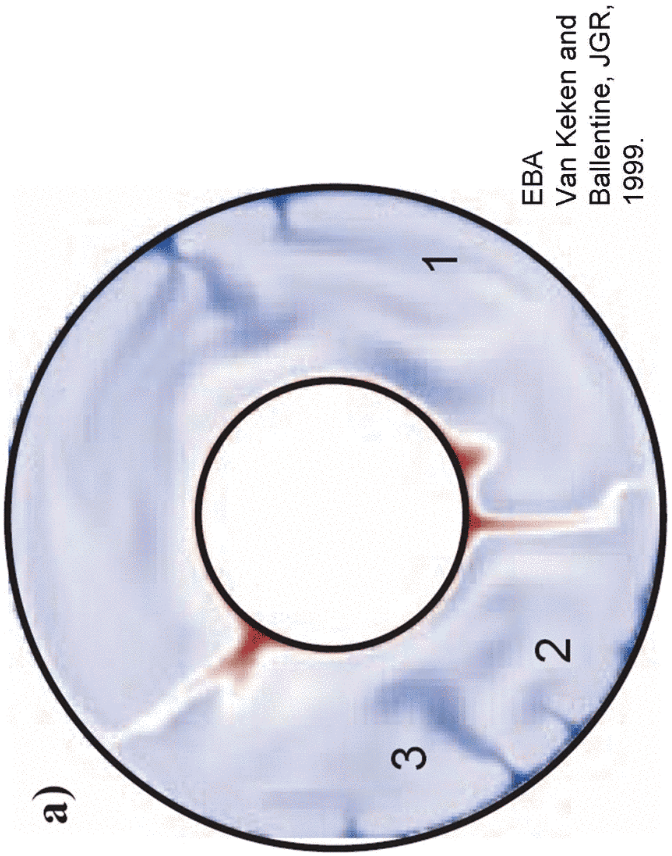
- Following Christensen and Yuen, 1985 (see also Nakagawa & Tackley, PEPI, 2004).

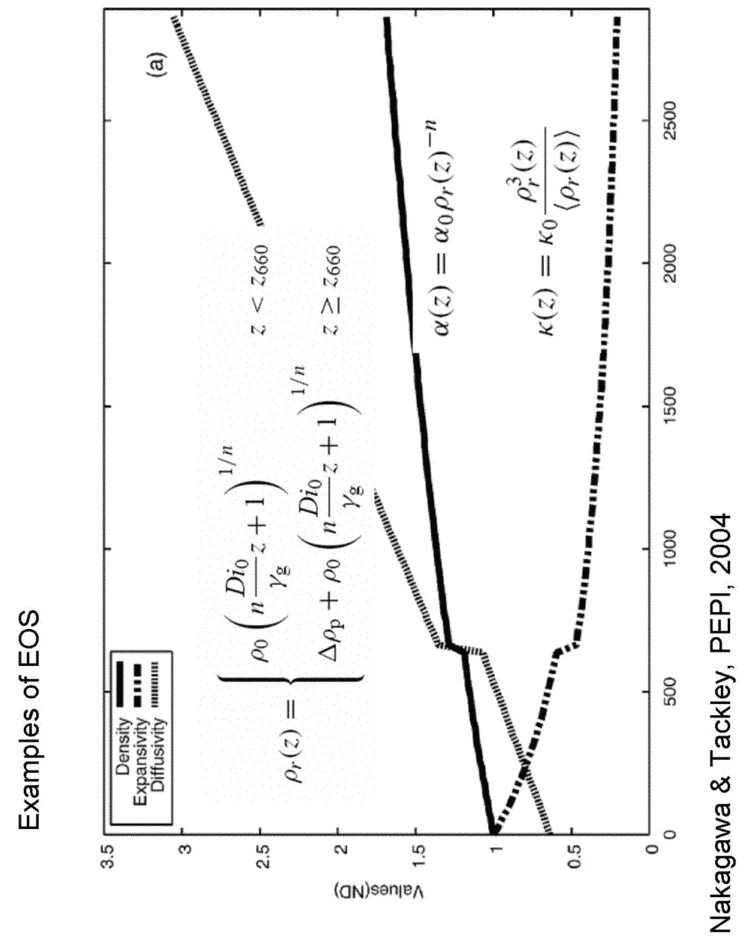
$$\Gamma = \frac{1}{2}\left[1 + \tanh\left(\frac{\pi}{d_{\mathrm{ph}}}\right)\right]$$

$$\pi = r_{\mathrm{ph}} - r - \gamma_{\mathrm{c}} T$$

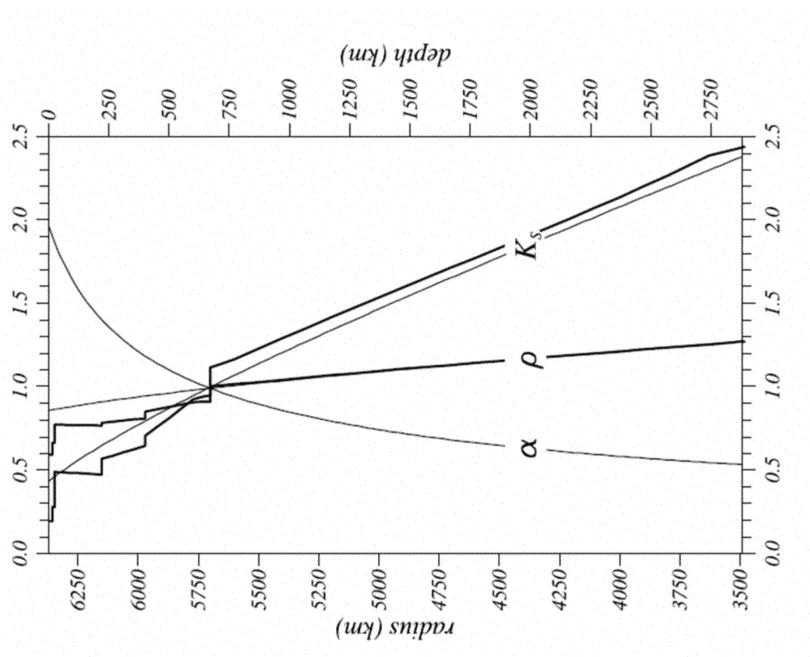
- Alpha and cp become locally influenced by phase change
- Extra buoyancy term reflects density increase at phase change

$$P = \gamma_{\mathrm{c}} \frac{\Delta\rho_{\mathrm{p}}}{\rho_0 \alpha_0 \, \Delta T} = \gamma_{\mathrm{c}} G$$


a)
1
2
3
EBA
Van Keken and Ballentine, JGR, 1999.

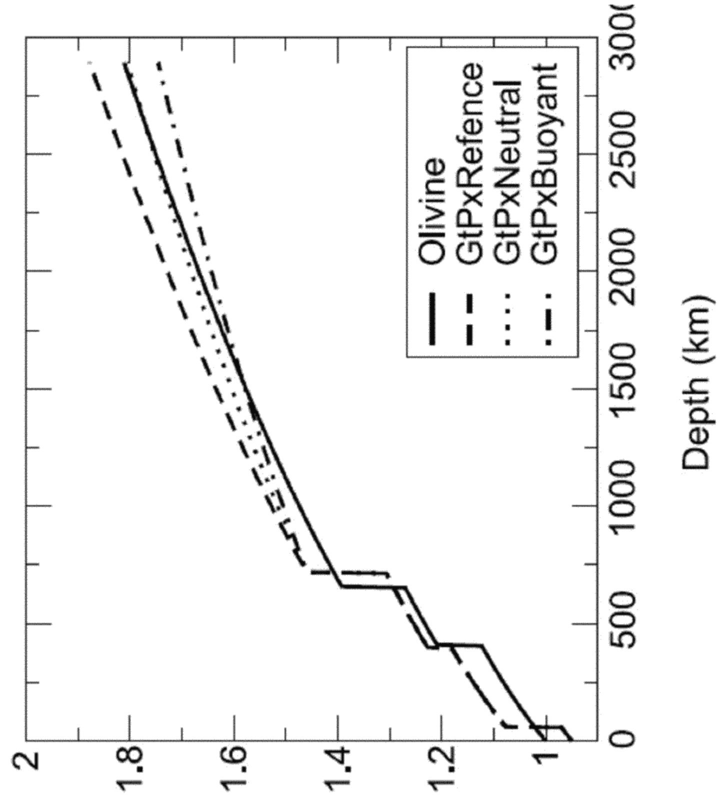

Examples of EOS
(a)
Density
Expansivity
Diffusivity
$\rho_r(z) = \begin{cases} \rho_0 \left(n \frac{Di_0}{\gamma_g} z + 1 \right)^{1/n} & z < z_{660} \\ \Delta\rho_p + \rho_0 \left(n \frac{Di_0}{\gamma_g} z + 1 \right)^{1/n} & z \geq z_{660} \end{cases}$
$\alpha(z) = \alpha_0 \rho_r(z)^{-n}$
$\kappa(z) = \kappa_0 \frac{\rho_r^3(z)}{\langle \rho_r(z) \rangle}$
Values(ND)
Nakagawa & Tackley, PEPI, 2004

