Hydrodynamics and transport in integrable and nearly integrable models

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Partly pedagogical, but includes results from

R. Vasseur, C. Karrasch, JEM, PRL 2015
V. B. Bulchandani, R. Vasseur, C. Karrasch, JEM, PRL 2018 and PRB 2018
T. Scaffidi, N. Nandi, B. Schmidt, AP Mackenzie, JEM, PRL 17

SIMONS FOUNDATION

Outline

Conventional thermodynamics rests on the assumption that initial states thermalize to a "Gibbs ensemble", determined by the conserved quantities (e.g., energy and particle number).

Other possibilities: MBL. Others?

1. What about systems with infinitely many conservation laws (picks up where Essler talk ended)?
2. Real systems are not exactly integrable and there is no "KAM theorem". Does any of this survive in slightly perturbed systems?
3. There are an increasing number of experiments on clean 2D materials with (possibly) hydrodynamical electrons. What can emerge?
4. Some open questions.

More background on transport: Linear response theory

Einstein's theory of motion of Brownian particles:
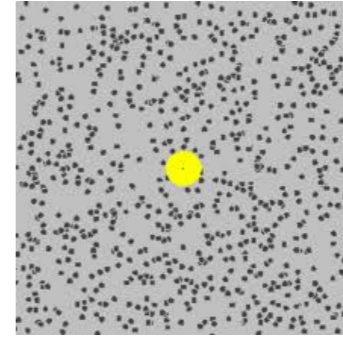
the diffusion constant D that appears in Fick's law
(which is the restoration to equilibrium from a density perturbation)

$$\mathbf{j} = -D\nabla n$$

is given by the dynamical correlation function of velocity *at equilibrium*:

$$D = \frac{1}{3}\int_0^\infty \langle \mathbf{v}(0)\cdot\mathbf{v}(t)\rangle_T\, dt \approx v^2\tau$$

Philosophy: how a system returns to equilibrium is independent of whether it was *driven* away or *fluctuated* away

Kubo formula for electrical conductivity in metals: dynamical correlation function of electrical current

Standard quantum transport theory

In principle, we want to calculate the Kubo formula for electrical conductivity

$$\sigma_{\alpha\beta} = \frac{ine^2}{m\omega}\delta_{\alpha\beta} + \frac{1}{\hbar\omega V}\int_0^\infty \langle [\hat{j}_\alpha(t)\hat{j}_\beta(0)]\rangle\, dt$$

using the full many-body current operator *j*.

Challenges: for this to be finite, need current to relax (e.g., by disorder scattering, but full quantum theory of disorder scattering is complicated…).

Can do perturbation theory in interactions and disorder (Feynman diagrams). Approximations give Boltzmann transport theory, which we often use in practice as a simple semiclassical picture.

$$\partial_t f_1 + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_1 = \int w'(f_1' f_2' - f_1 f_2)d\mathbf{p}_2\, d\mathbf{p}_1'\, d\mathbf{p}_2'.$$

(This is Boltzmann for collisions of *quasiparticles* in a gas with no external force; in a solid, usually add electron-impurity collisions and applied fields.)

A modern reference: J. Rammer, Quantum Transport Theory

Unusual hydrodynamics in 1D

1. "Old" history: two-reservoir quench in 1D

Simple cases: free bosons; CFTs

Often the leading corrections to criticality preserve integrability.
(Hubbard model, XXZ, …). So what?

In interacting integrable models: there are exact results for some quantities even arbitrarily far from equilibrium; can compare to DMRG simulations for XXZ (Vasseur, Karrasch, JEM PRL 2015)
Background to hydrodynamical/kinetic theory approaches for soliton gases, Lieb-Liniger, XXZ.

2. *Test of hydrodynamical kinetic theory predictions in more general cases: expansions from smooth initial conditions.*

When is hydrodynamics (i.e., Bethe-Boltzmann equation) valid? Can compare to *microscopic simulations at nonzero T,* when hydrodynamics should be generic. (At T=0, coarse-graining length diverges at least in CFTs)

Standard hydrodynamics (0th order)

The "zeroth-order" hydrodynamical equations in three dimensions, which neglect dissipative behavior such as viscosity, are

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0 \tag{1}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)\mathbf{u} + \frac{1}{\rho}\nabla P = \frac{\mathbf{F}}{m}. \tag{2}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)\tau + \frac{2}{3}(\nabla \cdot \mathbf{u})\tau = 0. \tag{3}$$

These come from the Boltzmann equation assuming local equilibrium.

Hydrodynamics: how does local equilibrium become global equilibrium?

Models to be studied

We will primarily focus on two examples of Yang-Baxter "integrable" systems:

the 1D Bose gas with delta-function interaction (Lieb-Liniger model);

the 1D "XXZ" spin chain.

$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

The latter has a more complicated Bethe ansatz formulation, but is easier to compare to microscopic DMRG numerics.

By adding a random field (last term), we can obtain a localized phase.

What makes these models special?

In attempting to construct coordinate Bethe ansatz wavefunctions for the ground state (basically combinations of plane waves with scattering phase shifts), one obtains a nontrivial constraint on the scattering.

This constraint appears already with 3 particles and leads to a deep mathematical structure (the *Yang-Baxter equation*). We start with the Lieb-Liniger model:

We treat a collection of bosonic atoms as neutral, with a repulsive δ-function interaction between to model Pauli exclusion from overlapping atoms. We use the model:

$$H = -\sum_i \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i \neq j} \delta(x_i - x_j).$$

For $c = 0$, this models free Bosons. For $c = \infty$, this gives free fermions in the thermodynamic limit. Various cases of this were solved by Lieb-Liniger, Yang & Yang, and Tonks-Giradeau. This is tied into the Thermodynamic Bethe Ansatz (TBA). To use the Bethe ansatz, we can't work in second quantization, and we return instead to first quantization.

The Bethe wavefunction is for N identical bosons $x_1 \ldots, x_n$. It's good enough to define everything for $x_1 \leq x_2 \leq \cdots \leq x_n$; the other values are determined by symmetry. So we write

$$\psi(x_1, x_2, \ldots, x_n) = \sum_{P \in S(N)} A(P) \exp\left(i \sum_{\ell} k_{P_\ell} x_\ell\right) \qquad \text{(Bethe Wavefunction)}$$

where $P \in S(N)$, permutations on N elements.

Let's do this in the $c = \infty$ case and see why this is like free fermions. The hard part is to figure out A. If $c = \infty$, then we need $\psi(\cdots, x, \cdots, x, \cdots) = 0$ whenever two particles are on top of each other. Let's try $N = 2$ with momenta k_1 and k_2. So

$$\psi = A(12)e^{ik_1 x_1 + ik_2 x_2} + A(21)e^{ik_2 x_1 + ik_1 x_2} \text{ if } x_1 \leq x_2$$

Therefore $\forall k_1, k_2$,

$$0 = \psi(x_1 = x, x_2 = x) = [A(12) + A(21)] \, e^{ik_1 x + ik_2 x} \implies A(12) + A(21) = 0.$$

WLOG, $1 = A(12) = -A(21)$. So then

$$\psi(x_1, x_2) = \det \begin{bmatrix} e^{ik_1 x_1} & e^{ik_1 x_2} \\ e^{ik_2 x_1} & e^{ik_2 x_2} \end{bmatrix},$$

which looks like a Slater determinant. Note that we're still working with bosons, but our energetics alone gave us this anti-symmetry. Another way to this about this is that exchanging $(12) \leftrightarrow (21)$ is like a π-phase shift, which is exactly the phase shift from δ-function scattering in $1D$. We can also compute that $H\psi = E\psi$ for $E = \sum_{i=1}^{2} k_i^2$. So we've found a good wavefunction for two particles. We want to see how this actually works for any number of particle — this is the special part.

So we are now in the case $N = 3$, $0 \leq c \leq \infty$. We have the condition

$$\frac{A(123)}{A(213)} = \frac{k_1 - k_2 + ic}{k_1 - k_2 - ic}; \quad \frac{A(123)}{A(132)} = \frac{k_2 - k_3 + ic}{k_2 - k_3 + ic}.$$

We have 6 ratios, one for each pair of permutations related by two-cycles, but because of normalization we only get 5 constraints. In general, there is no solution that will even fix these 5 pairs. What's special about this situation?

This satisfies the **Yang-Baxter Equation** or the **Star-Triangle Relation**. Define some notation

$$Y_{12} = \frac{A(123)}{A(213)}; Y_{23} = \frac{A(123)}{A(132)}, \ldots.$$

So we can go between different A's by various equivalent paths, where Y's act on the right:

$$A(321) = Y_{12}Y_{13}Y_{23}A(123) = Y_{23}Y_{13}Y_{12}A(123).$$

In general, scattering depends on what scatters off what first. However, for this particular problem, everything is independent of the order of scattering. This always works for δ function scattering, but mathematicians also write down other systems where scattering works like this.

Let's write this as a picture:
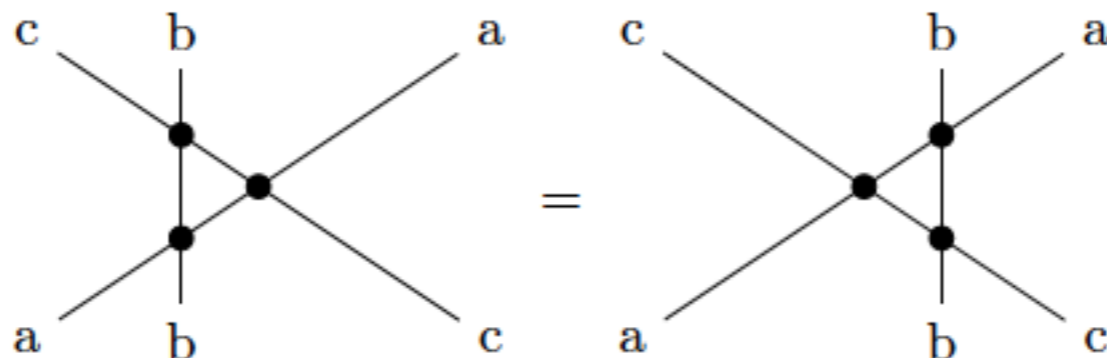
An important consequence for dynamics and thermalization is that there exists an infinite number of conserved quantities, although these become quite complicated for XXZ.
There the conserved *charges* and *currents* are (notation from T. Prosen), after spin and E,

$$q_{[3]} = -\Delta\sigma^{xyz} + \Delta\sigma^{yxz} - \Delta\sigma^{zxy} + \Delta\sigma^{zyx} + \sigma^{xzy} - \sigma^{yzx},$$

$$\begin{aligned}
q_{[4]} =\ & -2\Delta^2(\sigma^{zxxz} + \sigma^{zyyz}) - (2\Delta^2 + 2)(\sigma^{xx00} + \sigma^{yy00}) + 2\Delta(\sigma^{x0x0} + \sigma^{xyxy} \\
& - \sigma^{xyyx} + \sigma^{xzxz} + \sigma^{y0y0} - \sigma^{yxxy} + \sigma^{yxyx} + \sigma^{yzyz} + \sigma^{z0z0} + \sigma^{zxzx} + \sigma^{zyzy} \\
& - 2\sigma^{zz00}) - 2\sigma^{xzzx} - 2\sigma^{yzzy},
\end{aligned}$$

$$\begin{aligned}
q_{[5]} =\ & (4\Delta^3 + 14\Delta)(\sigma^{xyz00} - \sigma^{yxz00} + \sigma^{zxy00} - \sigma^{zyx00}) + 6\Delta^2(-\sigma^{x0yz0} + \sigma^{xyxxz} \\
& + \sigma^{xyyyz} + \sigma^{y0xz0} - \sigma^{yxxxz} - \sigma^{yxyyz} - \sigma^{zx0y0} + \sigma^{zxxxy} - \sigma^{zxxyx} - \sigma^{zxzyz} \\
& + \sigma^{zy0x0} + \sigma^{zyyxy} - \sigma^{zyyyx} + \sigma^{zyzxz}) + (10\Delta^2 + 8)(-\sigma^{xzy00} + \sigma^{yzx00}) \\
& + 6\Delta(\sigma^{x0zy0} - \sigma^{xy0z0} - \sigma^{xyxzx} - \sigma^{xyyzy} + \sigma^{xz0y0} - \sigma^{xzxxy} + \sigma^{xzxyx} + \sigma^{xzzyz} \\
& - \sigma^{y0zx0} + \sigma^{yx0z0} + \sigma^{yxxzx} + \sigma^{yxyzy} - \sigma^{yz0x0} - \sigma^{yzyxy} + \sigma^{yzyyx} - \sigma^{yzzxz} \\
& - \sigma^{z0xy0} + \sigma^{z0yx0} + \sigma^{zxzzy} - \sigma^{zyzzx}) - 6\sigma^{xzzzy} + 6\sigma^{yzzzx},
\end{aligned}$$

$$P_{[2]} = -2q_{[3]},$$

$$\begin{aligned}
P_{[3]} =\ & 2\Delta^2(\sigma^{zxxz} + \sigma^{zyyz}) - (2\Delta^2 + 2)(\sigma^{0xx0} + \sigma^{0yy0}) + 2\Delta(-2\sigma^{0zz0} - \sigma^{xyxy} \\
& + \sigma^{xyyx} - \sigma^{xzxz} + \sigma^{yxxy} - \sigma^{yxyx} - \sigma^{yzyz} - \sigma^{zxzx} - \sigma^{zyzy}) + 2\sigma^{xzzx} + 2\sigma^{yzzy},
\end{aligned}$$

$$\begin{aligned}
P_{[4]} =\ & (4\Delta^3 + 4\Delta)(\sigma^{0xyz0} - \sigma^{0yxz0} - \sigma^{zxy00} + \sigma^{zyx00}) + 4\Delta^2(-\sigma^{xyxxz} - \sigma^{xyyyz} \\
& + \sigma^{yxxxz} + \sigma^{yxyyz} + \sigma^{zx0y0} - \sigma^{zxxxy} + \sigma^{zxxyx} + \sigma^{zxzyz} - \sigma^{zy0x0} - \sigma^{zyyxy} \\
& + \sigma^{zyyyx} - \sigma^{zyzxz}) + (4\Delta^2 + 4)(-\sigma^{0xzy0} + \sigma^{0yzx0} + \sigma^{zxy00} - \sigma^{yzx00}) \\
& + 4\Delta(2\sigma^{0zxy0} - 2\sigma^{0zyx0} + \sigma^{xy0z0} + \sigma^{xyxzx} + \sigma^{xyyzy} - 2\sigma^{xyz00} - \sigma^{xz0y0} \\
& + \sigma^{xzxxy} - \sigma^{xzxyx} - \sigma^{xzzyz} - \sigma^{yx0z0} - \sigma^{yxxzx} - \sigma^{yxyzy} + 2\sigma^{yxz00} + \sigma^{yz0x0} \\
& + \sigma^{yzyxy} - \sigma^{yzyyx} + \sigma^{yzzxz} - \sigma^{zxzzy} + \sigma^{zyzzx}) + 4\sigma^{xzzzy} - 4\sigma^{yzzzx}, \qquad (82)
\end{aligned}$$

That is one reason we call Bethe-ansatz solvable models "integrable".

