Transport in 1D lattice models

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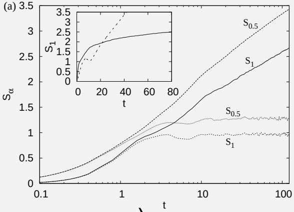
Outline

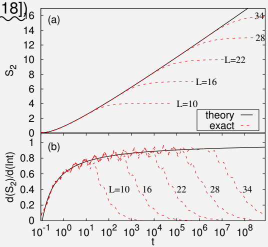
1. Transport:
   (a) Phenomenology: Fourier's law
   (b) The Goal (of studying transport)
   (c) How to study transport?
2. Open systems:
   (a) Describing driving
   (b) Lindblad master equation
   (c) Local driving
   (d) Exact Lindblad solutions
3. Transport in the Heisenberg XXZ model:
   (a) Numerics: density operators vs. pure states
   (b) Clean
   (c) Random potential
   (d) Quasiperiodic potential

Other work: *Log growth of entanglement in MBL*



- MBL: Entropy $S_\alpha \sim \log(t)$
  Anderson: $S_\alpha \to \text{const.}$
  (Žnidarič, Prosen & Prelovšek, arXiv:0706.2539 [PRB '08])
  (also Bardarson, Pollmann & Moore, PRL '12)
- due to exponential interaction:
  (Serbyn, Papić & Abanin PRL '13; Vosk & Altman, PRL '13; Kim, Chandran, Abanin '14)
- Can be explained with the l-bit H (Serbyn et al. '14; Huse et al. '14)
  $H = \sum_k J_k^{(1)} \sigma_k^z + \sum_{k<l} J_{k,l}^{(2)} \sigma_k^z \sigma_l^z + \cdots$

- $S \sim \log t$ is not exact: (M.Ž arXiv:1803.02874 [PRB'18])



Exact result: logarithmic correction

$$S_2(t) = \xi \ln t + A - \log_2 [B + \xi \ln t]$$

1a) Fourier's law

Jean Baptiste Joseph Fourier (1768-1830)

$$j = -\kappa \nabla T$$

Bonetto, Lebowitz & Rey-Bellet, (2000)

"Fourier law: A challenge to theorists"

"We present a selective overview of the current state of our knowledge (more precisely of our ignorace)... empirically well tested...There is however at present no rigorous mathematical derivation of Fourier's law... for any system with a Hamiltonian microscopic evolution."

Even simpler task: For given *H* determine transport type (e.g., diffusive, ballistic,...)

1b) The Goal (of studying transport)

$$\boxed{j = -\kappa\, \nabla U}$$

How does j (or κ) scale with linear size L?

For fixed ΔU in general $\quad j \sim 1/L^{\gamma}$

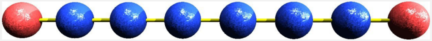
Possible types (local H):

1. localization: $j \sim \exp(-\alpha L), \quad \gamma \to \infty$.
2. subdiffusion: $j \sim 1/L^{\gamma}, \quad \gamma > 1$.
3. diffusion (a.k.a. normal transport): $j \sim 1/L, \quad \gamma = 1$.
4. superdiffusion: $j \sim 1/L^{\gamma}, \quad 0 < \gamma < 1$.
5. ballistic: $j \sim 1/L^{0}, \quad \gamma = 0$.

1c) How to study transport?

nonequilibrium: $j = -\kappa\, \nabla U$

- Indirectly:
  1. Green-Kubo linear resp.: $\kappa \sim \frac{1}{T} \int_0^\infty \mathrm{d}t \lim_{L \to \infty} \frac{1}{L} \langle J(t) J \rangle_{\mathrm{eq.}}$
  2. Evolution of inhomogeneous states: $\sigma \sim t^\beta$
- Directly:
  3. Driven system: Nonequilibrium steady states (NESS)



[simplest nonequilibrium setting]

have to deal with density operator $\rho(t)$ instead of $\psi(t)$.

2a) Describing driving

Derivation of master equation from H: complicated!

We do not want a specific bath, just κ.

In thermodynamic limit bath should not matter!

Take the simplest possible framework.

1. Our **system** described by $\rho(t)$.
2. **Linear** generator $\mathcal{L}$, $\rho(t) = \mathrm{e}^{\mathcal{L}\cdot t}\rho(0)$.
3. $\mathcal{L}$ preserves "density matrices" :

   $$\rho(0) \geq 0 \longrightarrow \rho(t) \geq 0, \qquad \text{CPTP map}$$

4. Bath described only effectively: as "random" as possible.

   Markovian bath – no memory

   $$\rho(t+\tau) = \mathrm{e}^{\mathcal{L}\cdot t}\rho(\tau) = \mathrm{e}^{\mathcal{L}\cdot t}\mathrm{e}^{\mathcal{L}\cdot \tau}\rho(0)$$

Any evolution satisfying the above conditions can be written in terms of **Lindblad equation**.

