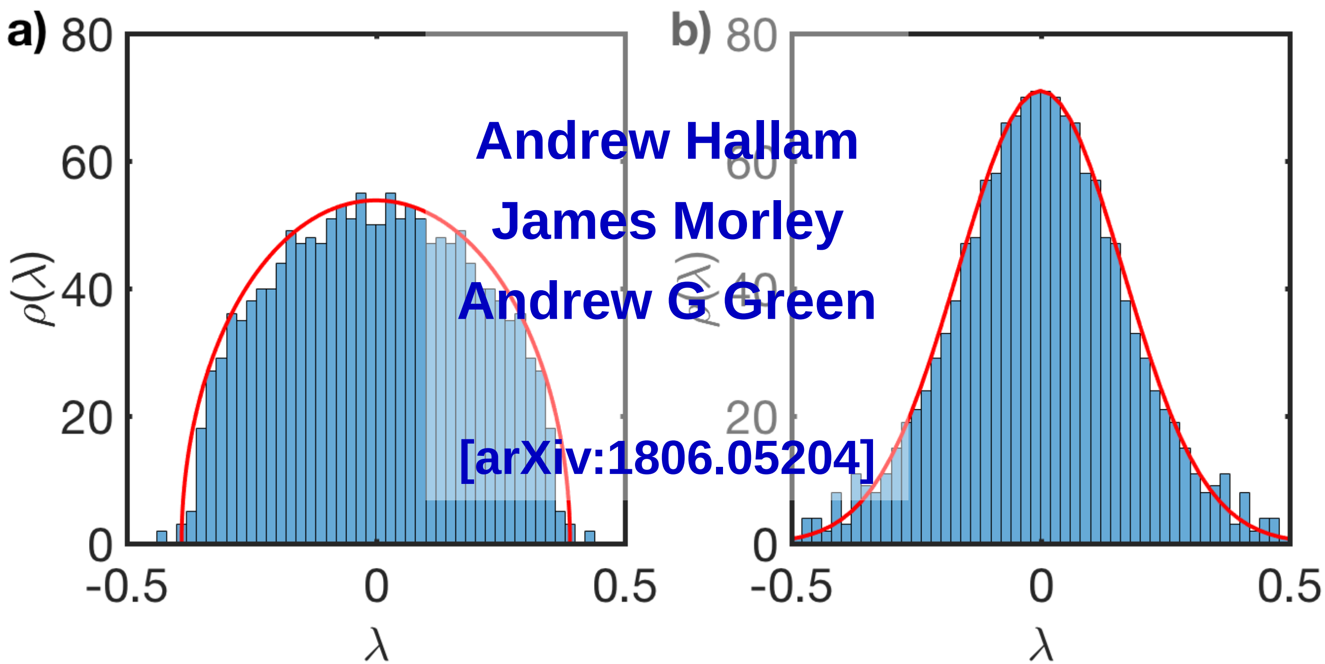
The Lyapunov Spectrum of Quantum Thermalisation



Andrew Hallam

James Morley

Andrew G Green

[arXiv:1806.05204]

What connects quantum and classical many-body thermalization?

- Mapping Quantum to Classical Dynamics
  - TDVP of wavefunction MPS
  - TDVP of thermofield MPS
- Extracting the Lyapunov Spectrum
- Results: Ising with tilted field
  - Entanglement growth vs Kolmogorov-Sinai
  - A semi-circle law for Lyapunov spectrum
- Discussion