In classical physics with a finite number of degrees of freedom:

if we have enough independent integrals of motion, the system is not chaotic but rather integrable. (Example: a set of harmonic oscillators)

We said that Bethe ansatz models have an infinite number of conservation laws.

Are there "enough" to change the dynamics dramatically?

Turns out that even for XXZ, the answer was unclear until ~2011.

We will build up in these lectures to a quite general approach to use the conservation laws to study hydrodynamical behavior.

Non-equilibrium energy transport in XXZ

$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

1. Create two different temperatures in two disconnected, infinite 1D "leads".
2. Connect them by a finite region (e.g., one bond).
3. Evolve in time for as long as possible.

Is a steady-state heat current reached?

Is non-equilibrium (finite bias) thermal transport determined by linear-response thermal conductance?

We observe different outcomes, depending on integrability of the leads and whether the connected system is homogeneous.

Stefan-Boltzmann picture

Idea: the right lead is prepared at one temperature and the left lead at a different temperature.
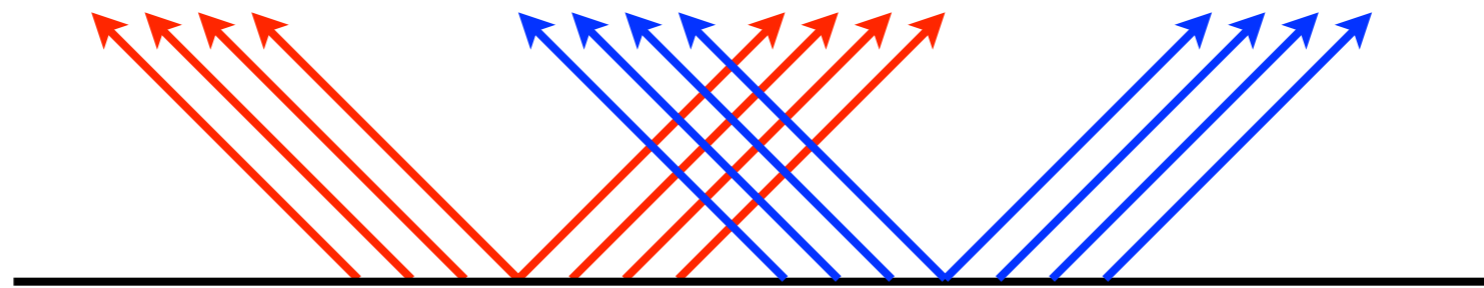
In a ballistic system like a CFT, there is no local temperature at x=0 at later times; rather the right-movers are at a different temperature than the left-movers. The thermal current is the difference between total radiation from left and right.
(Sotiriadis-Cardy, Bernard-Doyon)

Warmup: free bosons

We compute the right-moving energy current from a lead at temperature $T + dt$ and subtracting the left-moving energy current from a lead at T. Assume one-dimensional free bosons as in the Schwab et al. experiment mentioned above. Using k for momentum, we have that the total energy current (units of energy per time) is

$$J_E = J_E^R - J_E^L = \int_0^{\pi/a} \frac{dk}{2\pi} \left[f_{T+dt}(\hbar\omega_k) - f_T(\hbar\omega_k) \right] \hbar\omega_k v_k. \tag{1}$$

Here $v_k = d\omega_k/dk$ and $f_T(E)$ is the Bose factor $(e^{E/k_B T} - 1)^{-1}$. So

$$J_E = (dt) \int_0^{\omega_{max}} \frac{d\omega}{2\pi k_B T^2} \frac{e^{\hbar\omega/k_B T}}{(e^{\hbar\omega/k_B T} - 1)^2} \hbar^2 \omega^2. \tag{2}$$

Here ω_{max} is the highest phonon frequency. If we assume that the temperature is small compared to this, so that $x = \hbar\omega/k_B T$ runs from 0 to infinity, then we obtain (note that we need to multiply by $(k_B T/\hbar)^3$)

$$J_E = \frac{k_B{}^2 T}{2\pi\hbar} (dt) \int_0^\infty dx \, \frac{x^2 e^x}{(e^x - 1)^2}. \tag{3}$$

The dimensionless integral gives $\pi^2/3$, so

$$G_0 = \frac{J_E}{dt} = \frac{\pi^2 k_B{}^2 T}{3h}. \tag{4}$$

An interesting fact about the thermal conductance G_0 is that it is the same for bosons or fermions (or indeed anyons), unlike charge transport. The Schwab et al. experiment observed one thermal conductance quantum G_0 for each low-temperature phonon mode.

Dissipationless transport

When is there a nonzero Drude weight *D*?

$$\sigma(\omega) = D\delta(\omega) + \ldots$$

Two easy examples:

I. Superconductors (transport by condensate)

II. Part of the current is conserved: Mazur lower bound

$$D = \frac{1}{2LT} \lim_{t\to\infty} \langle J(t)J(0) \rangle \geq \frac{1}{2LT} \sum_k \frac{\langle JQ_k \rangle^2}{\langle Q_k{}^2 \rangle}$$

$$\sigma(\omega) = D\delta(\omega) + \dots$$

$$D = \frac{1}{2LT} \lim_{t \to \infty} \langle J(t)J(0) \rangle \geq \frac{1}{2LT} \sum_k \frac{\langle JQ_k \rangle^2}{\langle Q_k^2 \rangle}$$

What about "integrable" models with an infinite number of conserved local quantities, none of which gives a lower bound?

Actually this happens quite often in 1D--simplest case is spinless interacting fermions (XXZ model in zero magnetic field).

$$H = \sum_i \left[J_{xx}(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + \Delta S_i^z S_{i+1}^z + h S_i^z \right]$$

The Drude weight is easy to calculate and nonzero at T=0. 20+ years of efforts to calculate it (or even prove that it is nonzero) at T>0, h=0, by either analytical or numerical methods.

(cf. Sirker, Pereira, Affleck, PRB 2011)
(Thermal Drude weight is easier, for reason said later: found by Klumper and Sakai)

Drude weight progress, from 2011

$$\sigma(\omega) = D\delta(\omega) + \dots$$

$$D = \frac{1}{2LT} \lim_{t\to\infty} \langle J(t)J(0)\rangle \geq \frac{1}{2LT} \sum_k \frac{\langle JQ_k\rangle^2}{\langle Q_k{}^2\rangle}$$

Prosen: there is an iterative process to construct a nonlocal quantity that gives a lower bound that depends non-analytically on anisotropy, with cusps at Δ=cos(π/n). (PRL 2011)
(subsequent work generalizing this result: Ilievski-Prosen, …)

Karrasch-Bardarson-JEM: The Drude weight can be calculated numerically for all but the lowest temperatures at positive Δ, and essentially all temperatures at negative Δ.

The lower bound appears to saturate the full value at the cusps.

"Non-equilibrium expansions"

Almost everything that follows will be specific to 1D systems, where we have special analytical and numerical tools.

A. "Point current": A natural question about electrons is to compute the (charge or energy) current through a *point*.
Many beautiful works, especially in quantum impurity models. Older works on translation-invariant case: Sotiriadis-Cardy, Bernard-Doyon, Karrasch-Ilan-JEM
Solved (not too easily) by new hydrodynamical methods discussed later

$$T_1 \qquad\qquad T_2$$
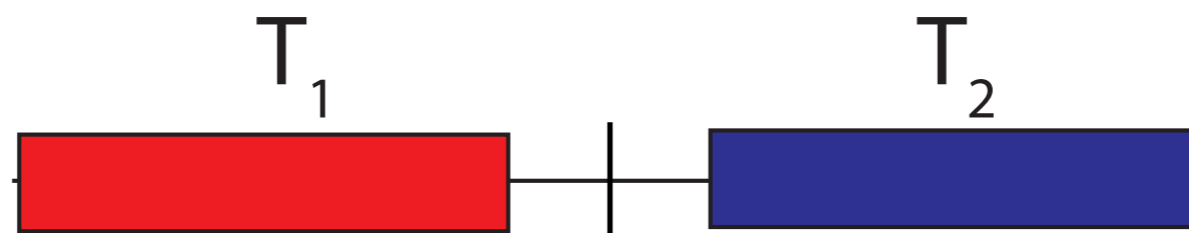
B. "Expansion": with atoms, it is more natural to *image* the full distribution of atoms (or conceivably energy). Two nice features:
1. The most natural model of 1D interacting atoms, the Lieb-Liniger model, is integrable.

2. For charge current in Lieb-Liniger, or energy current in the XXZ model, there is conservation of the spatially integrated current, which turns out to have remarkable consequences: expansion is controlled by a form of non-equilibrium thermodynamics.

"Non-equilibrium expansions"

Lieb-Liniger model = Bose gas with delta-function interactions

Thermodynamics (Yang and Yang) interpolates from free bosons to free fermions as interaction strength increases.

Tonks–Girardeau gas of ultracold atoms in an optical lattice

One experimental example:
Nature, 2004

Belén Paredes[1], Artur Widera[1,2,3], Valentin Murg[1], Olaf Mandel[1,2,3],
Simon Fölling[1,2,3], Ignacio Cirac[1], Gora V. Shlyapnikov[4],
Theodor W. Hänsch[1,2] & Immanuel Bloch[1,2,3]

Originally, it was only possible to measure momentum-space distributions; now several groups have achieved imaging of individual sites of an optical lattice (Greiner, Chin, Bloch, …).

How to quantify an expansion?

There is a great deal of theoretical work, especially on the Lieb-Liniger case (~100 papers; Stringari, Caux-Konik, Gangardt,...). Three time scales:

Short time: initial transient, which we ignore

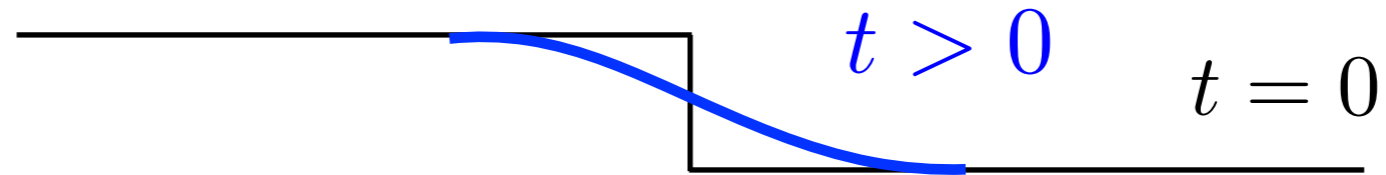Intermediate time: (becomes infinite if reservoirs are infinite)
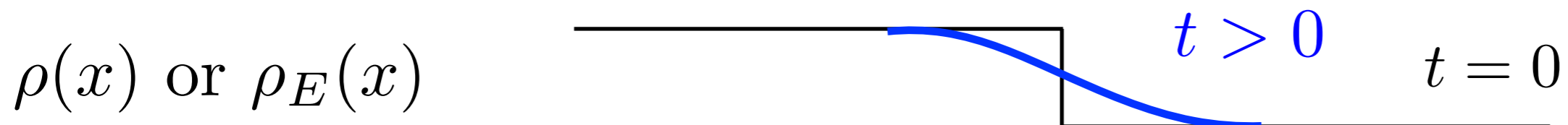
$\rho(x)$ or $\rho_E(x)$ $\qquad\qquad\qquad\qquad\qquad$ $t > 0$ $\qquad$ $t = 0$

In a ballistic (nonzero Drude weight system), the first moment increases quadratically in time.

Long time: long-time expansion into vacuum can be analyzed relatively simply in BA because asymptotic density is zero. (cf. Heidrich-Meisner)

How to quantify an expansion?

$\rho(x)$ or $\rho_E(x)$

$t > 0$

$t = 0$

In a ballistic (nonzero Drude weight system), the first moment increases quadratically in time.
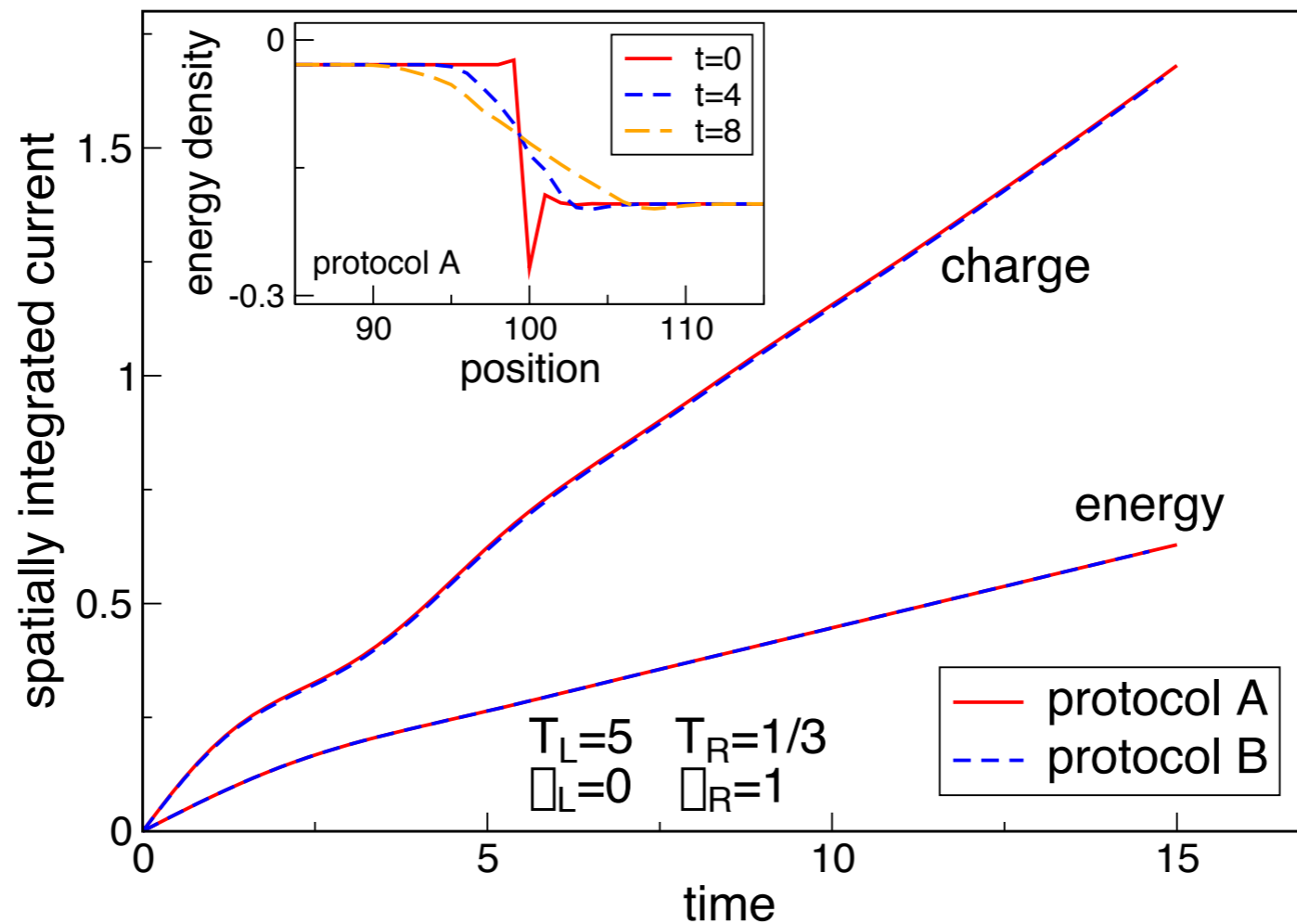
How to quantify an expansion?