(G. Lindblad; V. Gorini, A. Kossakowski & E.C.G. Sudarshan '76), history Chruscinski & Pascazio '17

2b) Lindblad equation (G. Lindblad; V. Gorini, A. Kossakowski & E.C.G. Sudarshan '76)

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \mathcal{L}(\rho(t)) = \mathrm{i}[\rho(t), H] + \mathcal{L}^{\mathrm{diss}}(\rho(t)),$$

$$\boxed{\mathcal{L}^{\mathrm{diss}}(\rho) = \sum_j [L_j\,\rho, L_j^\dagger] + [L_j, \rho\, L_j^\dagger].}$$

- The generator $\mathcal{L}$ (the Liouvillian) is non-hermitean.

steady state $\boxed{\rho_\infty = \mathrm{e}^{\mathcal{L}\cdot(\,t\to\infty)}\rho(0)}$.

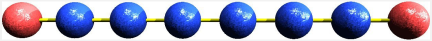
- Depending on L_j:
  - equilibrium steady state.
  - ρ_∞ is a **nonequilibrium steady state** (NESS).

- Books:
  Breuer & Petruccione, *The Theory of Open...* (2002);
  Alicki & Lendi, *Quantum Dynamical Semigroups...* (2007);

  U.Weiss, *Quantum Dissipative Systems*

2c) Local driving

Transport can be studied via local L_j acting on the boundary.



$$L_1 = \sqrt{(1+\mu)}\,\sigma_1^+, L_2 = \sqrt{(1-\mu)}\,\sigma_1^-,$$
$$L_3 = \sqrt{(1-\mu)}\,\sigma_L^+, L_4 = \sqrt{(1+\mu)}\,\sigma_L^-$$

- In the thermodynamic limit the choice of L_j should not matter.

  We do not want/need all superficial details of $H_{1,b}$!

- For small systems the choice of L_j can be important:

  (F. Barra, Sci. Rep. **5**, 14873 (2015); E. Arrigoni et al. PRB **89**, 165105 (2014))

We have simple framework for NESS – Lindblad.

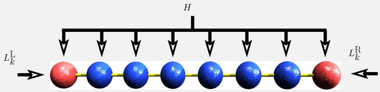
Having simple equation does not mean that NESS ρ_∞ is simple!

Are we able to efficiently find/simulate $\rho(t \to \infty)$?

2d) Exact Lindblad solutions

1. XX chain with dephasing:

$$H = \sum_{j=1}^{L-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y)$$



- boundary driving + dephasing $L_j = \sqrt{\gamma}\sigma_j^z$.
- quartic $\mathcal{L}$, nevertheless solvable (M.Ž. arXiv:1005.1271, JSTAT '10)
- diffusion: linear spin profile, $j \sim 1/L$.
- at $\gamma = 0$ noneq. phase trans. from ballistic to diffusive.
- entanglement in NESS: yes, $\gamma < 1/L$; no, $\gamma > 1/L$
  even at $T = \infty$ you can have 2-body entanglement in NESS
  (M.Ž. arXiv:1112.4415, PRA '12)
  (see also recent Gullans & Huse arXiv:1804.00010; Skinner, Ruhman & Nahum arXiv:1808.05953; Li, Chen & Fisher arXiv:1808.06134; Chan, Nandkishore, Pretko, Smith arXiv:1808.05949)

2. Maximally driven ($\mu = 1$) XXZ chain:
   (Prosen PRL '11, Karevski, Popkov, Schültz PRL '13,...)
   does not give generic bulk transport, but
   new quasilocal conserved quantity!
   Fractal ballistic spin transport for $\Delta < 1$.

3a) Numerics: Operators vs. pure states

- For time-evolution with $\mathrm{e}^{\mathcal{L}dt}$ we will use t-DMRG.
- Have to deal with $\rho(t)$, not just $\psi(t)$ (bigger object).

Not necessarily hopeless.

- For integrable systems even unitary evolution of local operators can be efficient, even if evolution of ψ is not. (Prosen & Žnidarič, arXiv:quant-ph/0608057, PRE '07).

Even better: infinite temperature equilibrium

- Pure states: random high-energy $|\psi\rangle$ well describes local expectations at $T = \infty$.

  But such $|\psi\rangle$ is complicated and highly entangled!
- Density operator: $\rho \sim \mathrm{e}^{-\beta H} \sim \mathbb{1} = \mathbb{1}_1 \otimes \mathbb{1}_2 \cdots \otimes \mathbb{1}_L$ is simple and separable (in operator space).

