## Thermal Structure of the Earth
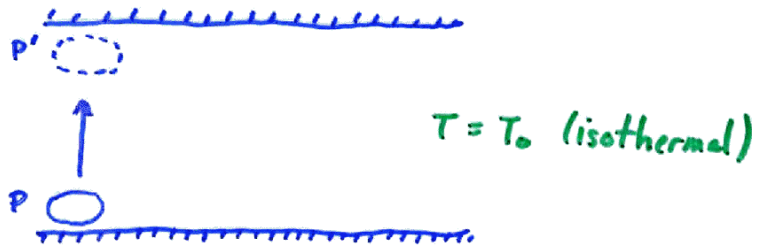
What is the temperature distribution in a vigorously convecting fluid?

Isothermal Fluid



$T = T_0$ (isothermal)

parcel at P is moved to P'

What happens to the temperature?

Conclusion: parcel is not in equilibrium with surroundings

## Thermodynamic Background

$1^{st}$ Law

$$dU = Tds - PdV$$

where $dU$ is the change in internal energy
     $Tds$ is the addition of heat
     $-PdV$ is the work done (on the system)

### MAXWELL RELATIONS

Let $U = U(x,y)$

$\therefore \quad dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$

Consider $U = U(s,v)$

$\therefore \quad dU = \frac{\partial U}{\partial s} ds + \frac{\partial U}{\partial v} dV$

It follows from $1^{st}$ Law that

$$\frac{\partial U}{\partial s} = T \quad , \quad \frac{\partial U}{\partial v} = -P$$

$\therefore \quad \frac{\partial^2 U}{\partial v \partial s} = \left(\frac{\partial T}{\partial v}\right)_s \quad \vdots \quad \frac{\partial^2 U}{\partial s \partial v} = -\left(\frac{\partial P}{\partial s}\right)_v$

## MAXWELL RELATIONS

Equality of $\partial^2 U / \partial v \partial s$ and $\partial^2 U / \partial s \partial v$

means

$$\left(\frac{\partial T}{\partial v}\right)_s = -\left(\frac{\partial P}{\partial s}\right)_v \qquad \text{MAXWELL Relation} \quad \#\,1$$

We can define other functions

$H = U + PV$        Enthalpy

$F = U - TS$        Helmholtz Free Energy

$G = U - TS + PV$    Gibbs Free Energy

The utility of these functions emerges from their differential form

→ gives other Maxwell relations

## MAXWELL RELATIONS

Example

$$dH = dU + Pdv + VdP$$
$$= Tds - Pdv + Pdv + VdP$$
$$= Tds + VdP$$

Consider $H = H(S, P)$

$$\therefore dH = \frac{\partial H}{\partial s}\,ds + \frac{\partial H}{\partial P}\,dP$$

It follows that

$$\frac{\partial H}{\partial s} = T \quad , \quad \frac{\partial H}{\partial P} = V$$

$$\frac{\partial^2 H}{\partial P \partial s} = \left(\frac{\partial T}{\partial P}\right)_s \quad \vdots \quad \frac{\partial^2 H}{\partial s \partial P} = \left(\frac{\partial V}{\partial s}\right)_P$$

This gives

$$\left(\frac{\partial T}{\partial P}\right)_s = \left(\frac{\partial V}{\partial s}\right)_P \qquad \text{Maxwell Relation} \quad \#\,2$$

### APPLICATION

We want to know how $T$ changes with $P$ such that no heat is exchanged with surroundings

$$\left(\frac{\partial T}{\partial P}\right)_S = \left(\frac{\partial V}{\partial S}\right)_P = \left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial T}{\partial S}\right)_P$$

By definition

$$\left(\frac{\partial V}{\partial T}\right)_P = \alpha V \qquad \alpha \text{ is coefficient of thermal expansion}$$

Now we evaluate $\left(\frac{\partial T}{\partial S}\right)_P$

From definition of $dH$ we have

$$dH = T dS \qquad \text{(at constant } P)$$
$$\uparrow \text{heat added}$$
$$= C_P dT \qquad \text{(at constant } P)$$

$$\therefore \quad \frac{dT}{dS} = \frac{T}{C_P}$$

Adiabat becomes

$$\left(\frac{\partial T}{\partial P}\right)_S = \frac{\alpha V T}{C_P} = \frac{\alpha T}{\rho C_P} \qquad \text{✳}$$