At $t = 0$, prepare two leads at (μ_1, T_1) and (μ_2, T_2). The initial state on the boundary between the two leads does not matter after some initial transient. We can quantify the expansion by the time dependence of the first moment of particle density (or similarly for energy)

$$M_1 = \int_{-\infty}^{\infty} n(x) x \, dx. \tag{1}$$

The continuity equation relates density and current:

$$\partial_t n + \partial_x j = 0. \tag{2}$$

Now

$$\partial_t M_1 = \int_{-\infty}^{\infty} x \partial_t n \, dx = -\int_{-\infty}^{\infty} x \partial_x j \, dx = \int_{-\infty}^{\infty} j \, dx, \tag{3}$$

where in the integration by parts we have assumed $j(x)$ vanishes rapidly at $x = \pm\infty$. We will make considerable use of the fact that in many problems of interest

$$\left[\int_{-\infty}^{\infty} j \, dx, H \right] = 0. \tag{4}$$

Now go back to basic ideas of equilibration...

Linear and non-linear response: point current

For the final H a homogeneous integrable model, there is numerically a "generalized Stefan-Boltzmann law" to high accuracy (to be defined in a moment), which led us to the idea that this picture can be made exact for expansions.

For final H homogeneous and non-integrable, we do not observe a steady state. We believe that the temperature gradient is decreasing and Fourier's law is setting in, but cannot access very long times.

For final H inhomogeneous, there can be a steady state if the leads are integrable and J is a function of both temperatures jointly.

We can see the onset of the nontrivial power-laws in tunneling between Luttinger liquids as temperature is lowered.

Linear and non-linear response

When the finite system is homogeneous and the leads have a nonzero Drude weight, we found a while ago:
(Karrasch, Ilan, Moore, PRB 2013)

1. there is a steady state;
2. there is approximately a function f such that

$$\lim_{t \to \infty} \langle J_E(n,t) \rangle = f(T_L) - f(T_R)$$

In other words, linear response $G = \partial_T f$
is sufficient to determine non-linear response.

For a CFT (Sotiriadis-Cardy, Bernard-Doyon), this was known, and f goes as T^2 for small T, 1/T for large T. ("1D black-body")

Linear and non-linear response

2. there is a function *f* such that

$$\lim_{t \to \infty} \langle J_E(n,t) \rangle = f(T_L) - f(T_R)$$

Makes testable predictions, e.g.,

$$J_E(T_1 \to T_3) = J_E(T_1 \to T_2) + J_E(T_2 \to T_3)$$
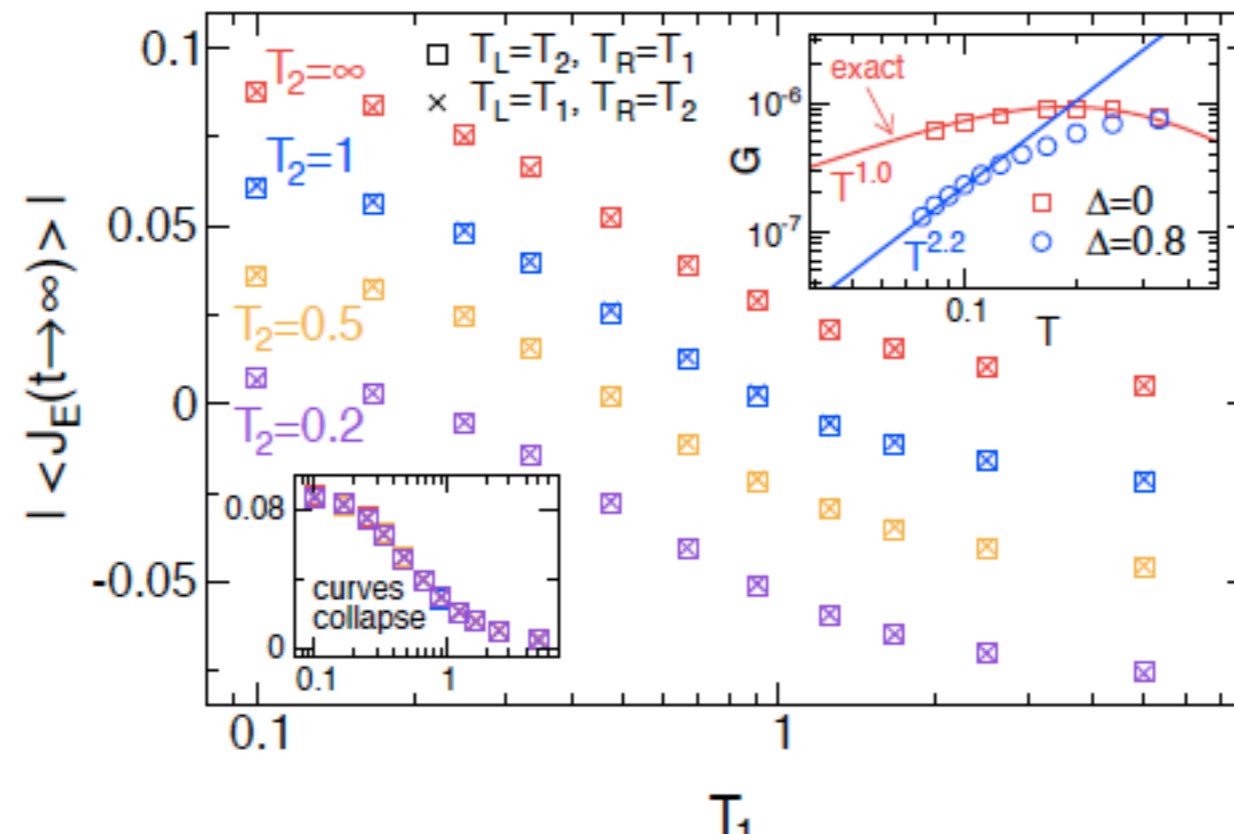
Alternative hydrodynamics: from more conservation laws

Energy transport in XXZ is special: because integrated energy current commutes with the Hamiltonian, we expect an additional continuity equation

$$\partial_t \rho_E + \partial_x j_E = 0, \quad \partial_t j_E + \partial_x Q_4 = 0$$

This is familiar from Lorentz-invariant models (cf. Bernard-Doyon): the energy current is itself a conserved density, by symmetry of the stress-energy tensor.

In other words, energy transport in XXZ is like that in a Lorentz-invariant model, and expansion occurs according to a expansion potential or generalized pressure Q4. ("cyclic law")

The "q-boson" model is a lattice example of similar physics for *charge* current rather than energy current.
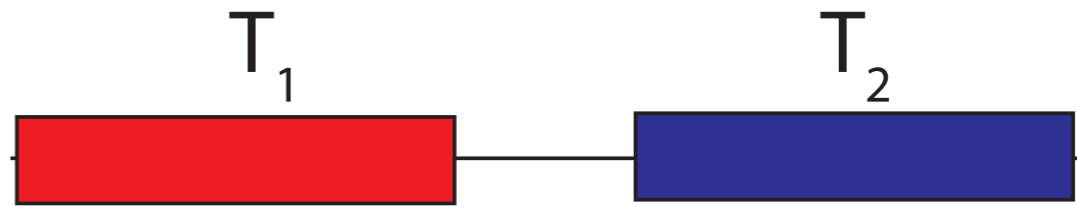
When the cyclic law is exact
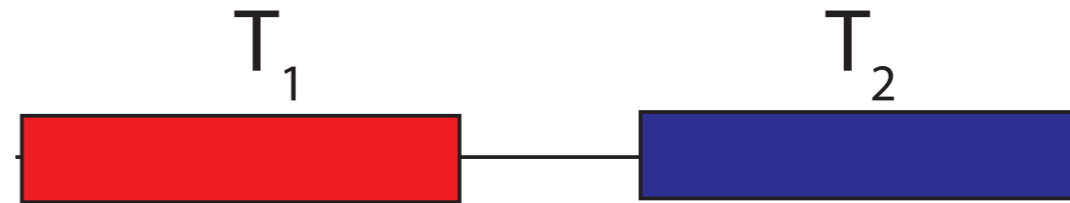
Global energy current conservation connects what happens at 3 interfaces
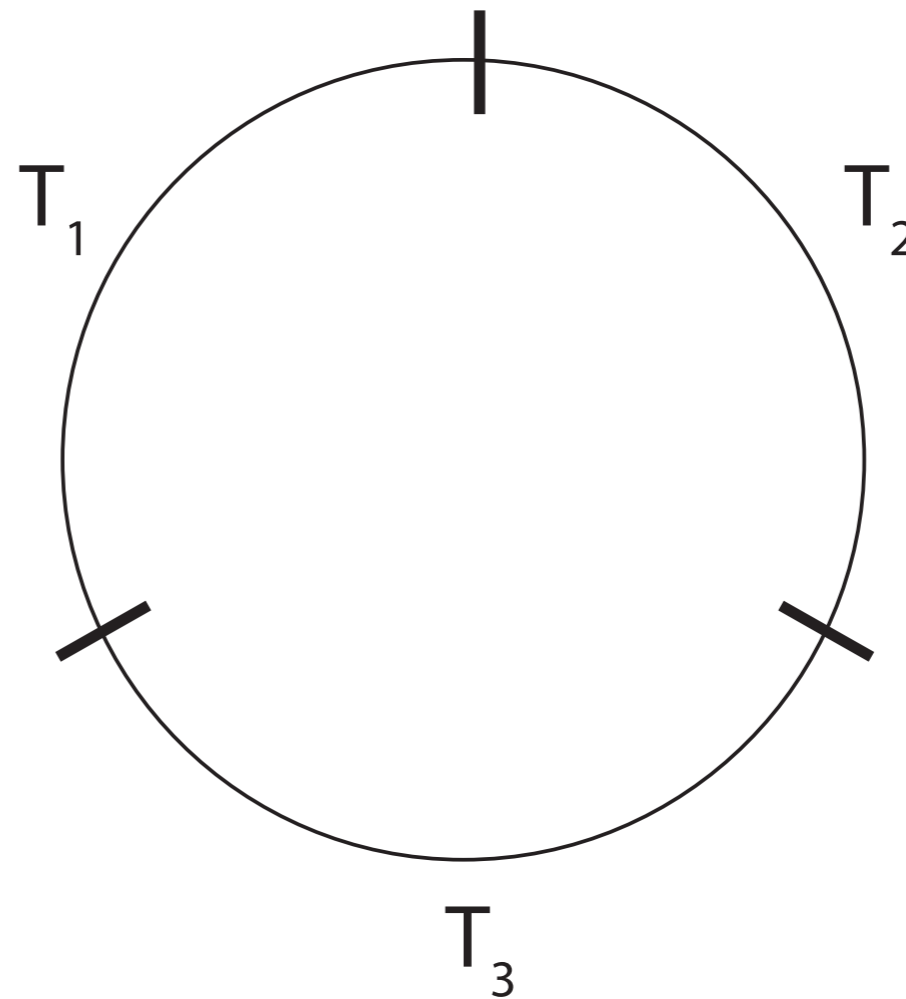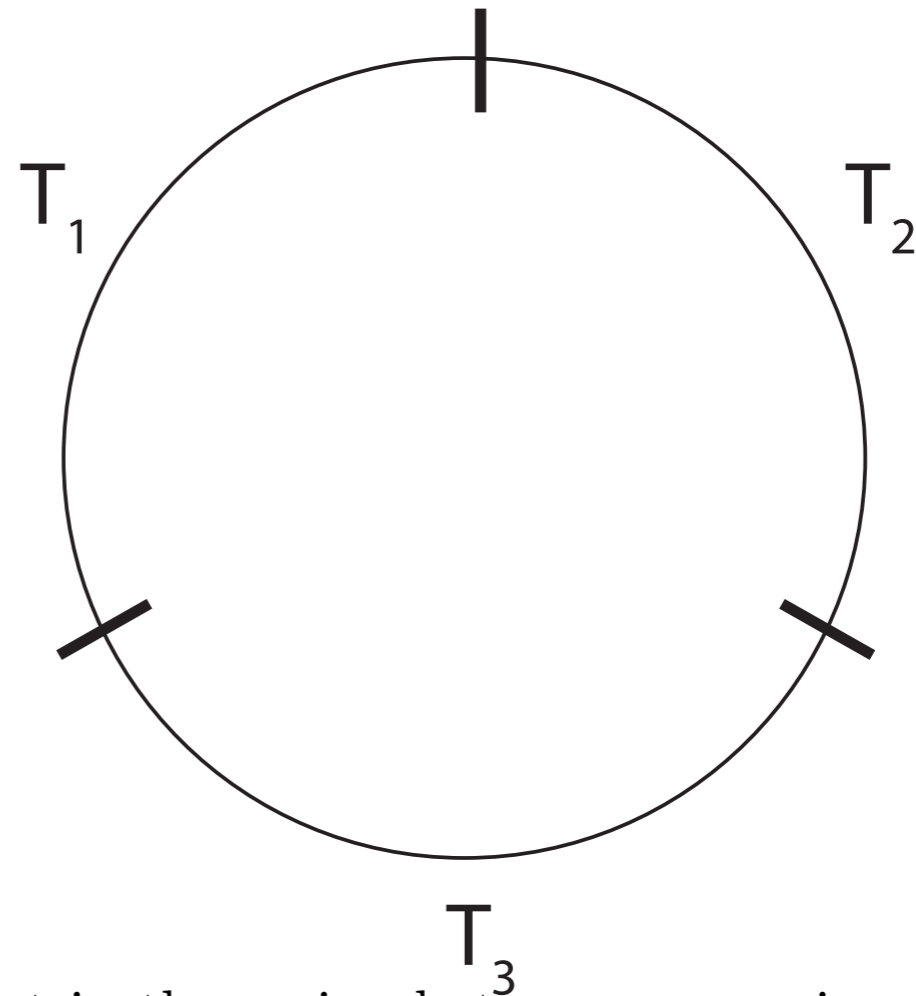
Cyclic form of existence of f-function:

$$j_E^\infty(T_1 \to T_2) + j_E^\infty(T_2 \to T_3) + j_E^\infty(T_3 \to T_1) = 0$$

When the cyclic law is exact

Global energy current conservation links spatial *integrals of current* across each interface, not steady-state current

In a CFT, probably same thing since there is a unique velocity, but they are not obviously connected in general.