At $T = \infty$ don't do ψ, do ρ!

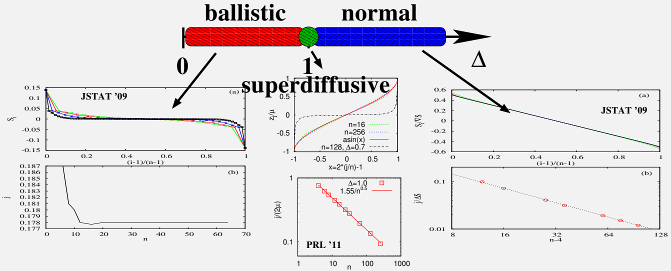
3b) Spin transport in **clean** Heisenberg XXZ

$$H = \sum \sigma_i^{\mathrm{x}} \sigma_{i+1}^{\mathrm{x}} + \sigma_i^{\mathrm{y}} \sigma_{i+1}^{\mathrm{y}} + \Delta \sigma_i^{\mathrm{z}} \sigma_{i+1}^{\mathrm{z}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \mathrm{i}[\rho(t), H] + \mathcal{L}^{\mathrm{diss}}, \quad \mathcal{L}^{\mathrm{diss}} = \sum [L_j \rho, L_j^\dagger] + [L_j, \rho L_j^\dagger]$$

Local L_j acting on the boundary, [half-filling, $T = \infty$]

$$L_1 = \sqrt{(1+\mu)}\,\sigma_1^+, L_2 = \sqrt{(1-\mu)}\,\sigma_1^-,$$
$$L_3 = \sqrt{(1-\mu)}\,\sigma_L^+, L_4 = \sqrt{(1+\mu)}\,\sigma_L^-$$



- Ballistic for $\Delta < 1$ understood (Zotos, Naef & Prelovšek PRB '97, Prosen PRL '11, Ilievski & De Nardis PRL '17,...).
- Theory for $\gamma = \frac{1}{2}$ ($\beta = \frac{2}{3}$) superdiffusion at $\Delta = 1$?
- Normal for $\Delta > 1$ and $T = \infty$ is strange (Sirker, Pereira & Affleck, '09)?

3c) Transport in a **disordered** Heisenberg chain

$$H = \sum \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z + 2h_i \sigma_i^z, \qquad h_i \in [-h, h]$$

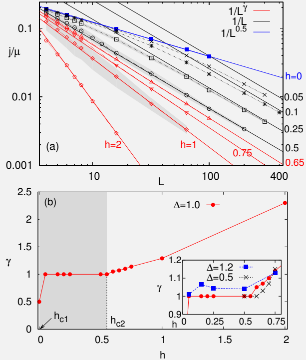
Localized phase $h > h_{c3} \approx 4$ no transport (insulator, $j \sim e^{-\mu L}$)

Ergodic phase, $h < h_{c3}$?

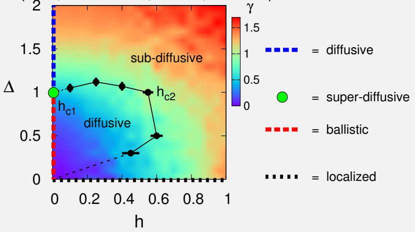
(small $h \to$ large scatt. length, need large L!)

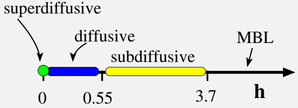
(Agarwal et al. PRL '15; Potter et al. PRX '15; Gopalakrishnan et al. PRB '16; Bar Lev et al. PRL '15; Luitz et al. PRB '16; Khait et al. PRB '16, Barišić et al. '16,...)

Numerics on $L \approx 400$ sites:



(M. Ž, Scardicchio, Varma, PRL '16)

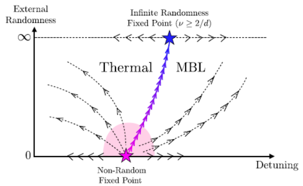




3d) Quasiperiodic potential – Theory/numerics:

Are quasiperiodic different?