Examples of EOS

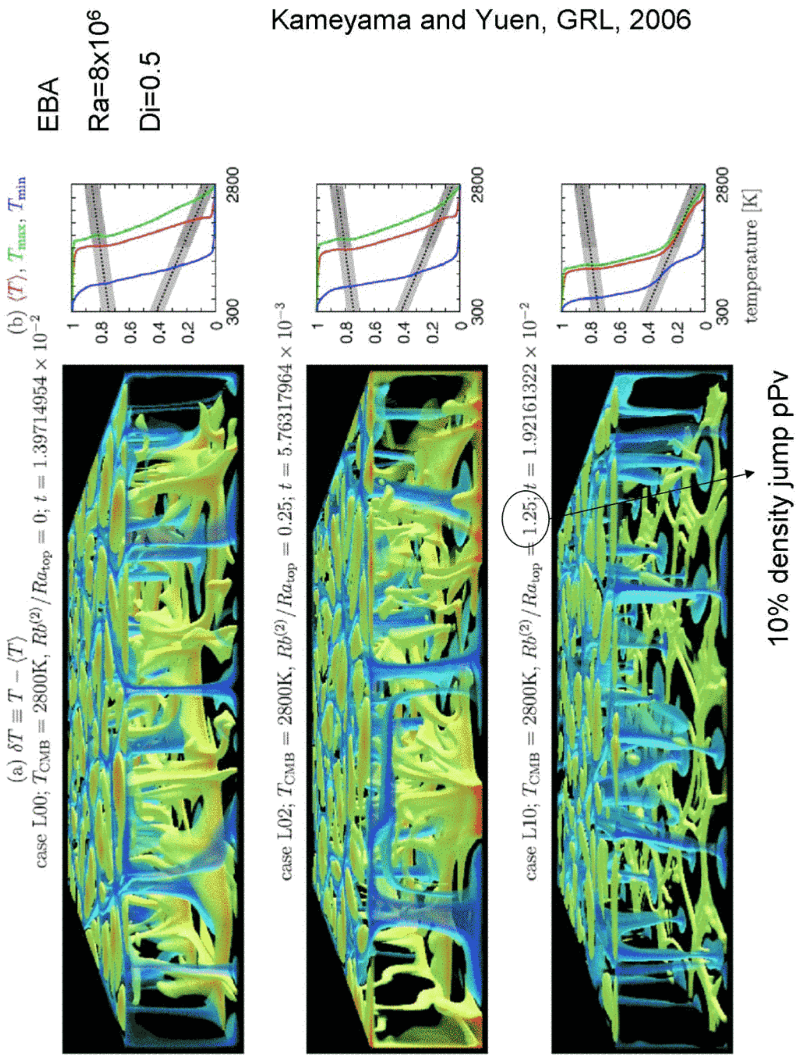


Monnereau & Yuen, EPSL, 2000

EOS:
Ol + GtPx

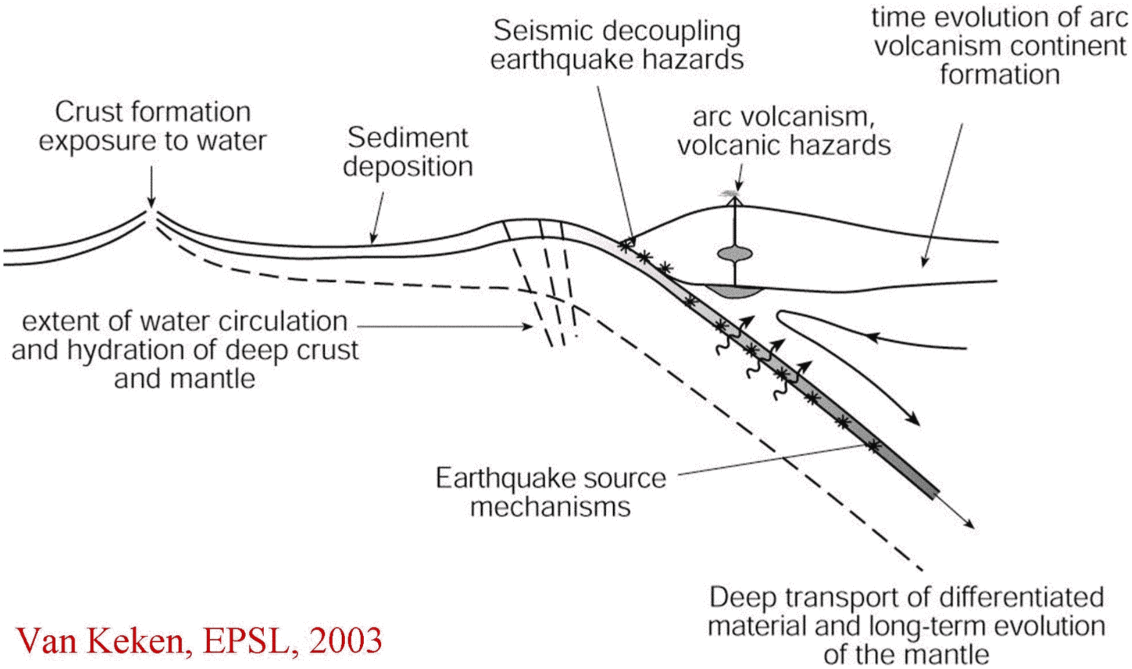


Xie and Tackley, JGR, 2004



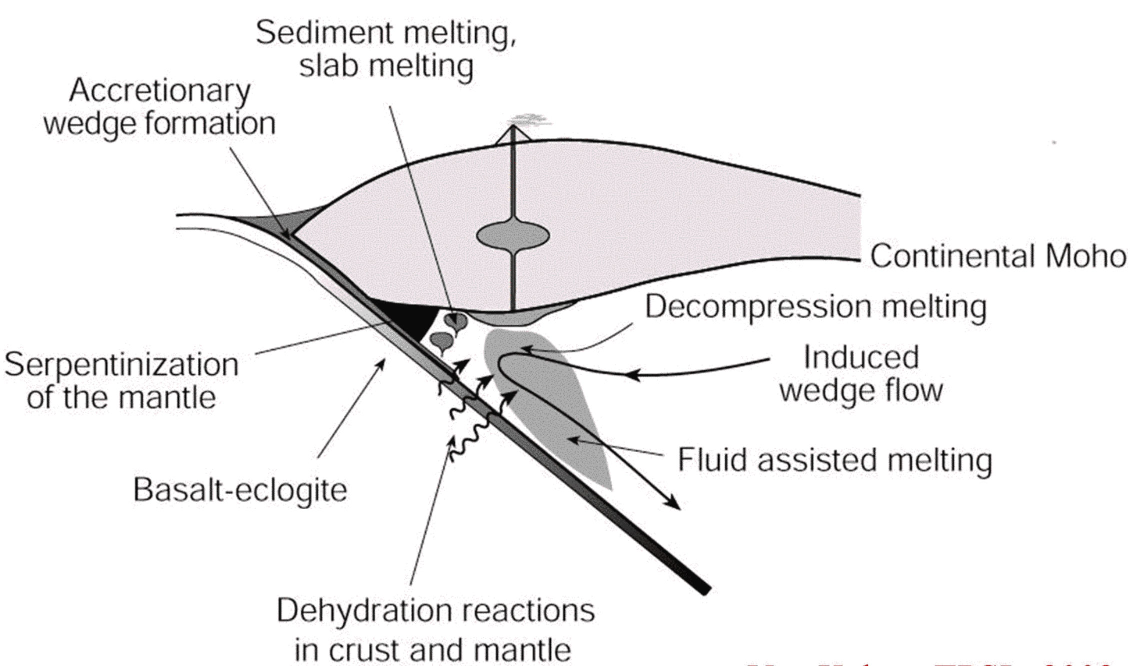
Subduction Zones

Subduction zone processes

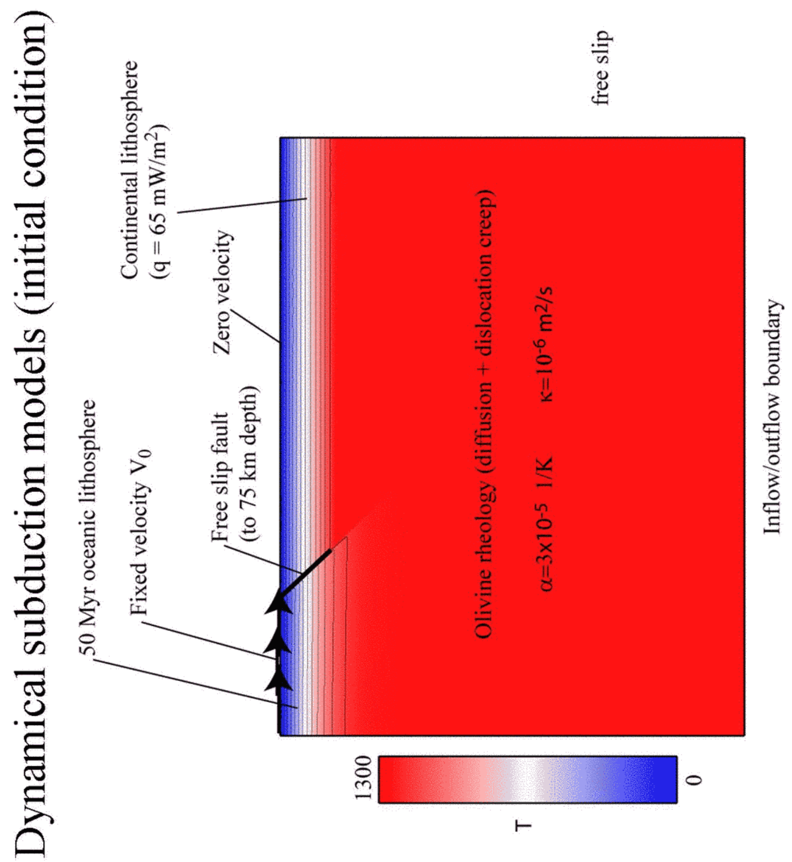