Let j_{12} be the spatially integrated current in the region between reservoirs 1 and 2. Then global current conservation means

$$[H, j_{12} + j_{23} + j_{31}] = 0. \tag{1}$$

But this cyclic law implies that at every time j_{12} is of the form $f(t, \mu_1, T_1) - f(t, \mu_2, T_2)$.

Linear response: Drude weight

The cyclic law means that linear-response is enough to predict non-equilibrium. The increase of the moment at linear-response can be related to the Drude weight: focusing for the moment on energy current and a purely thermal gradient, we find

$$\partial_t^2 M_1^{\text{th}} = \partial_t \int_{-\infty}^{\infty} j\, dx = G^{\text{th}} \times (\Delta T). \tag{1}$$

This can be checked numerically by comparing the rate of expansion to the thermal Drude weight of the XXZ model computed by Klümper and Sakai.

Actually this connection exists in LR even without current conservation: example is *charge* current in XXZ model
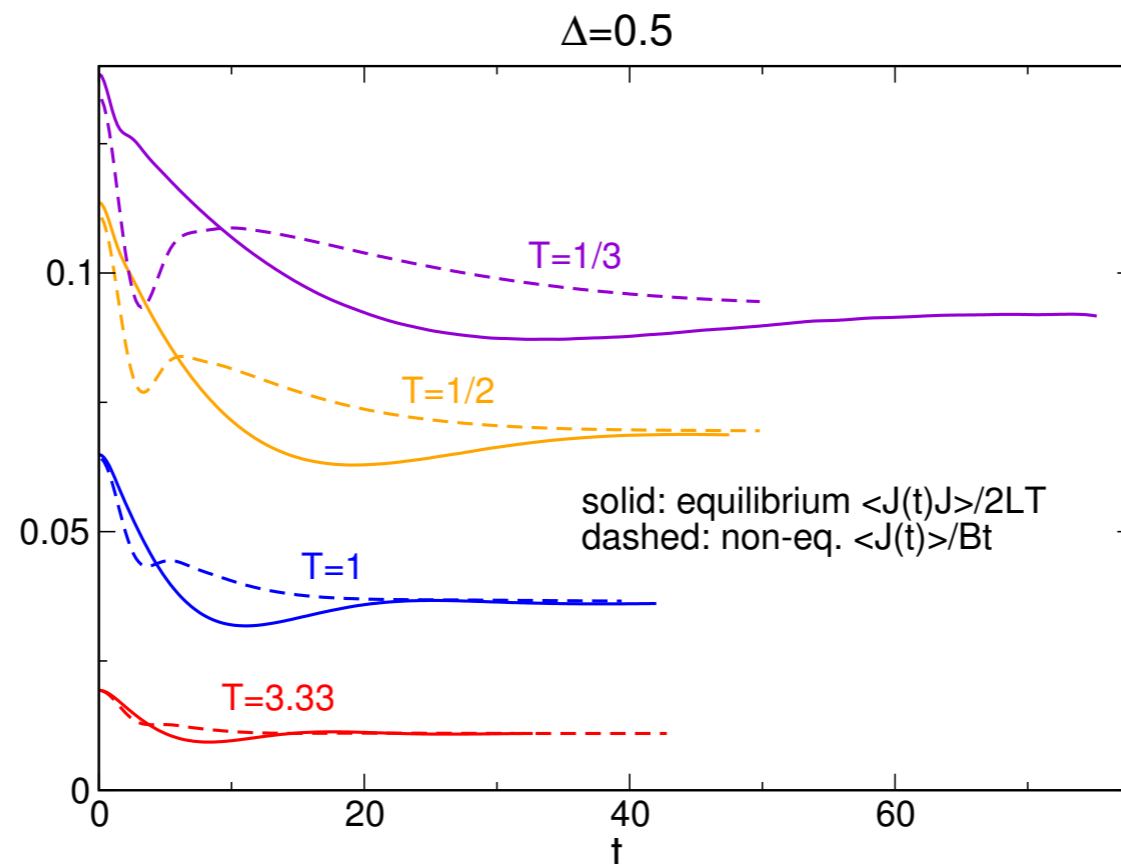
Exact expansions in XXZ

Claim: using some special properties of energy current in the XXZ model, it is possible to obtain exact far-from-equilibrium results for the two-reservoir quench.

Energy transport in XXZ is special: because integrated energy current commutes with the Hamiltonian, we expect an additional continuity equation

$$\partial_t \rho_E + \partial_x j_E = 0, \quad \partial_t j_E + \partial_x Q_4 = 0$$

This structure leads to a non-equilibrium version of Maxwell relations for "expansion potentials" (R. Vasseur, C. Karrasch, and JEM, PRL 2015).

Here, we use these exact results as a check on a more general hydrodynamical approach.

Exact far-from-equilibrium energy expansion in XXZ
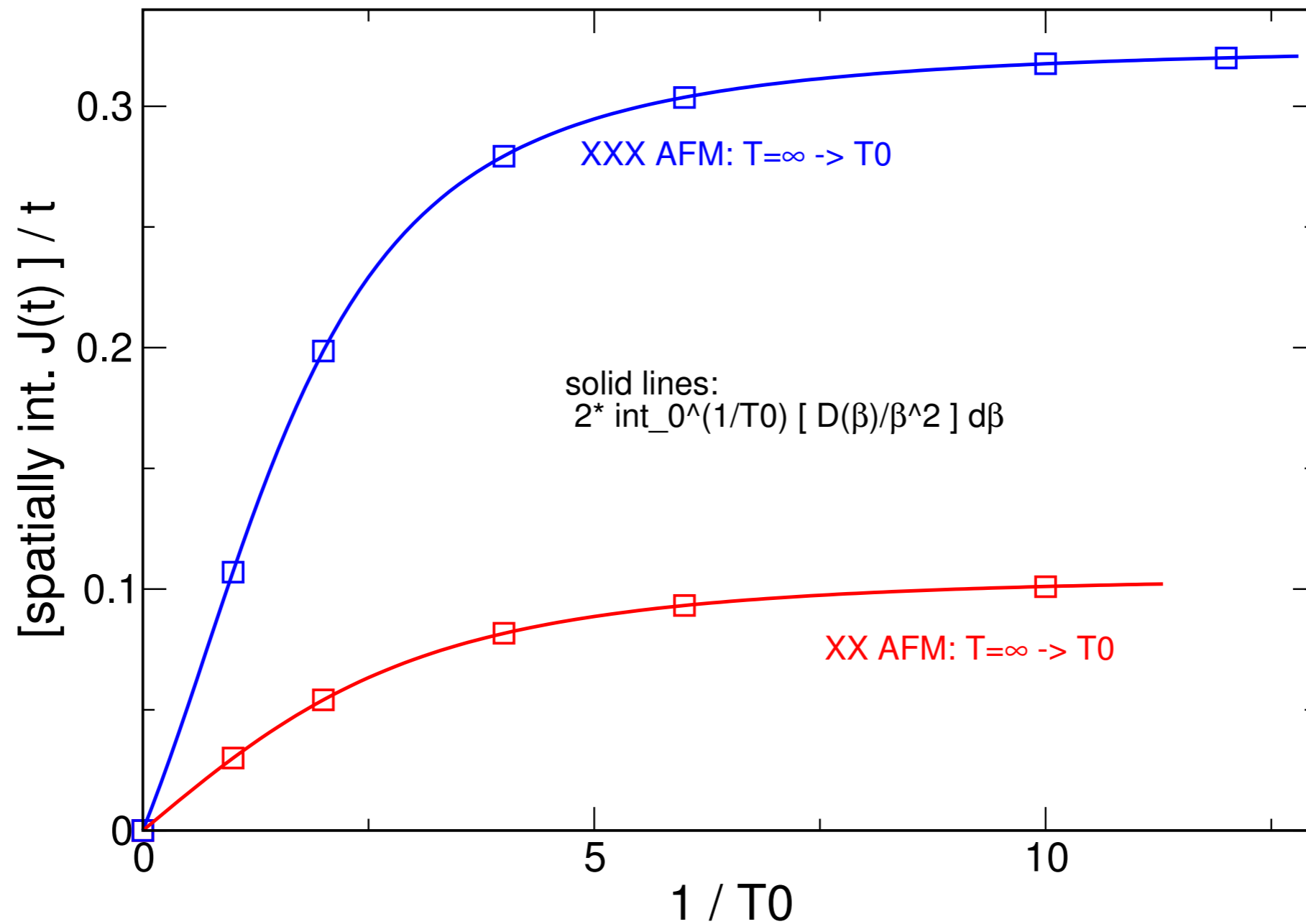
Comparison is rate of increase of energy current
versus temperature integral of Drude weight

Recent progress

The above is a very specialized trick to get some exact results for one model. Can one develop a more general approach for hydrodynamics in integrable models?

Yes: recent work on (almost certainly) asymptotically exact solutions for this initial condition by

> Castro-Alvaredo/Doyon/Yoshimura, PRX 2016 (Lieb-Liniger)
> Bertini/Collura/De Nardis/Fagotti, PRL 2016 (XXZ)

1. Key steps of approach (in one language)
   Physical picture of kinetic theory (Boltzmann equation): same *classical* spirit as El and Kamchatnov, PRL 2005

2. Does it pass XXZ numerical comparisons that previous similar ansatzes failed?

Our starting point: think of particles in an integrable model as streaming (with self-consistent velocity) but not colliding

"Bethe-Boltzmann equation"

$$\partial_t \rho(k, x, t) + \partial_x \left[v(\{\rho(k', x, t)\}) \rho(k, x, t) \right] = 0$$

No collision term since quasiparticles retain their identity; however, they modify each other's velocities via phase shifts

This type of equation was written down in various older contexts: I think the most relevant for the models here is

Kinetic Equation for a Dense Soliton Gas

G. A. El[1,*] and A. M. Kamchatnov[2,†]

[1]*Department of Mathematical Sciences, Loughborough University, Loughborough LE11 3TU, United Kingdom*
[2]*Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow Region, 142190, Russia*

We propose a general method to derive kinetic equations for dense soliton gases in physical systems described by integrable nonlinear wave equations. The kinetic equation describes evolution of the spectral distribution function of solitons due to soliton-soliton collisions. Owing to complete integrability of the soliton equations, only pairwise soliton interactions contribute to the solution, and the evolution reduces to a transport of the eigenvalues of the associated spectral problem with the corresponding soliton velocities modified by the collisions. The proposed general procedure of the derivation of the kinetic equation is illustrated by the examples of the Korteweg–de Vries and nonlinear Schrödinger (NLS) equations. As a simple physical example, we construct an explicit solution for the case of interaction of two cold NLS soliton gases.
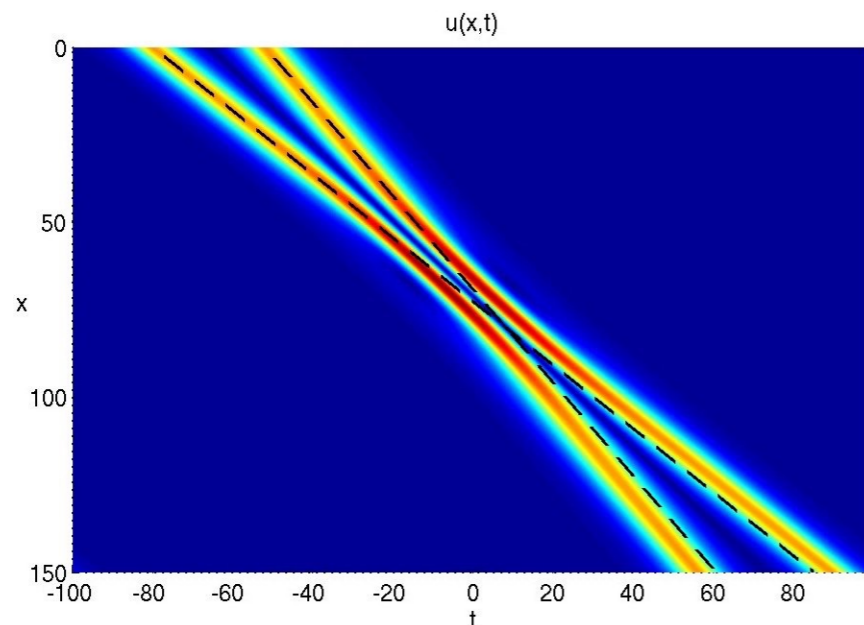
Why Boltzmann equation gets modified in (classical or quantum) integrable systems

Solitons/particles pass through each other even in dense system; no randomization of momentum and no collision term.

However, there is an interaction:

Classical

Solitons delay each other

u(x,t)

so velocity depends on other solitons at spacetime point

Quantum

Phase shift from Bethe equations

but semiclassically an energy-dependent phase shift is also just a time delay (Wigner)

$$\tau = 2\hbar \frac{d\delta}{dE}$$

This derives the equation directly for NLS and KdV solitons as a kinetic theory;
An alternate route is via hydrodynamical (Euler) equations.

Different integrable models just differ in the velocity form: three examples are

El and Kamchatnov for NLS solitons (dense generalization of Zakharov 1971):

$$s(\alpha, \gamma) = -4\alpha + \frac{1}{2\gamma} \int_{-\infty}^{\infty} \int_{0}^{\infty} \ln \left| \frac{\lambda - \bar{\mu}}{\lambda - \mu} \right|^2 f(\xi, \eta)$$
$$\times [s(\alpha, \gamma) - s(\xi, \eta)] d\xi d\eta.$$

Castro-Alvaredo/Doyon/Yoshimura for Lieb-Liniger

Bertini et al. for XXZ

arXiv:1605.09790 [pdf, other]
Transport in out-of-equilibrium XXZ chains: exact profiles of charges and currents
Bruno Bertini, Mario Collura, Jacopo De Nardis, Maurizio Fagotti

How do kinetic theory (Boltzmann equation) and
hydrodynamics (Euler equations) give the same description?