### ADIABAT

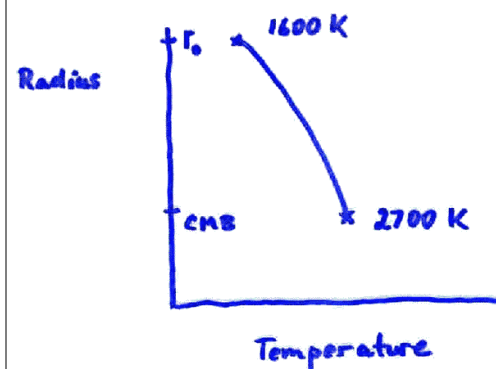The pressure in the mantle is nearly hydrostatic

$$\frac{dP}{dr} = -\rho g$$

$$\therefore$$

$$\frac{dT}{dr} = \left(\frac{\partial T}{\partial P}\right)_S \frac{dP}{dr} = -\frac{\alpha g T}{C_P} \qquad \begin{array}{l}\text{adiabatic}\\\text{gradient}\end{array}$$

Typical value $\sim 0.4 \ K/km$

Typical temperature gradient in crust $\sim 25 \ K/km$

Integrate for $T(r) = T(r_0) e^{\alpha g (r_0 - r)/C_P}$

(when $\alpha$, $C_P$, $g$ are constant)



Radius

$r_0$   1600 K

CMB   ✳ 2700 K

Temperature

## Complications

I. Phase transitions



Clapeyron Slope

$$\frac{dP}{dT} = \frac{\Delta S}{\Delta V} = \gamma$$

i) olivine → spinel　　$\Delta V < 0$, $\Delta S < 0$, $\gamma > 0$

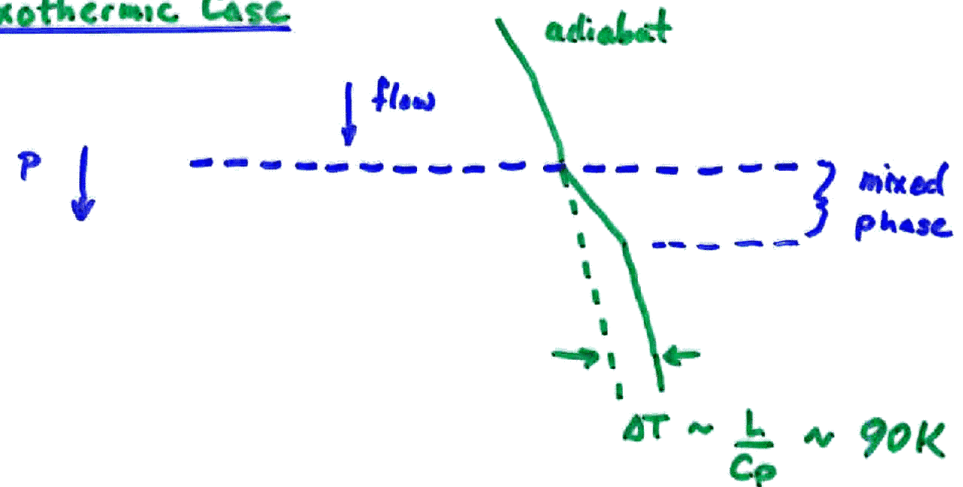ii) spinel → perovskite + oxides

$\Delta V < 0$, $\Delta S > 0$, $\gamma < 0$

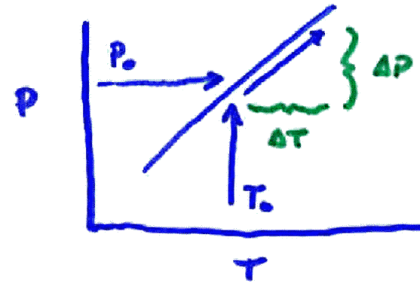Influence on Temperature is due to latent heat