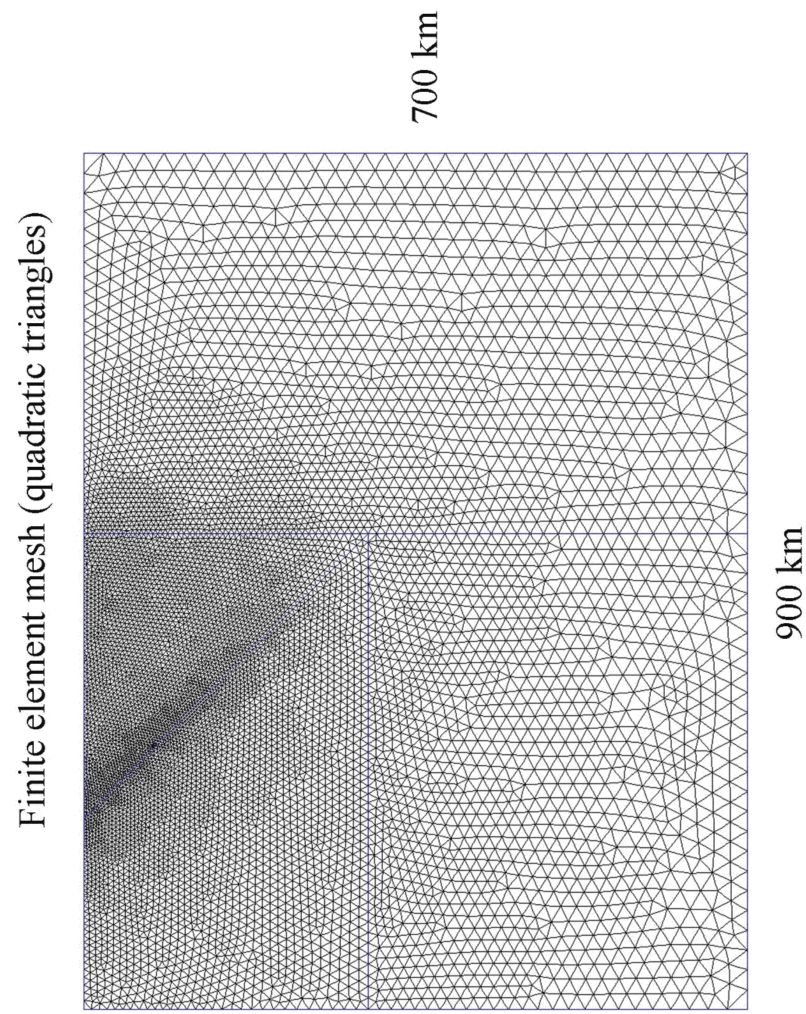
Van Keken, EPSL, 2003

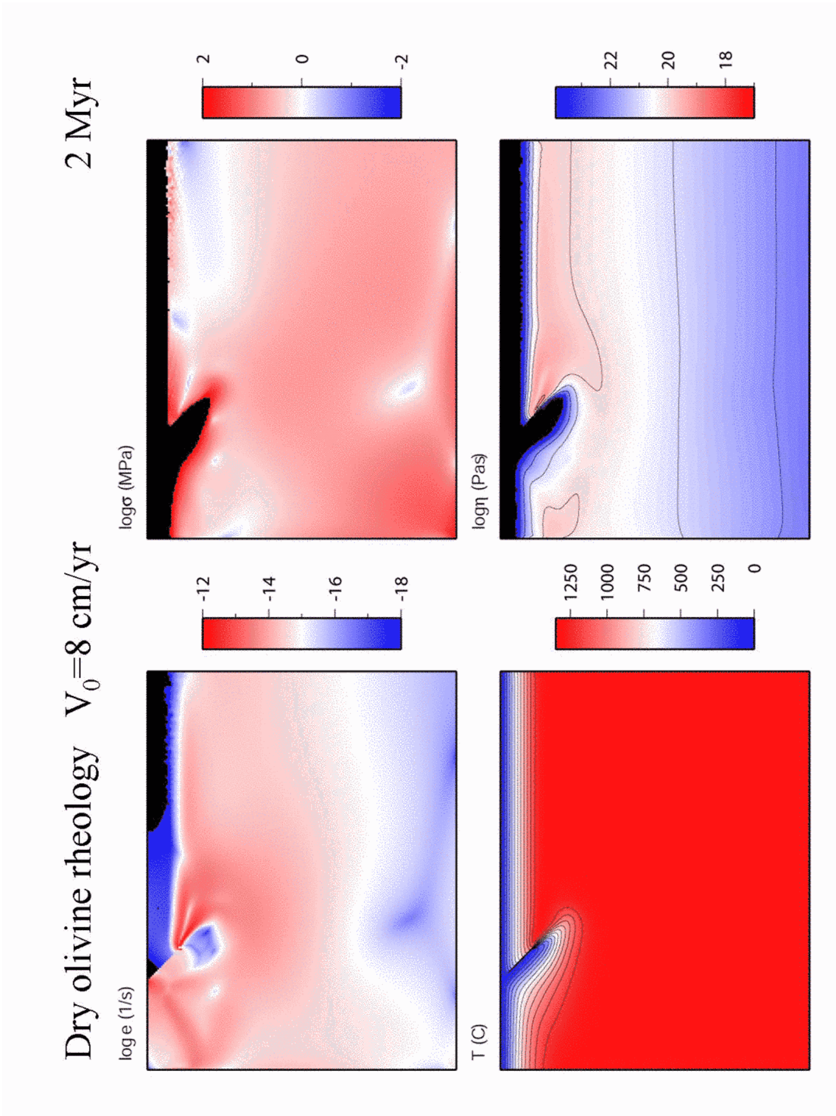
The mantle wedge

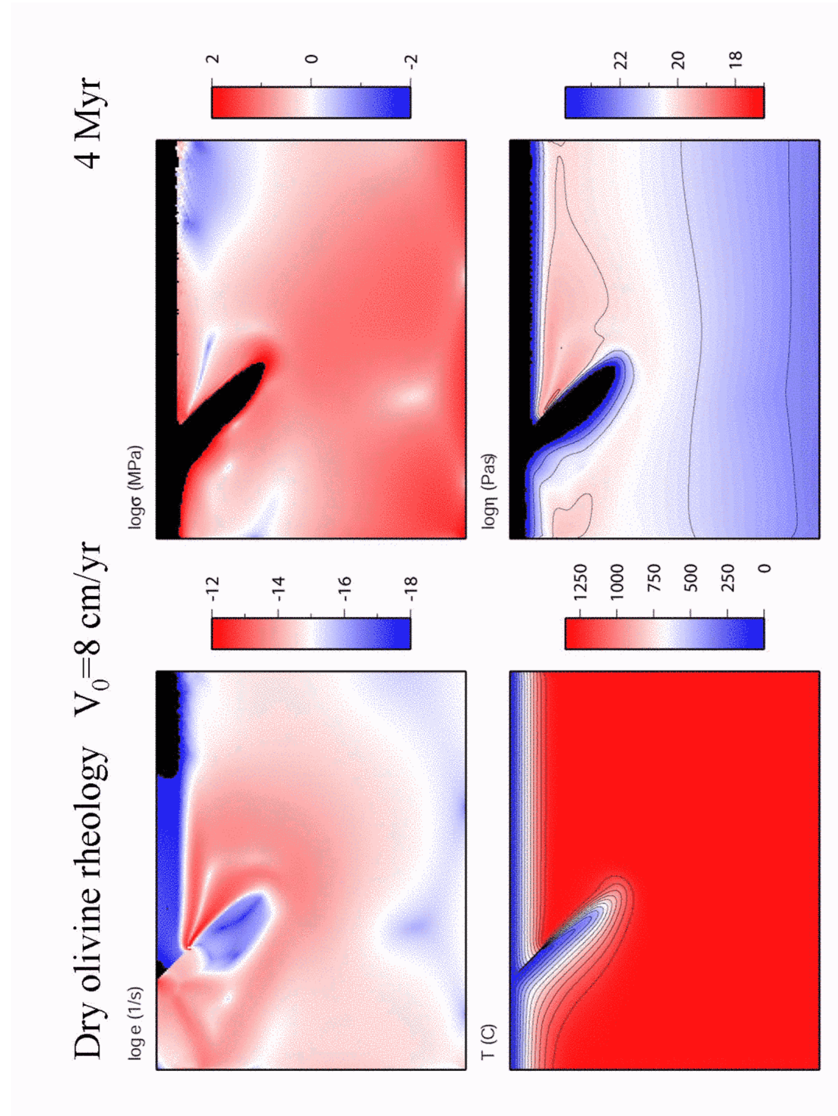


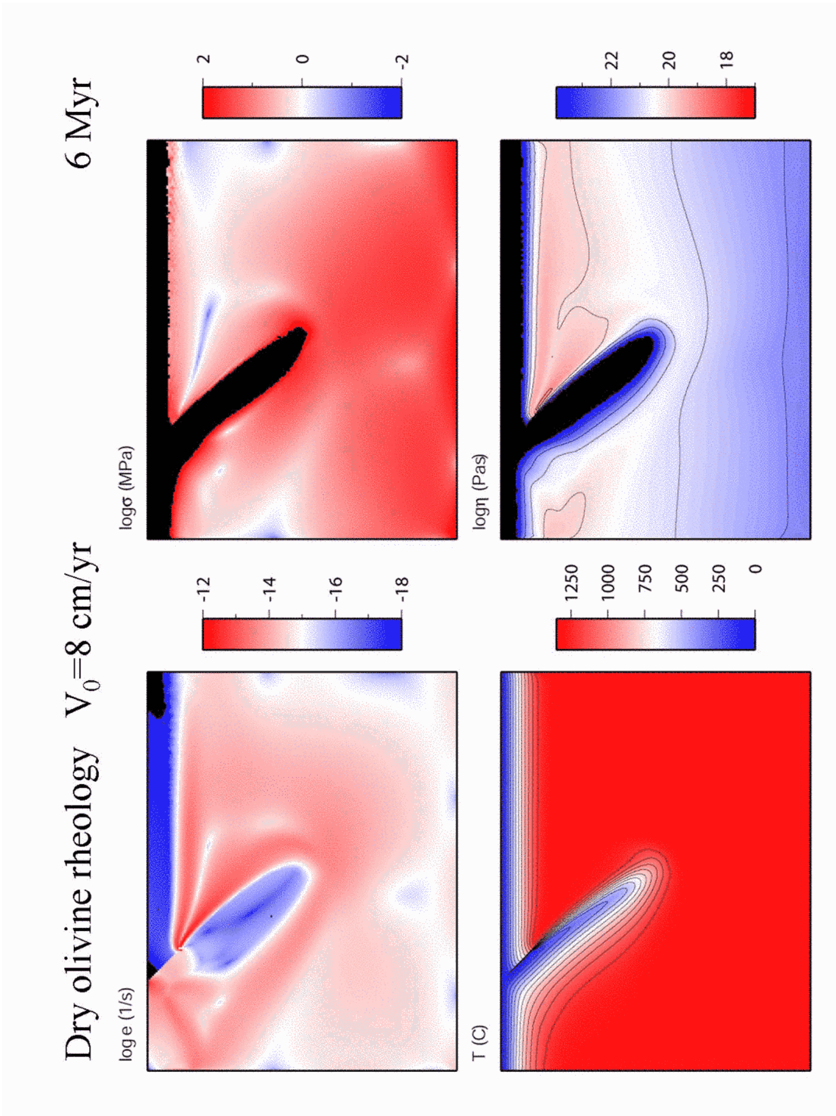
Van Keken, EPSL, 2003

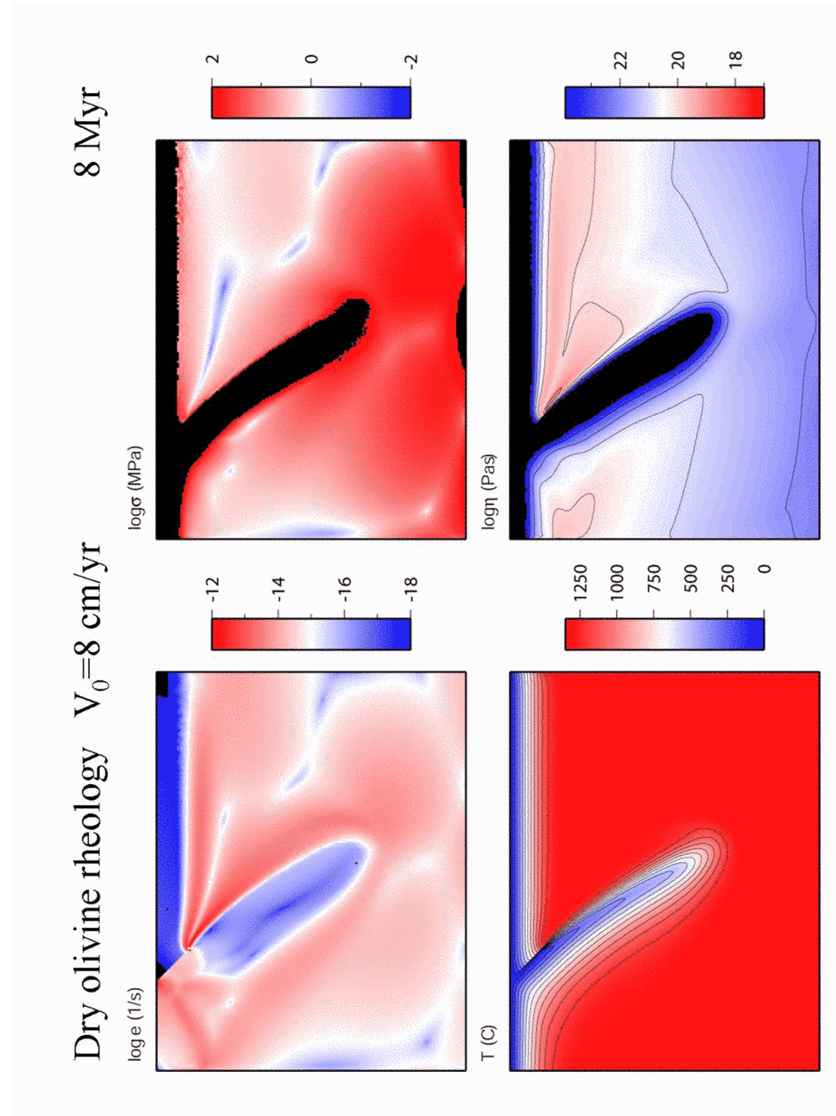

Dynamical subduction models (initial condition)
50 Myr oceanic lithosphere
Fixed velocity V_0
Free slip fault
(to 75 km depth)
Zero velocity
Continental lithosphere
(q = 65 mW/m^2)
free slip
Olivine rheology (diffusion + dislocation creep)
α=3x10^{-5} 1/K κ=10^{-6} m^2/s
Inflow/outflow boundary
1300
0
T