Integrable hydrodynamics

Simplest case is Bose gas with delta-function repulsion.

GGE = Generalized Gibbs Ensemble = includes an infinite number of conservation laws:

$$\int \rho(k, x, t)\, dk = n(x, t)$$

$$\int k\rho(k, x, t) = mv(x, t)$$

$$\int k^2 \rho(k, x, t) = 2m\epsilon(x, t)$$

$$\vdots$$

$$\int k^n \rho(k, x, t)$$

Kinetic theory: might as well work with

$$\rho(k, x, t)$$

instead of its moments.

GGE (conserved quantities) is equivalent to distribution function, rather than containing less information.

Somewhat surprising for XXZ, where the charges are quite complicated; somehow Takahashi's old TBA and Bertini et al. backflow leads to Drude weight, i.e., it "knows about" the deep quasilocal charges.

Summary of when this is useful

Normal fluid:

Initial state → Local equilibrium → Hydrodynamics

Integrable fluid:

Initial state → Local GGE → Boltzmann/hydrodynamics

So, for non-local-GGE initial conditions, still need to solve difficult "quench" problem, at least locally.

Two-reservoir problem already solved in 2016 papers: solution is function of one variable (x/t).

Let's look for full (x,t) solutions: are quantum dynamics really describable by these classical particle equations?

Mathematical properties of solutions ("semi-Hamiltonian structure"): Bulchandani, 2017, arXiv, as for NLS

Take XXZ in zero magnetic field. Make a spatial variation of initial temperature. Watch the energy spread out in time.

Note: nonzero temperature is required for coarse-graining time to be finite, according to basic principle that systems can't relax faster than hbar/kT. (Hence more physically generic than T=0 or Bethe-Bethe comparisons.)
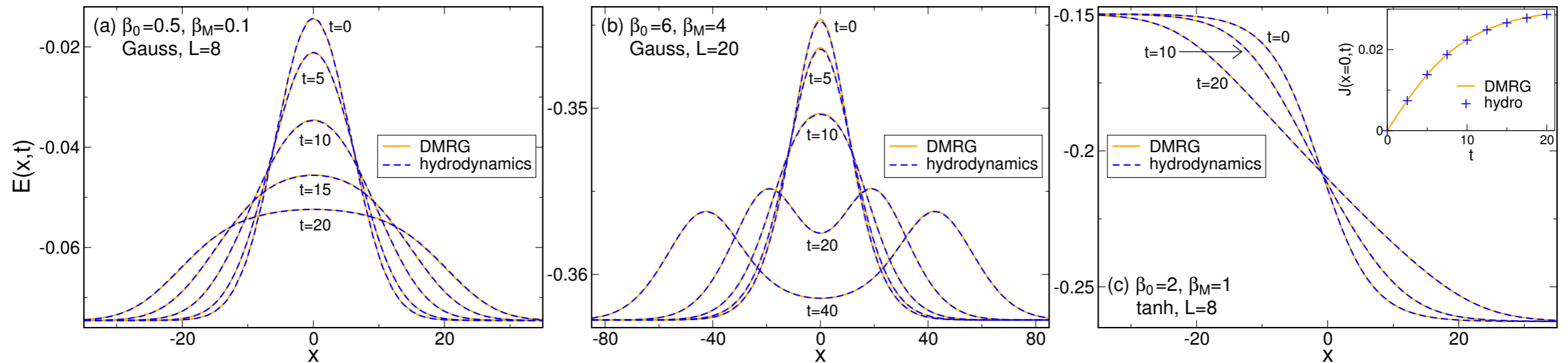
These are comparisons for interacting spinless fermions (XXZ) between backwards Euler solution of Bethe-Boltzmann and microscopic DMRG simulations.
(figure from "Solvable quantum hydrodynamics", V. Bulchandani, R. Vasseur, C. Karrasch, and JEM, arXiv April 2017)

(b) $\beta_0=6$, $\beta_M=4$
Gauss, L=20

t=0

t=5

t=10

DMRG
hydrodynamics

t=20

t=40
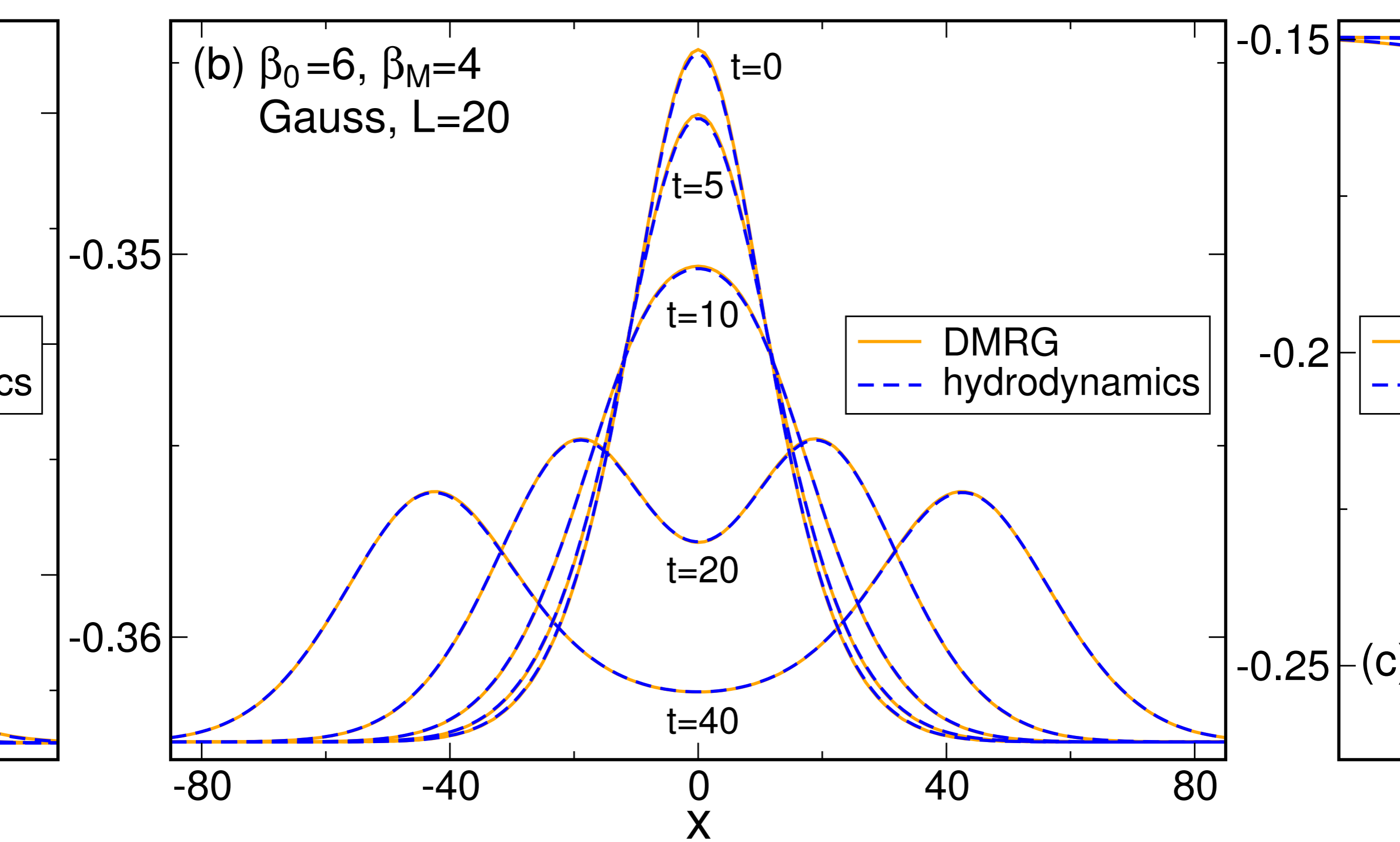
Breaking integrability

1. The generic state of 1D metals is a "Luttinger liquid". We can realize this by adding integrability-breaking terms that retain lattice translati invariance. Conclusion: there are at least *two* different mechanisms for adjustable power-laws in LL transport.

2. We can break translation invariance and integrability by adding a harmonic trap to an integrable system (leaving aside those like Calogero that retai integrability in a trap; see Abanov-Wiegmann, Kulkarni-Polychronakos).

3. What about electrons in real materials and d>1?

1D strategy: start with simple models

$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

(Clean XXZ chain + random z-directed Zeeman field)

$$h_i \in (-\Delta, \Delta)$$

Equivalent by Jordan-Wigner transformation to spinless fermions with nearest-neighbor interactions.

$$H = -t \sum_i \left(c_{i+1}^\dagger c_i + h.c. \right) + V \sum_i n_i n_{i+1} + \sum_i \mu_i n_i$$

Advantages:

1. "Solvable" (integrable) without random field. Can add a staggered field to break integrability while keeping translation invariance.
2. Can check predictions with DMRG/matrix product numerics.

One example of the Luttinger liquid idea

Consider the XXZ model when it describes a gapless, linearly dispersing system. ($|\Delta| < 1$).

Some things are *independent of the precise value of interaction:*

for example, the free energy at finite temperature is

$$f = \frac{F}{L} = f_0 - \frac{\pi}{6}c(kT)^2\hbar v$$

with central charge c=1 everywhere along this line.

Actually interactions are marginal, not irrelevant, and there is a line of critical points that differ in several transport measurements.
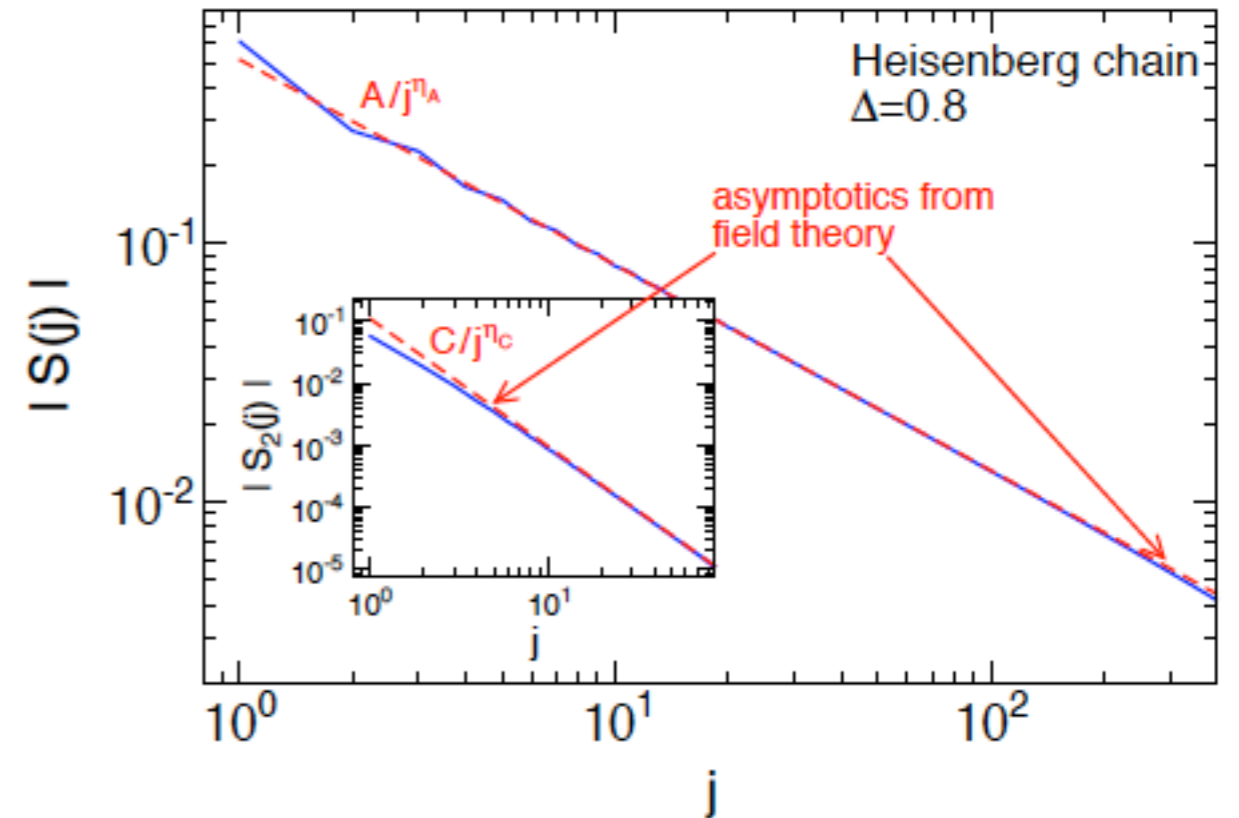
These are Luttinger liquids with varying Luttinger parameter.