- MBL possible (Iyer, Oganesyan, Refael, Huse, PRB '13)
- no "rare regions"
- transition point (Khemani, Sheng, Huse, PRL '17)



(MBL phase more stable in quasiperiodic)

- localization at $T = 0$ (Mastropietro, CMP '17)
- Michal et al. PRL '14, Varma & Pilati PRB '15, Bera et al. Ann. Phys. '17, Lee et al. PRB '17, Setiawan et al. PRB '17, Szabo & Schneider '18,...
- experiments are done with quasiperiodic (Bloch's group)

3d) Heisenberg in **quasiperiodic** potential

Chain: interacting AAH (Aubry-Andre-Harper)

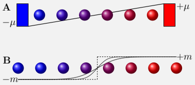
$$H = \sum \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + U \sigma_i^z \sigma_{i+1}^z + h_i \sigma_i^z, \qquad h_i = \lambda \cos(2\pi\beta i + \phi)$$

For $U = 0$:

- localized phase $\lambda > 2$, no transport.
- ballistic for $\lambda < 2$.
- Fractal NESS at $\lambda = 2$ (V. Varma et al., PRE '17)

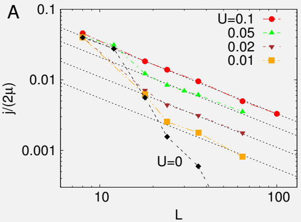
Transport for small interaction U

numerics on $L \approx 100 - 800$ sites (M. Ž, M. Ljubotina arXiv:1801.02955, PNAS'18)
(NESS and domain-wall spreading)



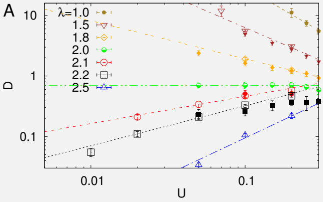
NESS current $j(L)$, $\lambda = 2.2$:

high-T, half-filling

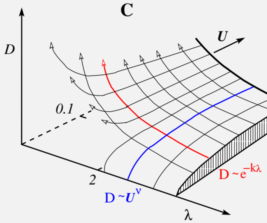


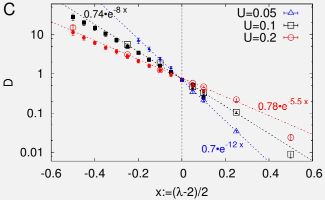
3d) ...cont. (interacting AAH)

Diffusion constant $D(U)$:



Diffusion constant $D(U)$: $D \sim U^{\nu} \asymp \exp\left(-4e^{1/2\xi}\ \ln \frac{1}{U}\right)$

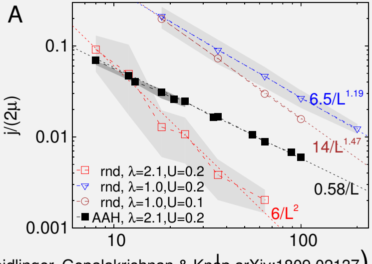




3d) Quasiperiodic vs. random

For small U:

- quasiperiodic: diffusion
- random: subdiffusion



(MZ & ML PNAS '18 see also recent Weidlinger, Gopalakrishnan & Knap arXiv:1809.02137)

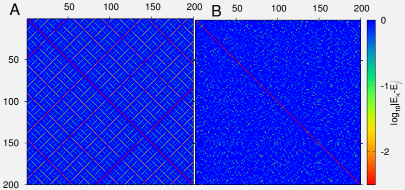
Quasiperiodic completely different: immediate local. breakdown

Single-particle resonances:

$$R_{l,k} = \frac{\langle \psi_{l-1}, \psi_l | V | \psi_{l-1}, \psi_k \rangle}{E_l - E_k}, \qquad V = \frac{1}{4} \sum_r \sigma_r^z \sigma_{r+1}^z$$

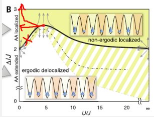
AAH Anderson

$E_j - E_k$:



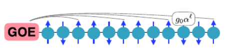
Question: MBL transition point for $U \to 0$?

- It seem one can not have $\lambda_c(U \to 0) = 2$: diffusion around

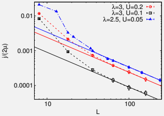


Schreiber et al., Science '15

- Possibilities:
  - finite $\lambda_c(U \to 0) > 3$
  - $\lambda_c(U \to 0) \to \infty$ (diffusion exponentially small at $\lambda \gg 2$)
- Rare regions instability: (Luitz, Huveneers & De Roeck PRL'17)



  - claim: (universal) instability of localization for $\xi > \frac{1}{\ln 2} \approx 1.44$ (in probability).
  - we see: diffusion (at least) upto $\lambda = 2.5 - 3$, where single-particle $\xi = 2.3 - 1.24$ (in probability); we should see signs of localization, but don't.



Open problems

1. Analytical understanding of superdiffusion for XXX at half-filling and high T.
2. Explain diffusion for XXZ and $\Delta > 1$ (is it real?).
3. Quasiperiodic pot. and localization for $U \ll 1$ and $\lambda \gg 1$?
   - explain quasiperiodic instability
4. Do boundary Lindblad give the same as linear response?
   - mathematical equivalence of NESS and LR settings is not clear even for classical systems
   - for diffusive systems both do give the same κ (M.Ž. arXiv:1806.11050).