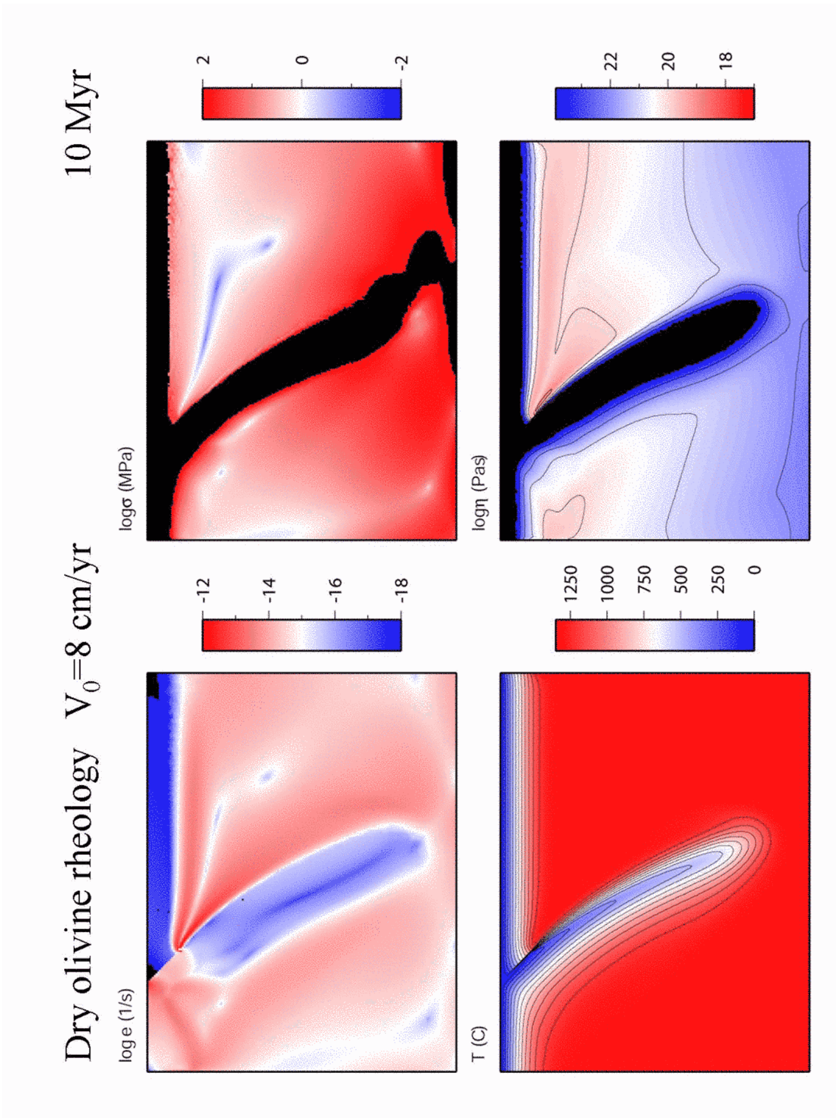

Finite element mesh (quadratic triangles)
700 km
900 km

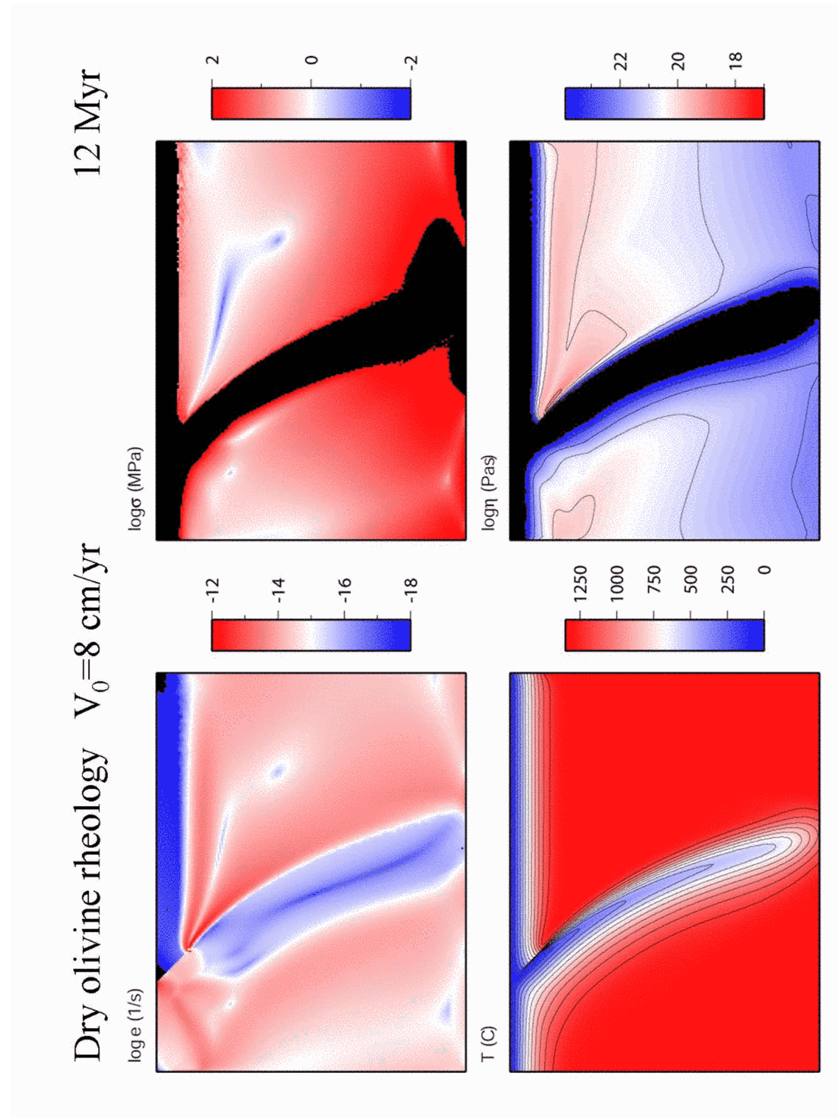

Dry olivine rheology V_0=8 cm/yr 2 Myr
loge (1/s)
-12 -14 -16 -18
logσ (MPa)
2 0 -2
T (C)
1250 1000 750 500 250 0
logη (Pas)
22 20 18

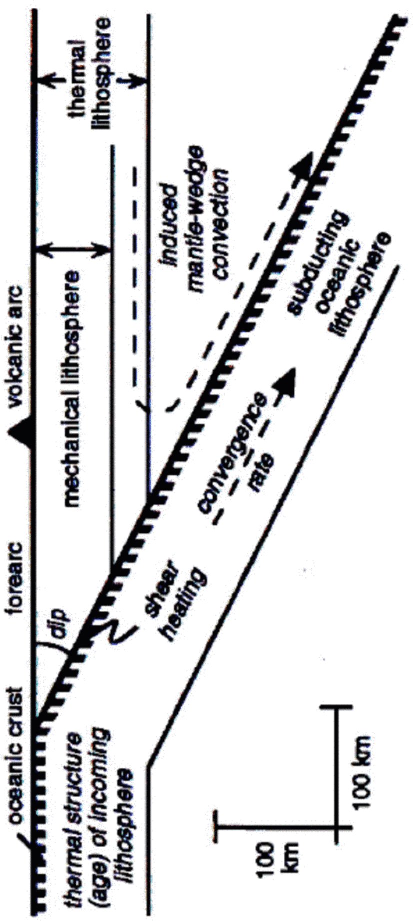

Dry olivine rheology V_0=8 cm/yr 4 Myr
loge (1/s)
-12 -14 -16 -18
logσ (MPa)
2 0 -2
T (C)
1250 1000 750 500 250 0
logη (Pas)
22 20 18


Dry olivine rheology V0=8 cm/yr
6 Myr
logε (1/s)
-12
-14
-16
-18
logσ (MPa)
2
0
-2
T (C)
1250
1000
750
500
250
0
logη (Pas)
22
20
18

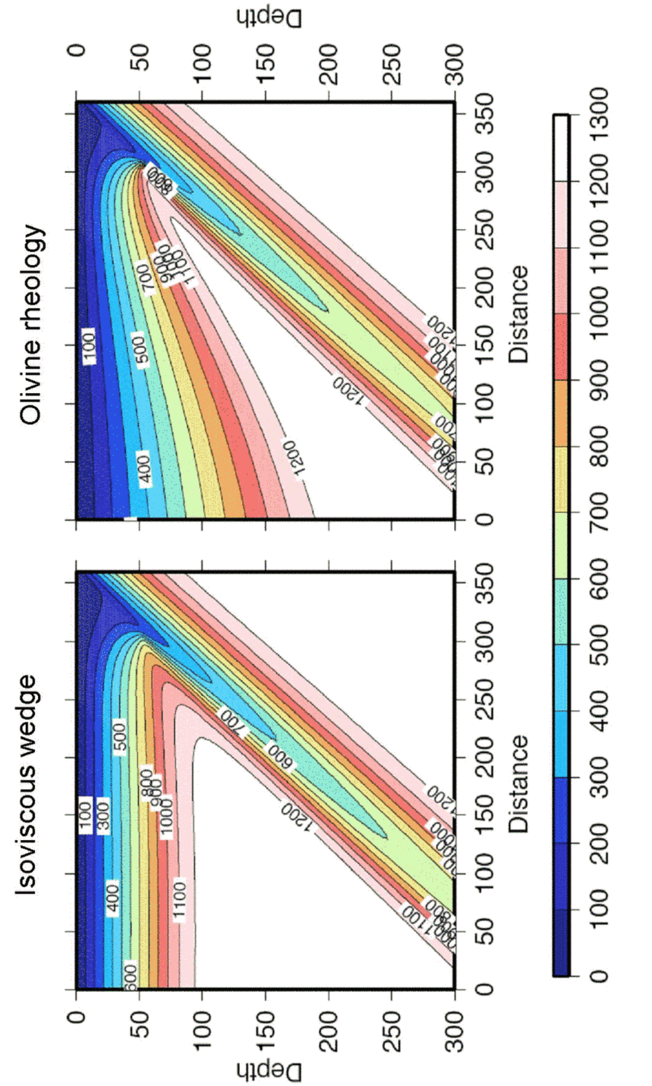

Dry olivine rheology V0=8 cm/yr
8 Myr
logε (1/s)
-12
-14
-16
-18
logσ (MPa)
2
0
-2
T (C)
1250
1000
750
500
250
0
logη (Pas)
22
20
18


Dry olivine rheology V_0=8 cm/yr
10 Myr
loge (1/s)
-12
-14
-16
-18
logσ (MPa)
2
0
-2
T (C)
1250
1000
750
500
250
0
logη (Pas)
22
20
18


Dry olivine rheology V_0=8 cm/yr
12 Myr
loge (1/s)
-12
-14
-16
-18
logσ (MPa)
2
0
-2
T (C)
1250
1000
750
500
250
0
logη (Pas)
22
20
18

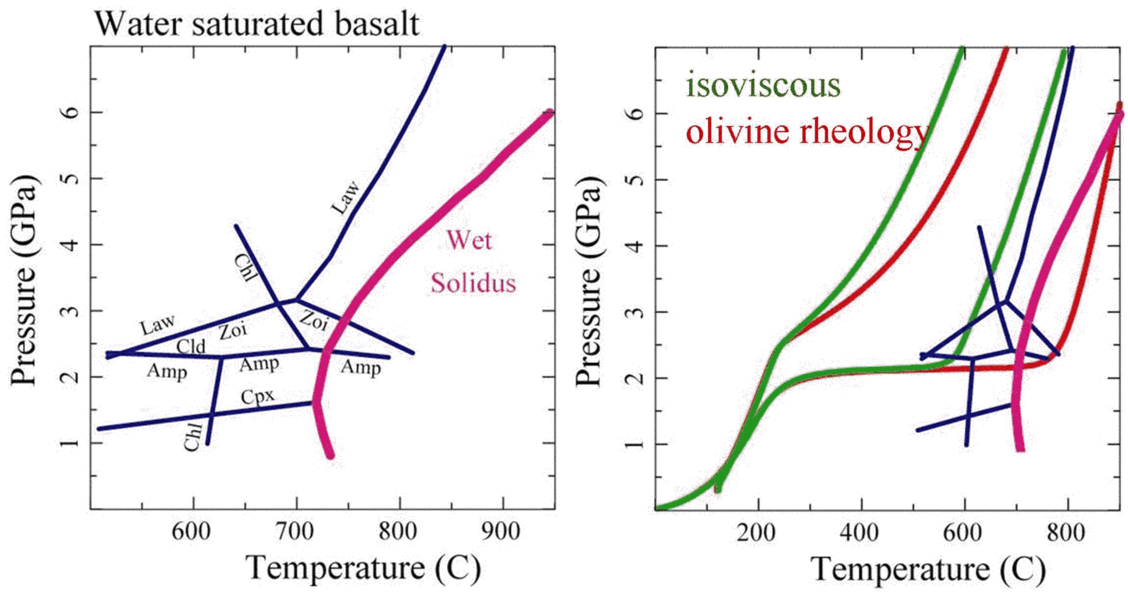


Peacock, 1996



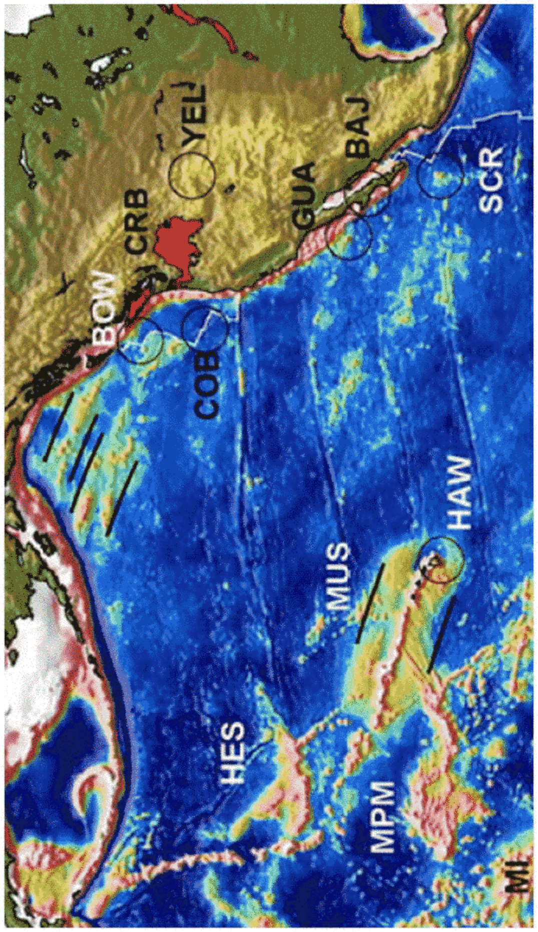
Conman solution to SZ benchmark (King, Treatise on Geophysics, submitted)