Tests of Luttinger liquid behavior in the XXZ model
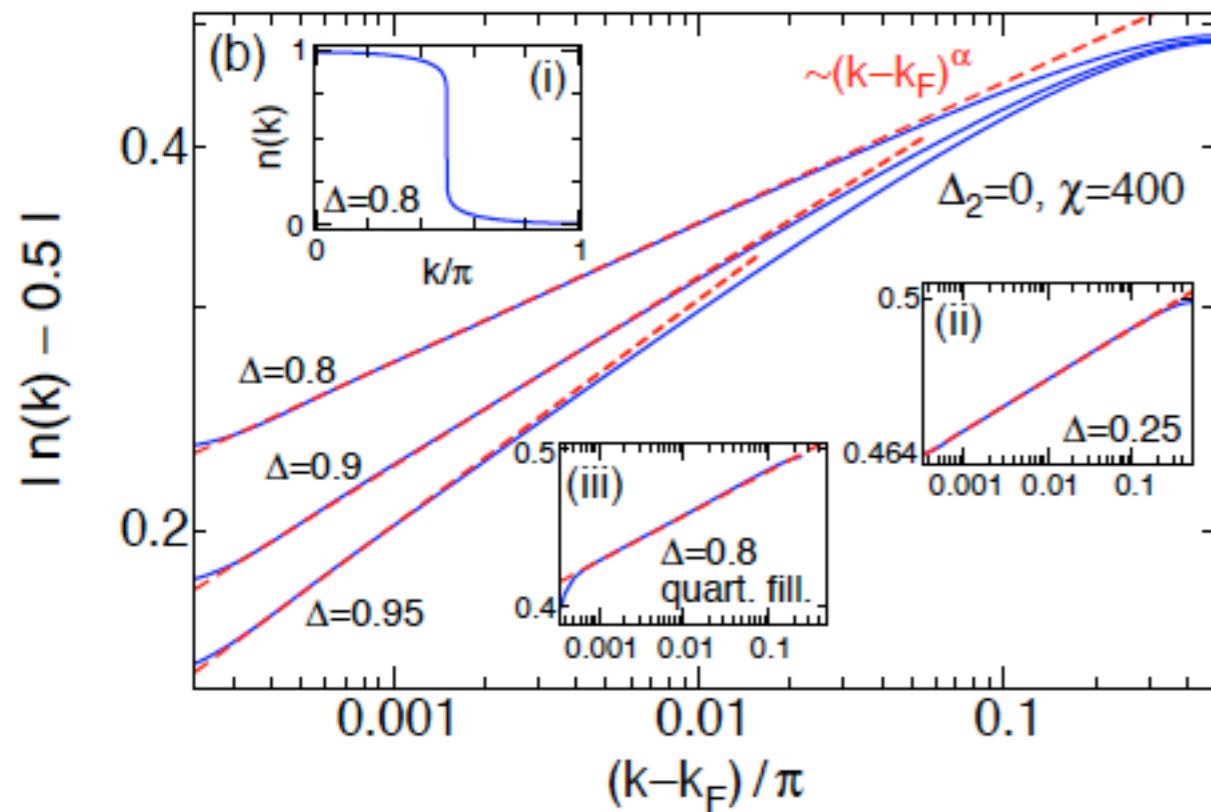
Momentum distribution n(k)

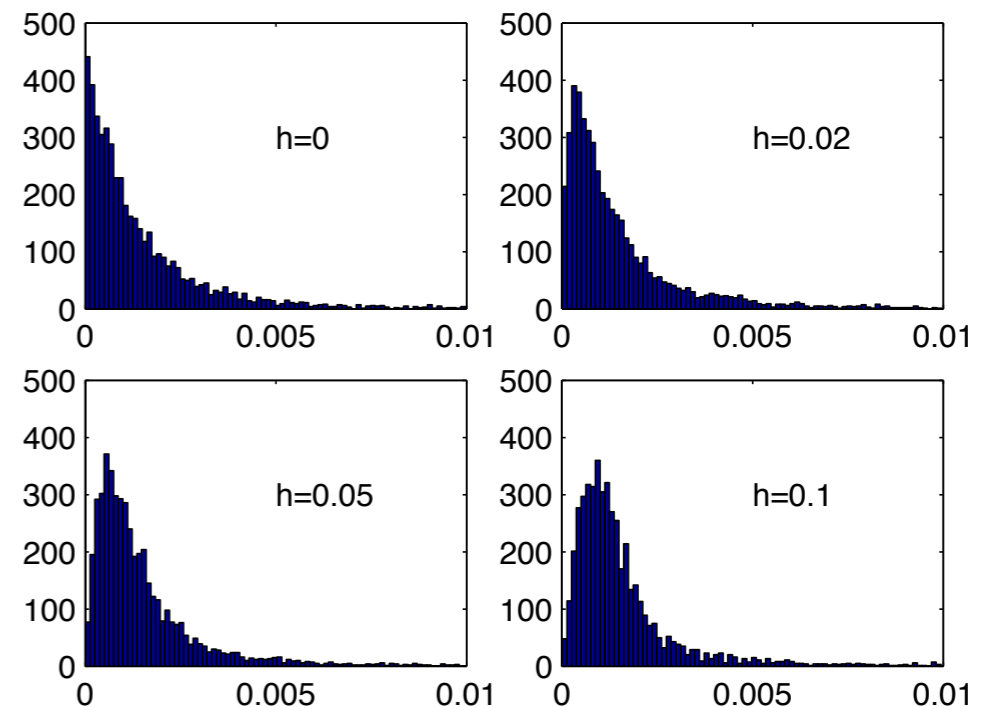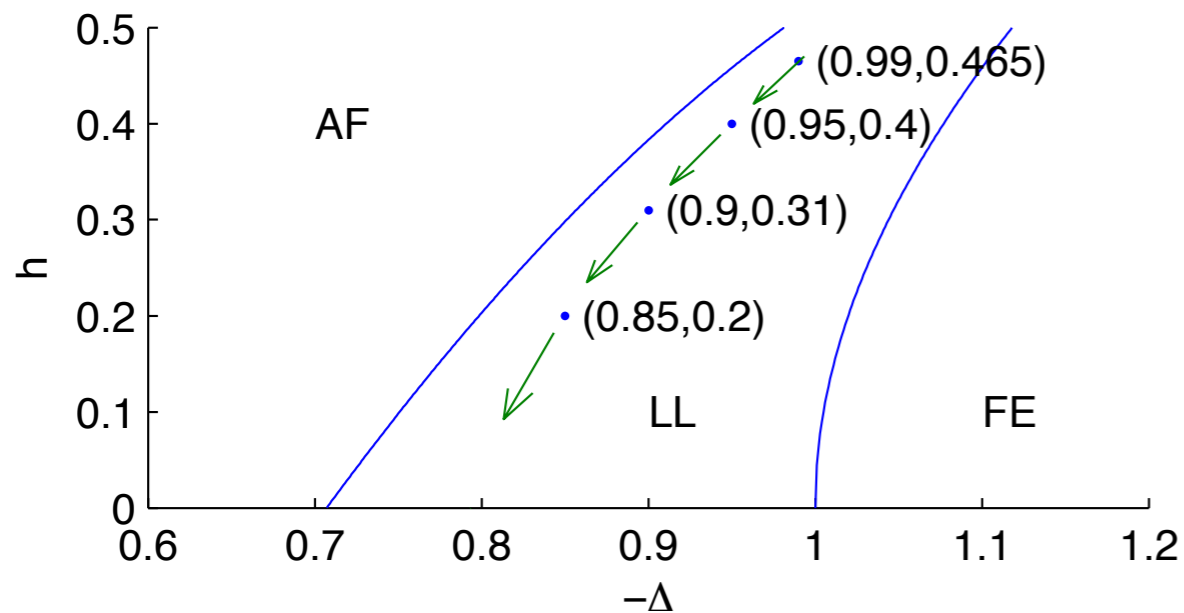Check of leading staggered and uniform correlators against Lukyanov and Terras

(C. Karrasch and JEM, PRB)

Current ground-state applications moving to 2D: FQHE, spin liquids, …
Next: try to solve an open problem of dynamical properties at finite temperature.

Staggered field non-integrability

$$H = \sum_{i=1}^{L} \left[S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z + (-1)^i h S_i^z \right]$$
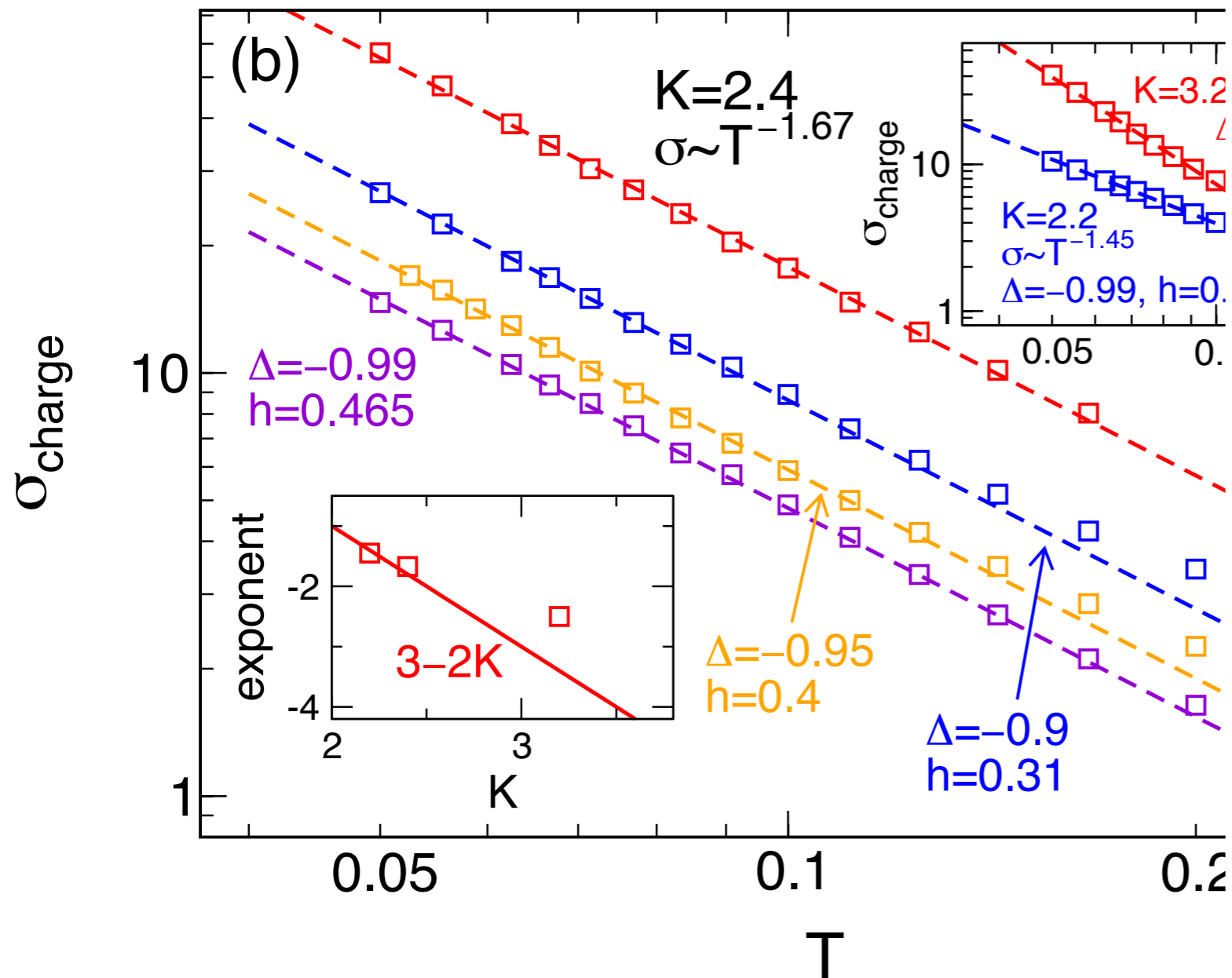

In one region, of the phase diagram, h is irrelevant (system remains Luttinger liquid), and we can track RG flow

Level statistics become Wigner-Dyson (level repulsion) rather than Poisson

Argument for Poisson statistics: two nearby states are likely to be in different symmetry sectors, and hence do not repel each other as they are not mixed by a perturbation.

"Conventional" conductivity scaling

$$\sigma = \lim_{t_M \to \infty} \lim_{L \to \infty} \frac{1}{LT} \operatorname{Re} \int_0^{t_M} \langle J(t) J(0) \rangle \, dt.$$

(b)

K=2.4
$\sigma \sim T^{-1.67}$

K=3.2

K=2.2
$\sigma \sim T^{-1.45}$
$\Delta = -0.99$, h=0.

$\Delta = -0.99$
h=0.465

σ_{charge}

exponent

3−2K

$\Delta = -0.95$
h=0.4

$\Delta = -0.9$
h=0.31

K

T

For K not too large, linear prediction is self-consistent and power-laws are observed that are consistent with bosonization predictions.

Conductivity diverges at low temperature as the integrability-breaking perturbation is irrelevant.

(Huang, Karrasch, Moore PRB 2013)

What if we break integrability by imposing an external global potential?

The hard-rod gas in a harmonic trap is equivalent to N one-dimensional harmonic oscillators with hard-core repulsive interaction. The Hamiltonian reads

$$H = \sum_{j=1}^{N} \left[\frac{1}{2} p_j^2 + V(x_j) \right] + \sum_{j<k} U(x_j - x_k) \qquad (1)$$

with

$$V(x) = \frac{1}{2} \omega^2 x^2 \,,\ U(\delta x) = \begin{cases} 0 & |\delta x| > a \\ \infty & |\delta x| \leq a \,, \end{cases} \qquad (2)$$

where $a > 0$ denotes the rod length, and x_j and p_j denote positions and momenta (we set $m = 1$). Upon re-scaling time as $t \to t\omega$, we may set $\omega = 1$ without loss of generality. Starting from a configuration such that $x_{j+1} - x_j \geq a$, $j = 1, \dots, N - 1$, the gas evolves as N decoupled oscillators, until the next collision (i.e., $x_{j+1} - x_j = a$ for some j) in which the rods j and $j+1$ exchange their velocities spontaneously. Such a dynamics can be efficiently and exactly simulated. There are two integrable limits. Upon removing the trap, one recovers the usual hard-rod gas. Its momentum distribution is conserved and its dynamics map to those of N independent particles. Meanwhile, in the limit of vanishing rod length $a = 0$, we obtain N decoupled harmonic oscillators. Yet, in the presence of both trap and interaction, we find no other conserved quantities besides the total energy and the center-of-mass energy which we set to 0.

A quantum Newton's cradle

Toshiya Kinoshita[1], Trevor Wenger[1] & David S. Weiss[1]

It is a fundamental assumption of statistical mechanics that a closed system with many degrees of freedom ergodically samples all equal energy points in phase space. To understand the limits of this assumption, it is important to find and study systems that are not ergodic, and thus do not reach thermal equilibrium. A few complex systems have been proposed that are expected not to thermalize because their dynamics are integrable[1,2]. Some nearly integrable systems of many particles have been studied numerically, and shown not to ergodically sample phase space[3]. However, there has been no experimental demonstration of such a system with many degrees of freedom that does not approach thermal equilibrium. Here we report the preparation of out-of-equilibrium arrays of trapped one-dimensional (1D) Bose gases, each containing from 40 to 250 [87]Rb atoms, which do not noticeably equilibrate even after thousands of collisions. Our results are probably explainable by the well-known fact that a homogeneous 1D Bose gas with point-like collisional interactions is integrable. Until now, however, the time evolution of out-of-equilibrium 1D Bose gases has been a theoretically unsettled issue[4–6], as practical factors such as harmonic trapping and imperfectly point-like interactions may compromise integrability. The absence of damping in 1D Bose gases may lead to potential applications in force sensing and atom interferometry.

To see qualitatively why 1D gases might not thermalize, consider the elastic collision of two isolated, identical mass classical particles in one dimension. Energy and momentum are conserved only if they simply exchange momenta. Clearly the momentum distribution of a

the prevailing density[14]. The collisions that we study satisfy this criterion well. Our observations extend from the Tonks–Girardeau regime, where only pairwise collisions can occur[15], to the intermediate coupling regime, where there can be three- (or more) body collisions[15–17]. In both regimes, atoms that are set oscillating and colliding in a trap do not appreciably thermalize during our experiment.