$$L = T \Delta S$$

i) olivine → spinel is exothermic

ii) spinel → perovskite is endothermic

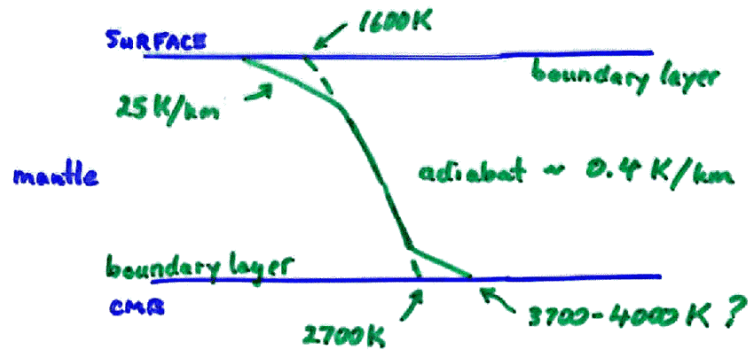How does latent heat alter temperature?

## Exothermic Case



adiabat

flow

mixed phase

$\Delta T \sim \frac{L}{C_p} \sim 90K$

temperature profile is controlled by clapeyron slope



$P_0$

$\Delta P$

$\Delta T$

$T_0$

for endothermic case $\Delta T \sim \frac{L}{C_p} \sim -70K$

## 2. Boundary Layers

SURFACE — 1600K

25 K/km

boundary layer

mantle

adiabat ~ 0.4 K/km

boundary layer

CMB

2700K    3700-4000 K ?
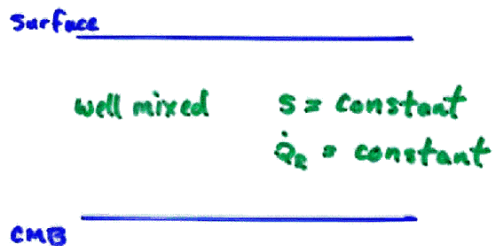
Temperature deviates from adiabat in vicinity of boundaries → not well mixed

Represents a transition from convective to conductive heat transport

## 3. Effect of Radioactive Heating $\dot{Q}_R$

Surface

well mixed    $S = $ Constant
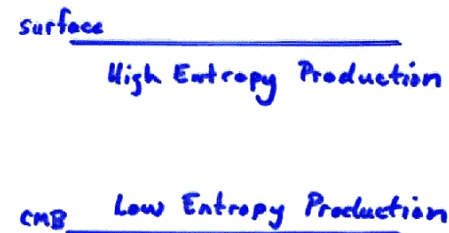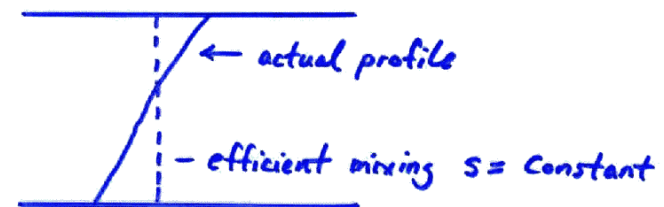           $\dot{Q}_R = $ constant

CMB

## Entropy Production

recall that $dQ = TdS$

$\therefore \dot{S} = \dfrac{\dot{Q}_R}{T}$

$\dot{S}$ is not uniformly distributed because $T(r)$ varies through the mantle

surface

High Entropy Production

$\left(\begin{array}{c}\uparrow\\\downarrow\end{array}\right)$ mixing tends to eliminate gradient

CMB    Low Entropy Production

In steady state

← actual profile

— efficient mixing $S = $ Constant

Well-mixed part of mantle should be thermally stratified (unavoidable!).

$\Delta T \propto \dot{Q}_R^{\frac{1}{2}}$

OBSERVATIONAL CONSTRAINTS

1. Production of Oceanic Crust    $T \sim 1600 K$

2. Depth of Phase Boundaries

   Olivine $\longrightarrow$ Spinel    $T \sim 1800 K$

   Spinel $\longrightarrow$ Perovskite   $T \sim 2000 K$

3. Mineral Assemblages of Mantle Xenoliths

4. Melting temperature of iron   $T \sim 5000 - 6000 K$