Temperature in subducted oceanic crust



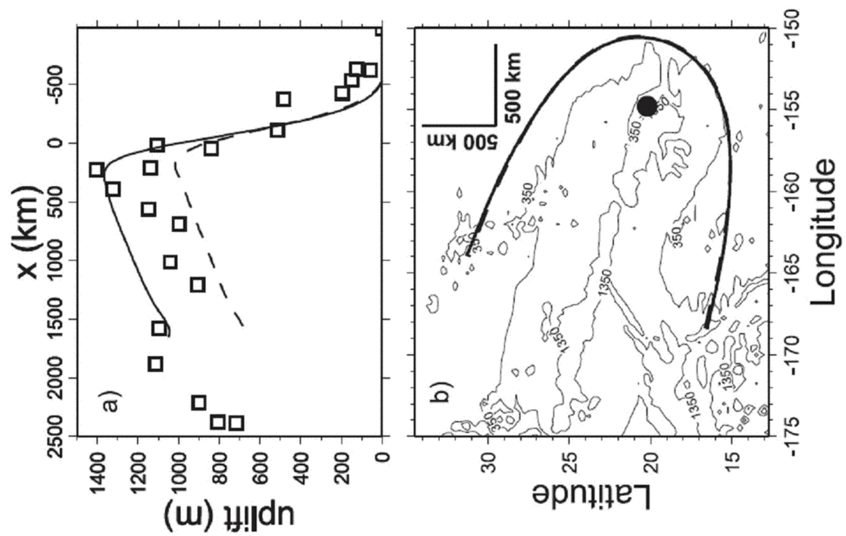
Van Keken et al., G^3, 2002: model for Honshu

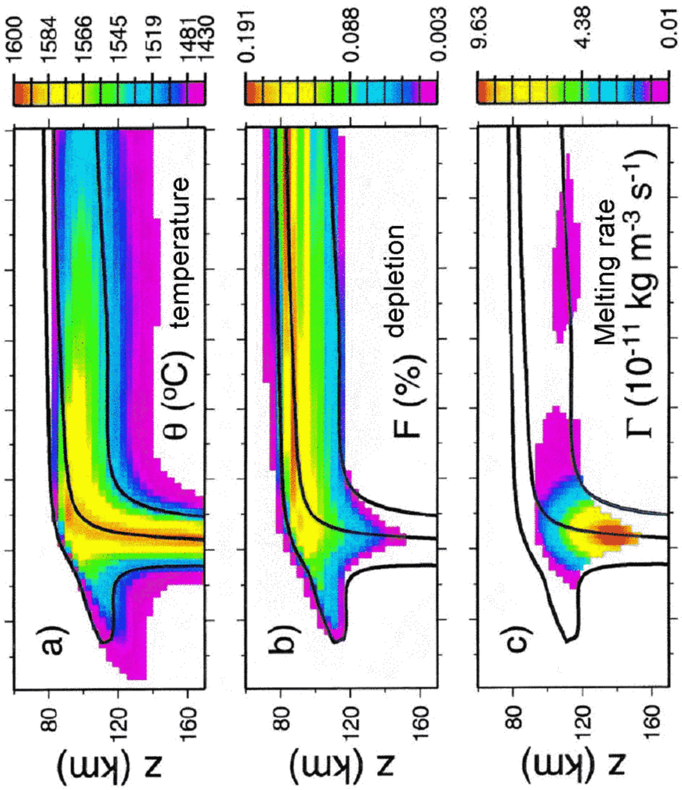
Plumes



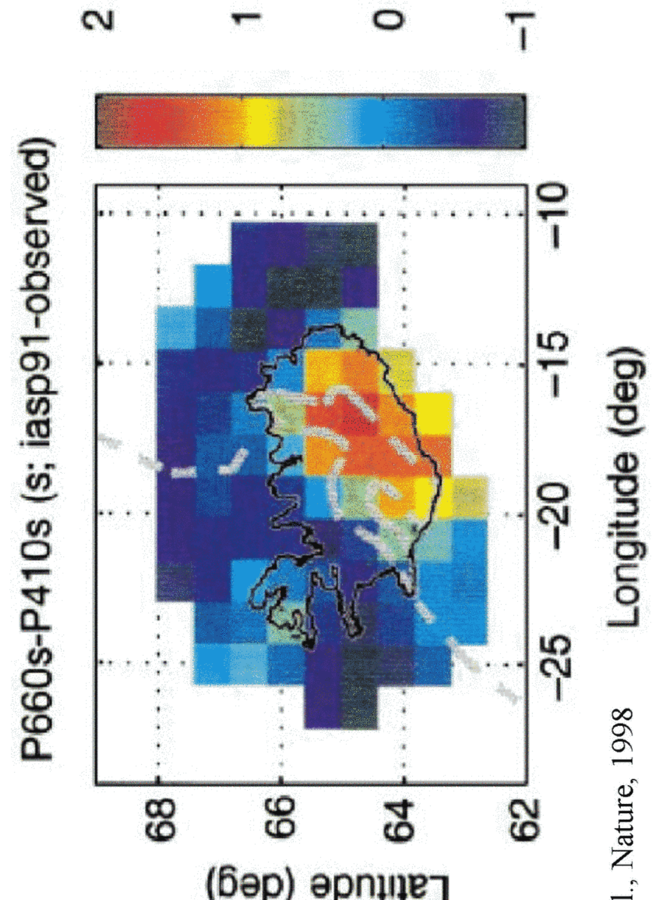
Ito and van Keken, Treatise on Geophysics, submitted