We start our experiments with a Bose–Einstein condensate (BEC) loaded into the combination of a blue-detuned two-dimensional (2D) optical lattice and a red-detuned crossed dipole trap (see Methods). The combination of light traps makes a 2D array of distinct, parallel Bose gases, with the 2D lattice providing tight transverse confinement and the crossed dipole trap providing weak axial trapping[11]. The dynamics within each tube of the 2D array are strictly 1D because the lowest transverse excitation, $\hbar\omega_r$ (where $\omega_r/2\pi = 67\,\text{kHz}$ is the transverse oscillation frequency), far exceeds all other energies in

a
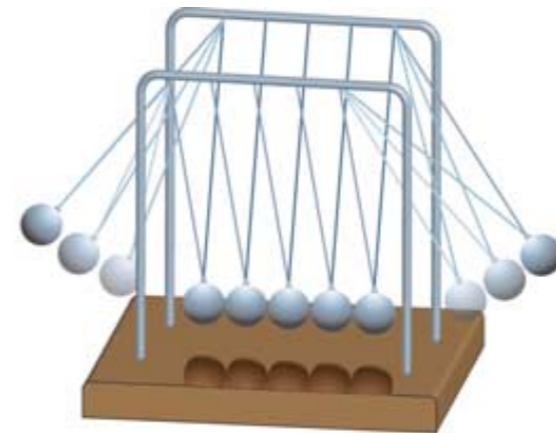
Integrability vs. thermalization in 1D Bose gas

Hard rods in 1D

This is a famous example of an integrable classical model where hydrodynamical approaches have a long history.

We study it because it has the same type of integrable kinetic theory (Boltzmann equation) as Lieb-Liniger or XXZ, but long-time numerics are <u>much</u> easier than in the quantum problems.

Simple guess:

at short times, the system will look like the integrable system;

at long times, the system will thermalize.

Three regimes, none thermal

1. integrable hydrodynamics
2. development of chaos
3. hydrodynamical (not thermal) steady state

Figures from Cao et al., PRL 2018

**Incomplete Thermalization from Trap-Induced Integrability Breaking:
Lessons from Classical Hard Rods**

Xiangyu Cao,[1] Vir B. Bulchandani,[1] and Joel E. Moore[1,2]
[1]*Department of Physics, University of California, Berkeley, Berkeley, California 94720, USA*
[2]*Materials Science Division, Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA*

Large-N limit: continuum hydrodynamics (Percus)

$$\partial_t \rho + \partial_x(v\rho) = 0, \, v[\rho](p) = p + \frac{a \int_{p'} (p - p')\rho(x, p')}{1 - a \int_{p'} \rho(x, p')}$$

Can add forcing from external potential to this Boltzmann-like equation.

Hydrodynamics works until a time scale determined by the initial density in units of rod length. Then non-integrability takes over and we see exponential separation of trajectories.

Strength of chaos: Lyapunov exponent observed to scale as $\gamma \sim N^{-0.25}$
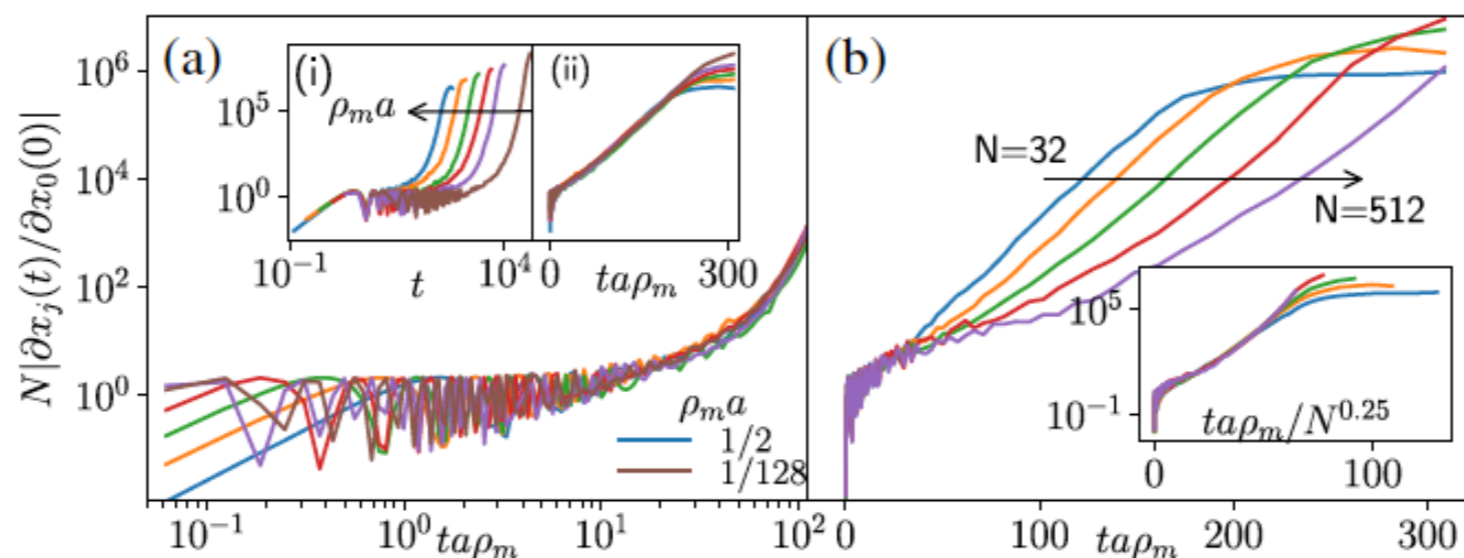
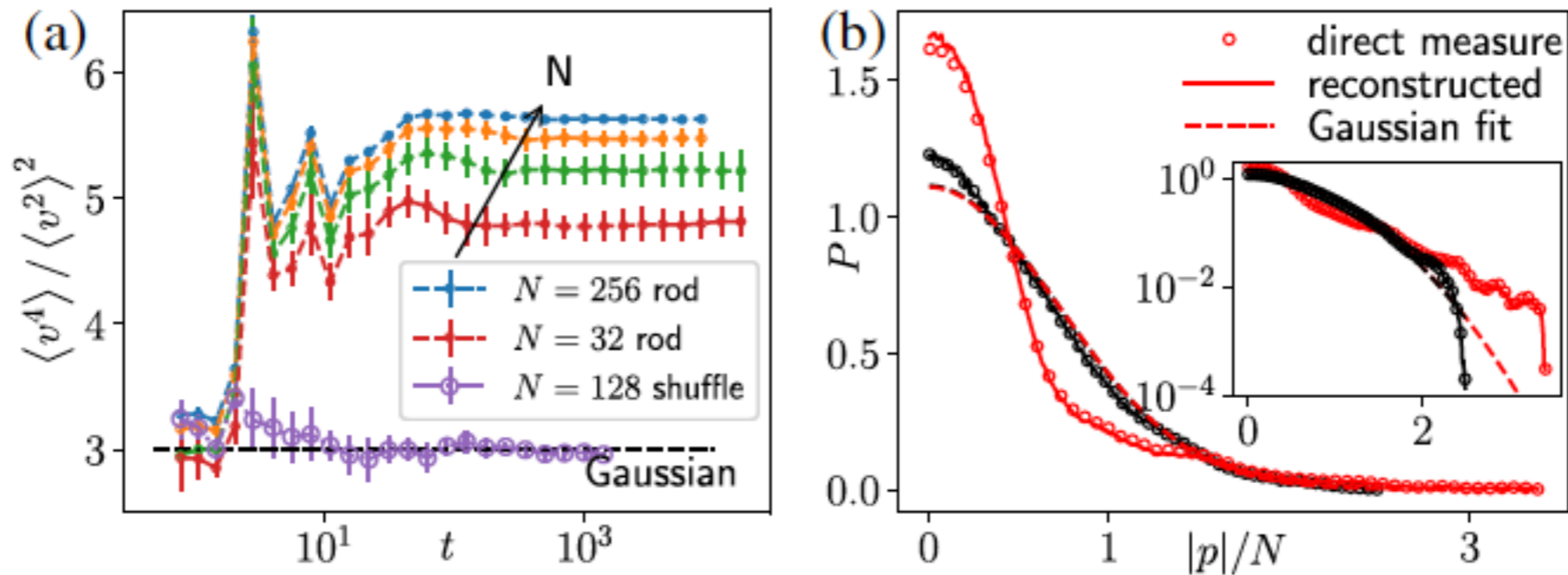# However, final state does not seem to be thermal (Maxwellian)

FIG. 5. (a) (Non)-Gaussianity of the velocity distribution of the time-averaged ensemble, as revealed by the moment-ratio test. (b) Comparing the velocity distribution with the reconstructed one from the density, assuming that the late-time ensemble solves the stationary THRE [Eq. (8)]. The two long-time ensembles are obtained from two squeezed (red) and circular (black) ICs, both with $\rho_m a = 1/2$.

Why?
Go back to three rods

(b)

$(p_2 - p_1)/(a\omega)$

1.0

0.5

0.0

-0.5 $p_3/(a\omega)$ 0.5
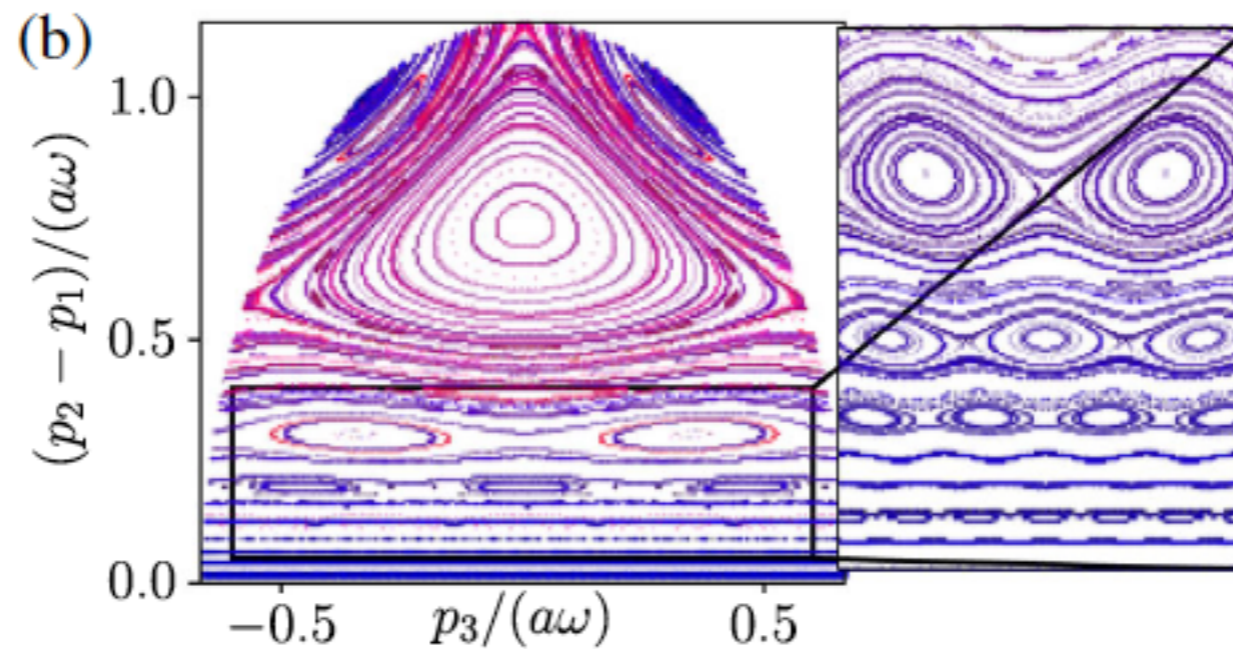
3D phase space: make Poincare map by looking just after a collision to reduce to 2 parameters. Orbits do not look integrable (since fractal structure) but also do not look micro canonical (ergodic over all possibilities of constant energy).

So the final state need not be thermal. What is it?

A constraint on final ensemble

We find that the final ensemble, for more than a few rods, is a solution of the steady-state hydrodynamical equation at every (x,p):

$$\partial_x (v[\rho]\rho) - \partial_x V \partial_p \rho = 0$$

We do not see additional thermalization on the accessible time scale (hundreds of thousands of periods for small number of rods).

So at least in this classical problem the hydrodynamical approach is not just useful for time evolution, but gives (partial) information about the final ensemble.

Electron hydrodynamics?

Why don't we normally treat electrons in a solid as fluid-like?

Real solids are not perfect: momentum is not a conserved quantity.

Solids are not generally isotropic, either—they can break spatial and time-reversal symmetries.

Finally, electrons are charged, which makes them a somewhat unusual fluid.

Hydrodynamics of 2D electrons

In materials that are very clean, momentum relaxation may take a relatively long time. It might be better to view electrons as a *fluid* rather than as independently scattering quasiparticles.

Thomas Scaffidi
(UCB/Toronto)

Solid-state electrons where fluid properties measured

2DEGs (Molenkamp & others, 1990s)
Graphene (P. Kim; A. Geim)
Layered crystals (A. Mackenzie)
…

Hydro in clean electron systems (slide from A. Mackenzie)

*R.N. Gurzhi, JETP **44**, 771 (1963); Usp. Fiz. Nauk **94**, 689 (1968)*

Key point introduced by Gurzhi: In solids, hydrodynamic effects can be parameterised in terms of the relationship between three length scales: momentum relaxing mfp ℓ_{MR}, momentum conserving mfp ℓ_{MC} and sample dimension (here *W*).
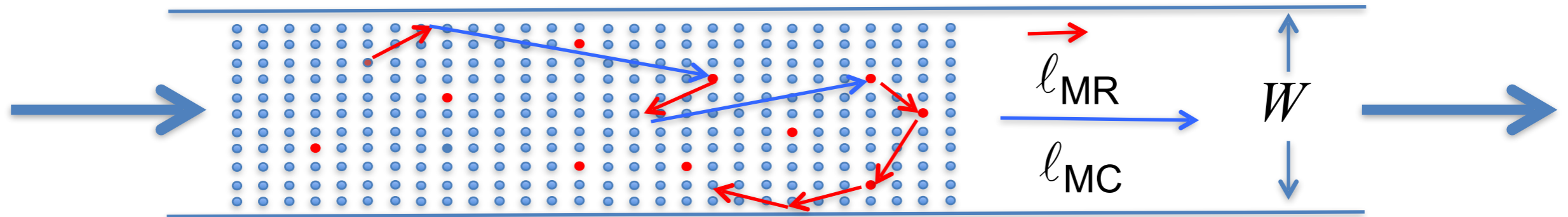
$$\ell_{MR} << \ell_{MC}, W \qquad\qquad\qquad \ell_{MC} << W << \ell_{MR}$$

Standard ohmic theory applies; *R* is determined entirely by solid resistivity ρ and usual geometrical factors

Hydrodynamic theory applies; *R* is determined entirely by fluid viscosity η, boundary scattering and 'Navier-Stokes' geometrical factors

Pioneering measurements on microfabricated narrow 2DEG channels:

*L.W. Molenkamp & M.J.M de Jong , Phys. Rev. B **49**, 5038 (1994)*

How are electron fluids different from normal fluids?