Ribe and Christensen, EPSL, 1999

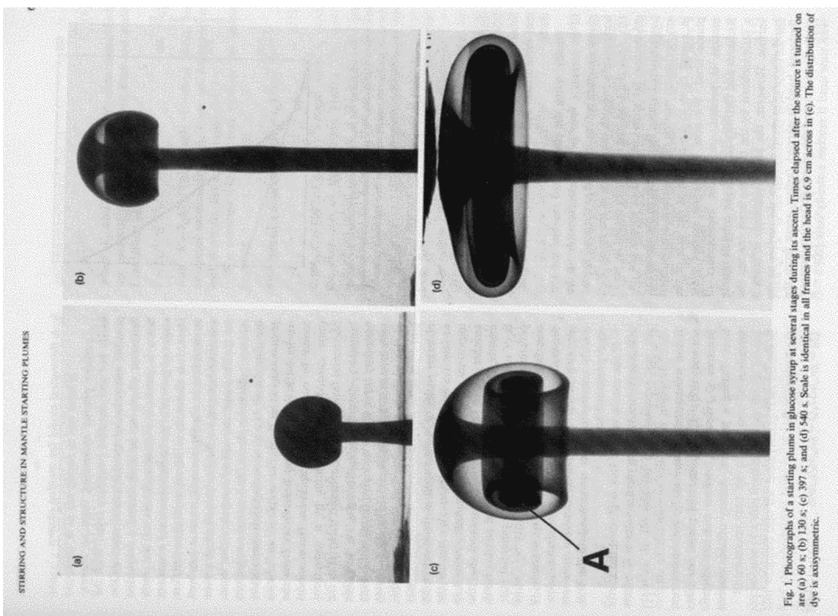




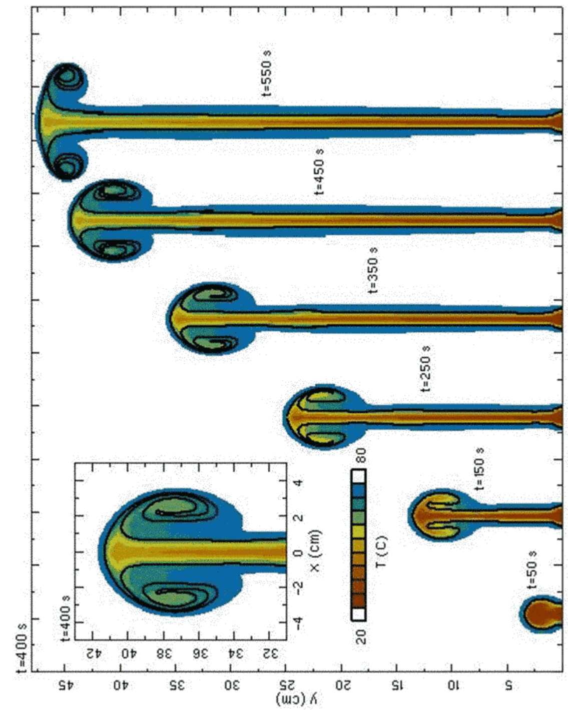
Ribe and Christensen, EPSL, 1999



Shen et al., Nature, 1998

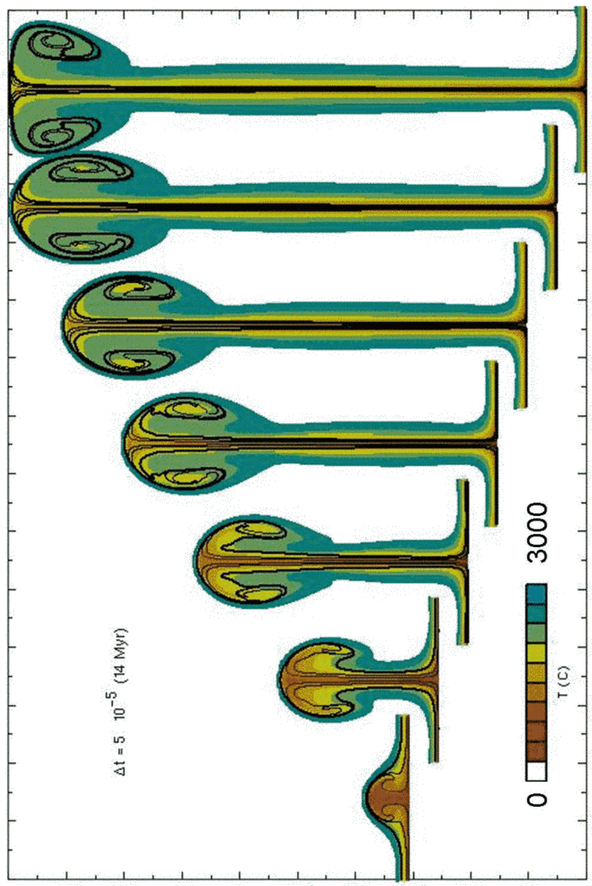


Campbell and Griffiths, EPSL, 1997



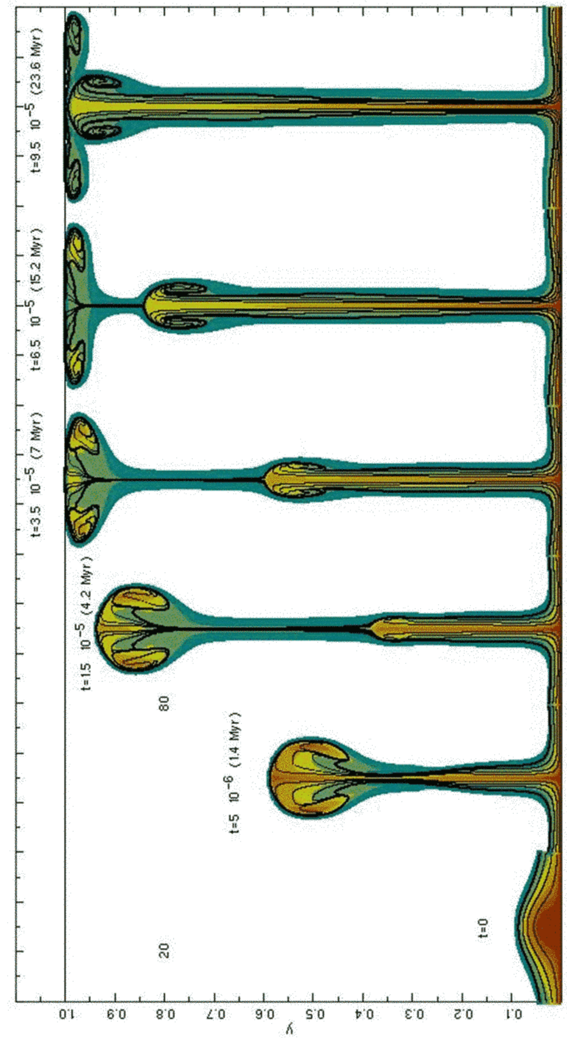
Corn syrup

Van Keken, EPSL, 1997



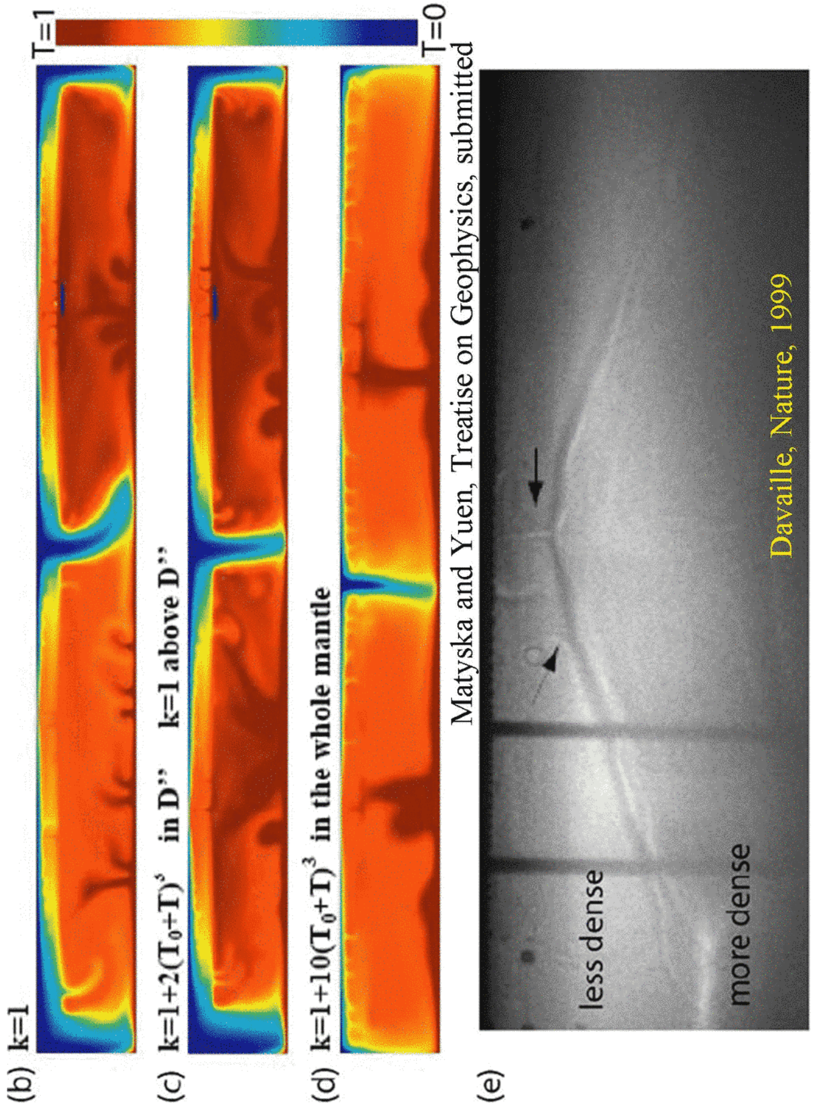
Olivine diffusion creep

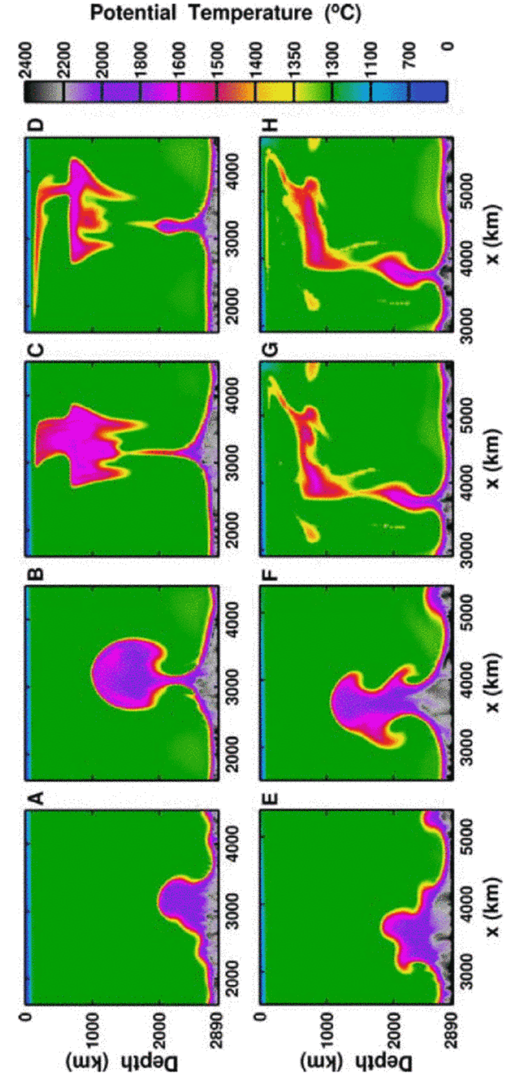
Van Keken, EPSL, 1997



Olivine rheology (diffusion+dislocation creep)

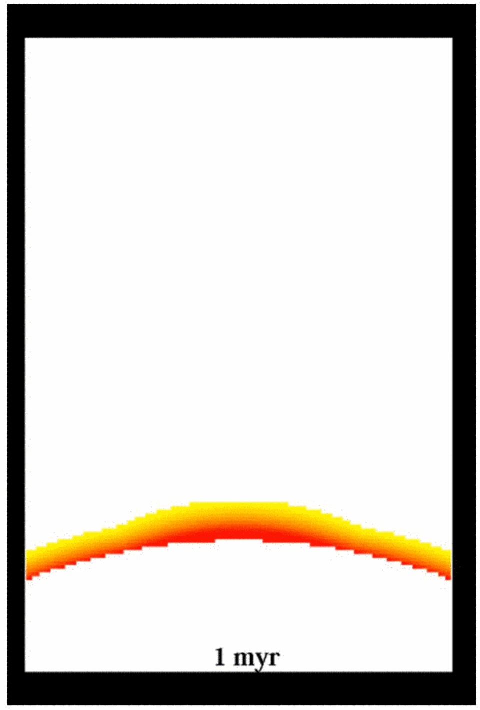
Van Keken, EPSL, 1997





Farnetani and Samuel, GRL, 2005

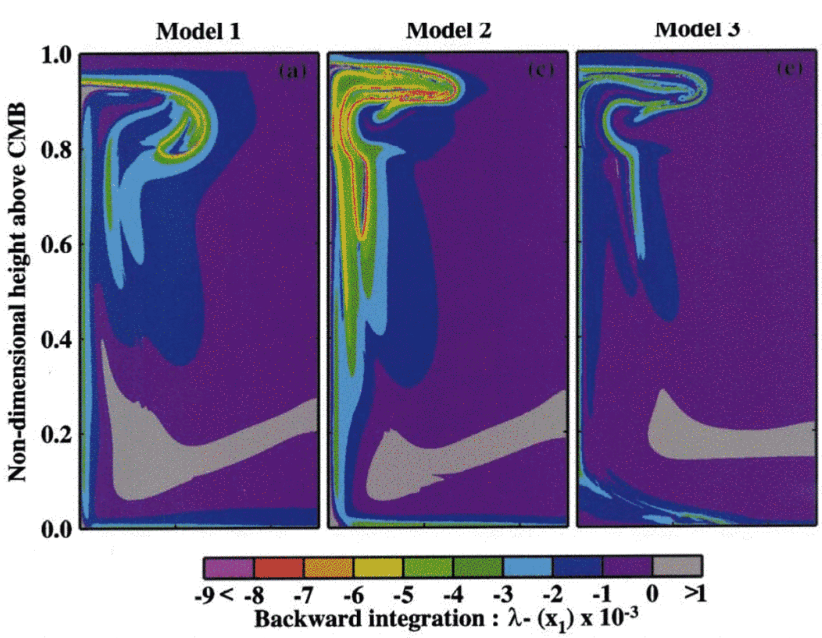
Prediction for multiple episodes of volcanism



$\Delta\eta=10^3$
$\Delta\rho=50$
d=100

Lin and Van Keken
Nature, 2005

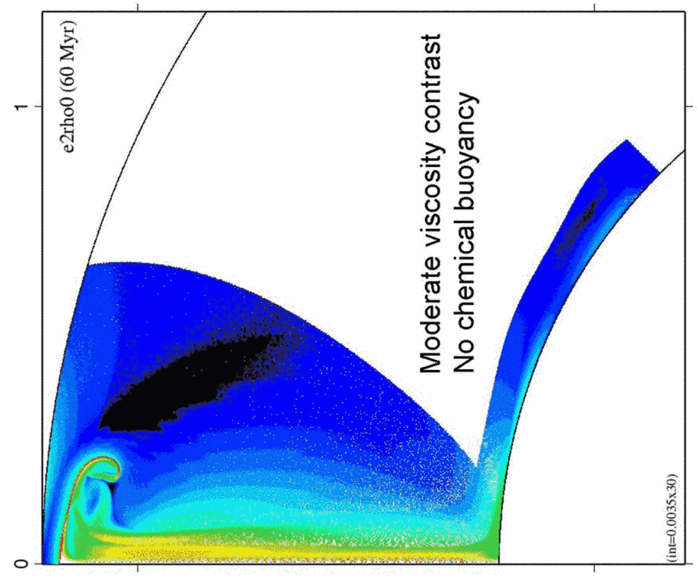
Mixing in plume heads and tails:
(finite time) Luyaponov exponent

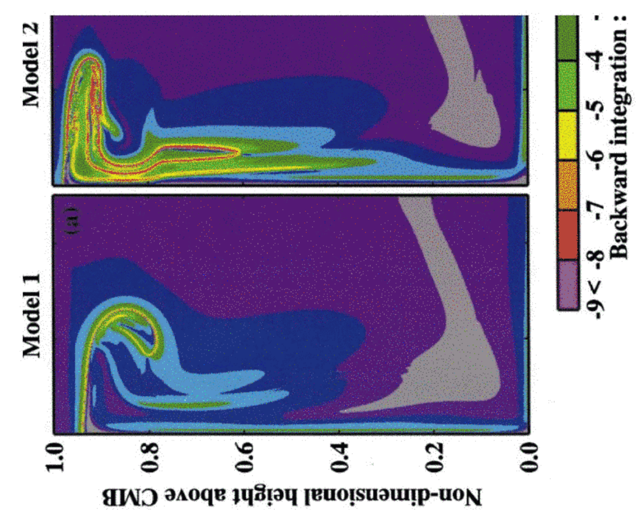


Farnetani et al., EPSL, 2002

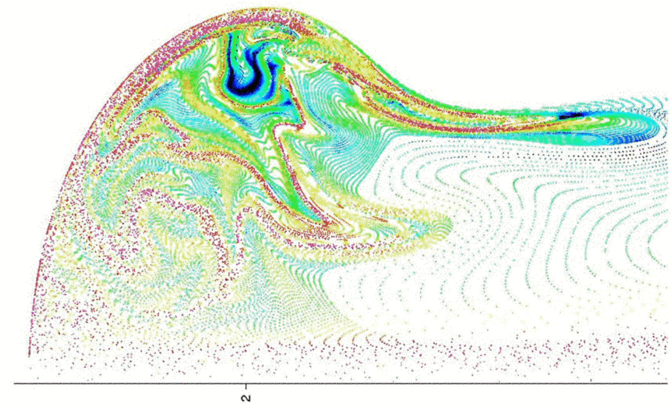
Mixing = stretching + folding

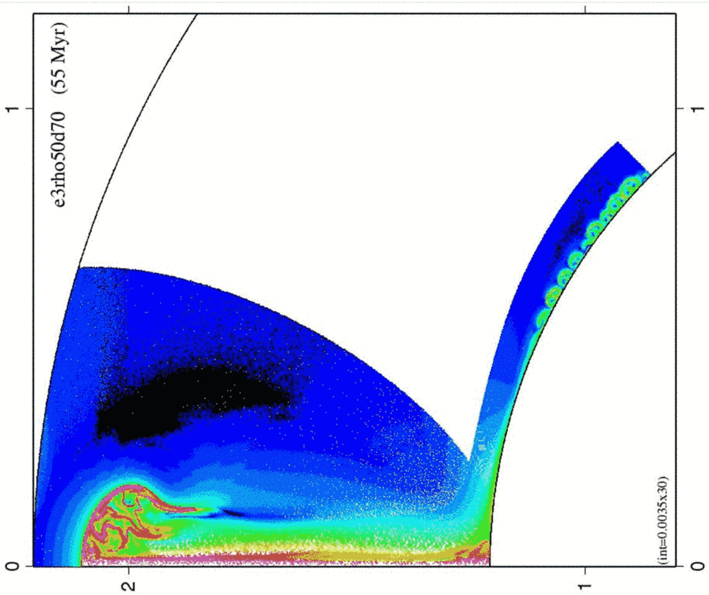
Mixing in plume heads and tails

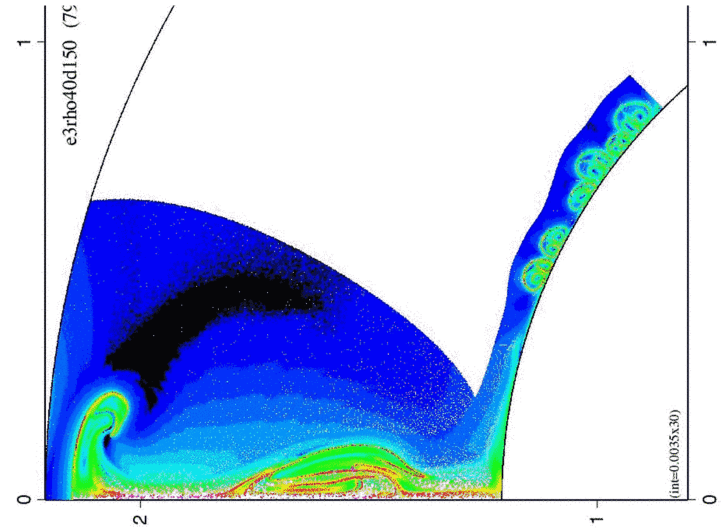




Farnetani et al., EPSL, 2002


2


e3rho50d70 (55 Myr)
(int=0.0035x30)
0
1
2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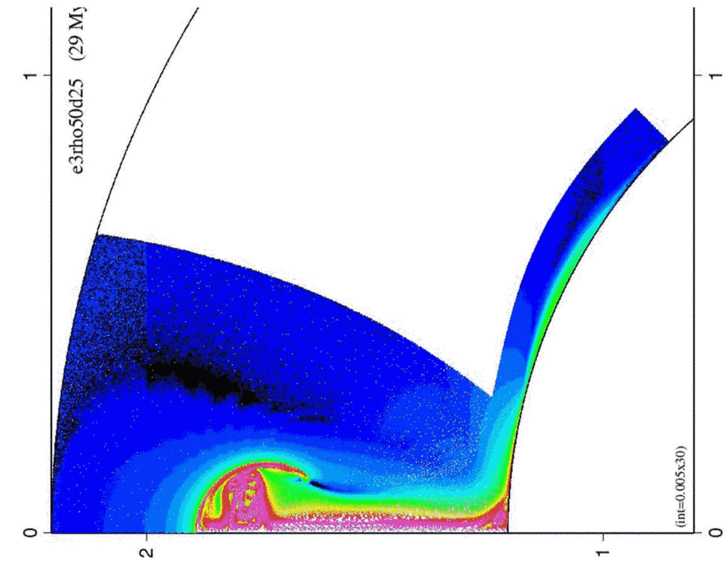
1
2


e3rho50d25
(int=0.005x30)
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1
2