One active area, originally motivated by AdS but derivable without gravity dual: (Kitaev, Maldacena-Shenker-Stanford, 2015; Sachdev-Ye-Kitaev models):

The Lyapunov exponent for short-time onset of chaos is bounded

$$\lambda_L \leq \frac{2\pi k_B T}{\hbar}$$

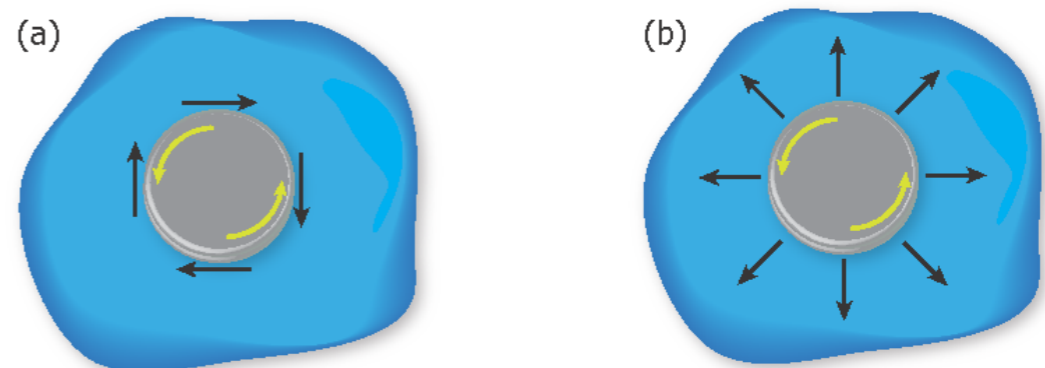**Other ways electron fluids have different hydrodynamics:**

"Hall viscosity" in topological states: (Avron; Read; Gurarie…) stress tensor is

$$T_{ij} = p\delta_{ij} - \lambda_{ijkl}\xi_{kl} - \eta_{ijkl}\dot{\xi}_{kl}$$

and T-breaking allows an odd contribution $\quad \eta^{(A)}_{ijkl} = -\eta^{(A)}_{klij}$

What is Hall viscosity in T-breaking gapless systems?
Allowed by symmetry…

(a)   (b)

From Hughes-Leigh-Parrikar

Hydrodynamics of electrons

What makes electron fluids different from classical fluids?

In 2D and 3D, can induce broken T by a magnetic field
and have a new kind of viscosity, "Hall viscosity"

Is significant, and could be observable, in simple *metals*:
T. Scaffidi, N. Nandi, B. Schmidt, AP Mackenzie, JEM, PRL 17

In the quantum Hall regime there are two contributions in the q^2
correction to Hall conductance

$$\sigma_{xy}(q) = \sigma_{xy}(0) + O(q^2)$$

that are comparable (Hoyos-Son): one from Hall viscosity and one from
(inverse) internal compressibility. In a metal, the internal compressibility part is
small and the Hall viscosity follows from a Boltzmann calculation.

Hydrodynamical calculation

$$\partial_t \vec{v} = \eta_{xx} \nabla^2 \vec{v} + \eta_{xy} \nabla^2 \vec{v} \times \vec{z} + \frac{e}{m}(\vec{E} + \vec{v} \times \vec{B})$$

Scaling in bulk, Alekseev:

$$\eta_{xx} = \eta \frac{1}{1 + (2\frac{l_{MC}}{r_c})^2}$$

$$\eta_{xy} = \eta \frac{2\frac{l_{MC}}{r_c}}{1 + (2\frac{l_{MC}}{r_c})^2}$$

Can incorporate boundary effects of diffuse scattering in a thin channel by going back to Boltzmann eq:

Prediction:

$$\rho_{xy} = \rho_{xy}^{\text{bulk}} \left(1 - 6\left(\frac{l_{MC}}{W}\right)^2\right) \text{ for } B \to 0$$

We want $l_{MC} \ll l_{MR}$ for hydrodynamics. Crucially, the hydrodynamic regime can happen only if W is somewhat larger than l_{MC}. In order to measure this effect, one needs to choose W to be as small as possible, but still larger than l_{MC}. If we have, say, $W = 5l_{MC}$, one then expects a relative change in the slope of the Hall resistivity at zero field of the order $\simeq 25\%$, which should be measurable.

Possibly just seen in graphene, Geim et al., arXiv:1806.01606.

# Ultrafast optics as a means to understand transport without contacts

An experimental challenge in DC transport is distinguishing ballistic from hydrodynamic behavior.

Ultrafast techniques, such as "transient grating" experiments, can allow measurement of nanoscale transport regimes without the complications of contacts or surface details.

This includes spin and energy transport as well as charge transport. An example of Coulomb-interaction hydrodynamics observed using *spin*:

$$S(q,\omega) \propto \frac{1}{i\omega - D(q,\omega)q^2}, \quad D(q,\omega) = \frac{v_F/2}{\sqrt{(i\omega/v_F - 1/l)^2 + q^2}}$$

C. P. Weber et al. (Orenstein, Awschalom), *Nature* 2005
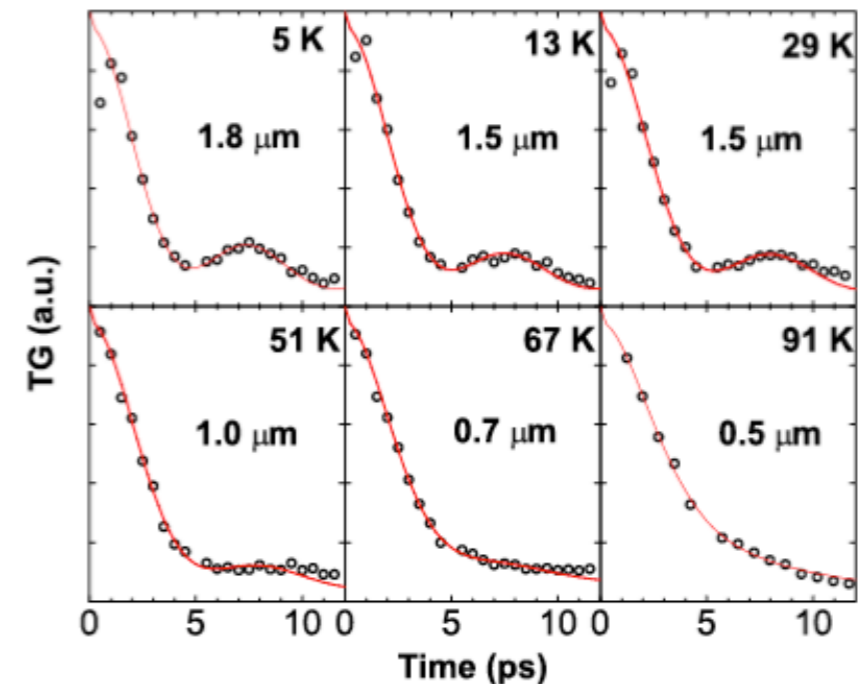
Crystal background modification to quantum transport theory

For the velocity in Boltzmann equation

$$\partial_t f_1 + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_1 = \int w'(f_1' f_2' - f_1 f_2) d\mathbf{p}_2 \, d\mathbf{p}_1' \, d\mathbf{p}_2'.$$

we should really use the semiclassical velocity of a wave packet:

$$\frac{dx^a}{dt} = \frac{1}{\hbar} \frac{\partial \epsilon_n(\mathbf{k})}{\partial k_a} + \mathcal{F}_n^{ab}(\mathbf{k}) \frac{dk_b}{dt}.$$

Second term is anomalous velocity or "Berry phase" piece.

This is just one example of how, even in a perfect crystal, the non-isotropy can modify the long-distance physics.

Active topic in quantum hydrodynamics above 1D: linear-response relaxation

An example of recent progress on a long-standing question:

Are there intrinisic limits on how fast a system can relax to equilibrium?

Related to conductivity via the Kubo formula: how rapidly does the current-current correlation decay in time?

Also related to existence of "hydrodynamical" regimes of electron transport where quasiparticle scattering is not the right picture.

Some past formulations:

Mott-Ioffe-Regel: mean free path must be at least the lattice spacing

Sachdev: $$\tau \geq \frac{\hbar}{k_B T}$$ quantum $e^{-iHt/\hbar} \leftrightarrow e^{-H/(k_B T)}$ thermal
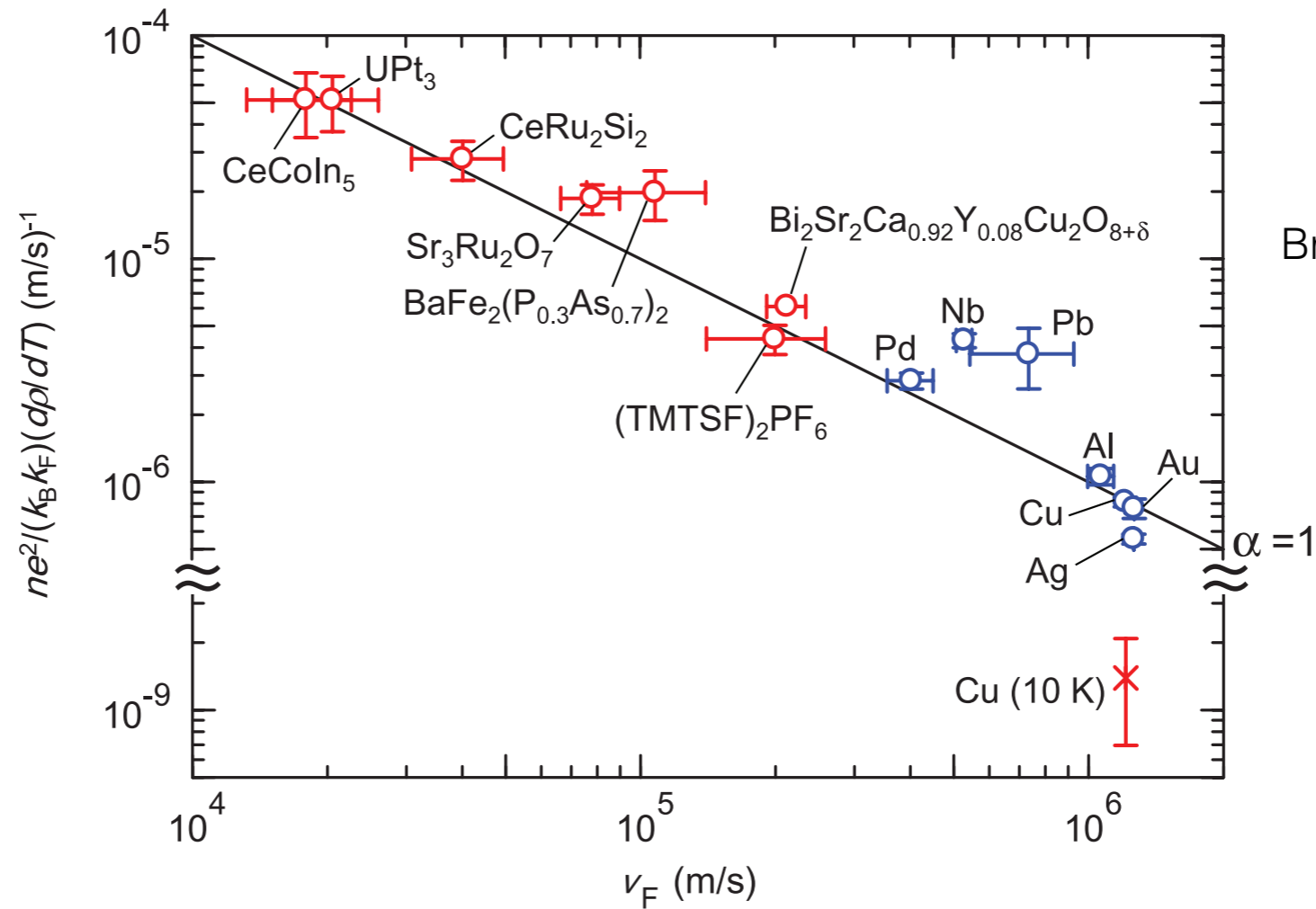
Kovtun-Son-Starinets: the viscosity is bounded below* $$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}$$

Hartnoll: reinterpret viscosity bound as a lower limit on diffusion constant

Near-equilibrium

Why these bounds matter: nature seems to contain such behavior

Bruin et al. (Mackenzie), Science 2013

$$\tau \geq \frac{\hbar}{k_B T}$$

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}$$

is satisfied (I believe) in all experimental liquids,
and within ~10 of saturation in helium and QGP,
but there exist violations in exotic theories

Personal list of some other questions and open problems

Entanglement dynamics (cf. Nahum talk in this program)

Diffusive pieces exist even in integrable models (J. Viti et al.;
S. Gopalakrishnan, D. Huse, V. Khemani, R. Vasseur)

XXZ gapped phase still perplexing (see Znidaric talk in this program)

What are the "universality classes" of quantum dynamics?

MBL (Hamiltonian, non-thermalizing).
Random unitary (non-Hamiltonian)
Luttinger liquid is non-generic in some ways (VBB, CK, JEM)

Other types of transport, e.g., drag effects (W. Berdanier et al.)

Connect electrons to HEP work on "eightfold way" for hydrodynamics, etc.

Some take-away messages

Lecture I: in integrable models, transport is ballistic and relaxation is to a "generalized Gibbs ensemble".

Flow between local GGEs seems to be well described by Bethe-Boltzmann equation, which can be checked against a few exact far-from-equilibrium results for XXZ, and against numerics.

Lecture II: the many-body-localized phase can be understood as a new type of integrable model. There are *local* conserved quantities (not sums of local densities).

The weak interactions between these lead to slow logarithmic dynamics that can be observed either in entanglement or in revival rate.

Lecture III: dynamics without integrability is hard. Transport in Luttinger liquids is controlled by leading irrelevant perturbations (a difference from higher dimensions).

Integrability breaking by a trap, in a classical problem, seems to lead to only partial thermalization on observable time scales. Are there quantum problems that retain some integrable features without being fully integrable?

Thank you for your participation!