Mapping to Classical Dynamics

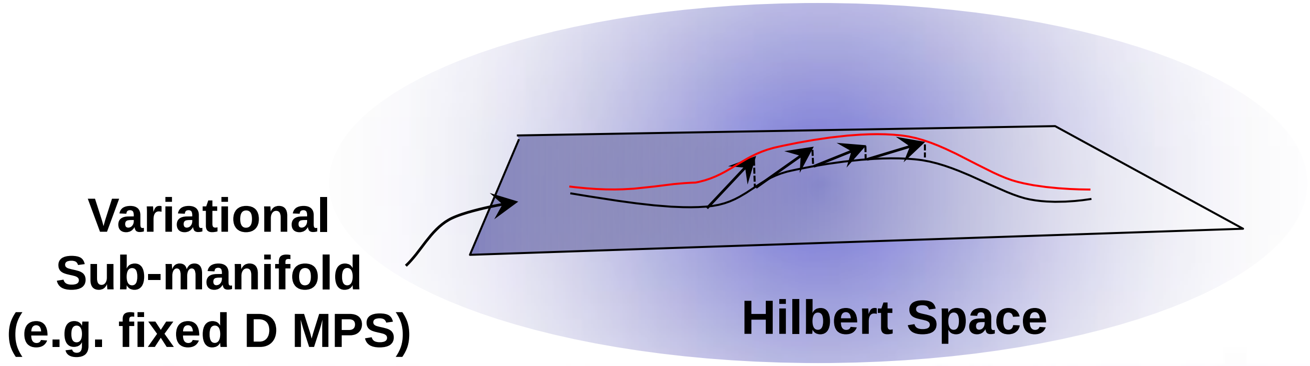
TDVP for Wavefunction

- **Variational parametrization** $|\psi(X)\rangle$

$$\langle \partial_{X_i}\psi|\partial_{X_j}\psi\rangle \dot{X}_j = i\langle \partial_{X_i}\psi|\mathcal{H}|\psi\rangle$$

- **Project dynamics / Optimize fidelity**

$$\mathrm{Max}_X |\langle \psi(\mathbf{X}+d\mathbf{X})|e^{i\hat{\mathcal{H}}dt}|\psi(\mathbf{X})\rangle|^2$$



[Dirac, Proc. Cam. Phil. Soc. 1930]

[Haegeman et al PRL 2011]

Mapping to Classical Dynamics

TDVP for Wavefunction

- **Variational parametrisation** $|\psi(X)\rangle$

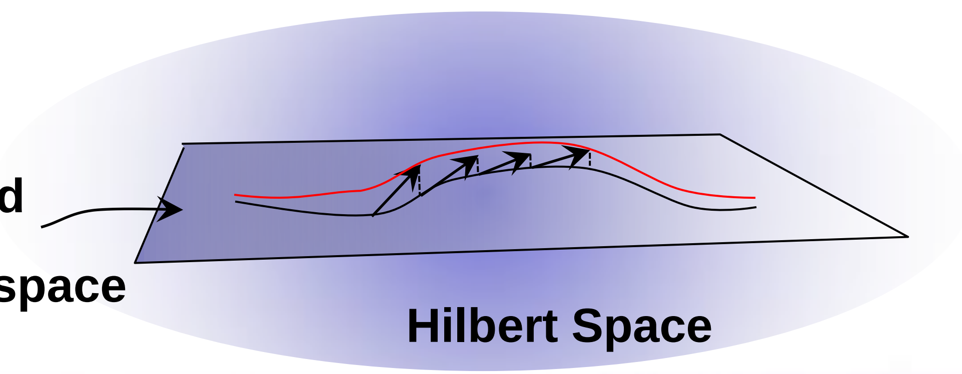
$$\langle \partial_{X_i}\psi | \partial_{X_j}\psi \rangle \dot{X}_j = i \langle \partial_{X_i}\psi | \mathcal{H} | \psi \rangle$$

g_{ij}

- **Classical Hamiltonian dynamics - Conserved quantities**

$$q_i = \sqrt{2}\mathcal{I}m\left[g_{ij}^{-1/2} X_j\right] \qquad p_i = \sqrt{2}\mathcal{R}e\left[g_{ij}^{-1/2} X_j\right]$$

Variational Sub-manifold $\equiv$ Classical phase space



Thermalisation => Dynamical Chaos

Mapping to Classical Dynamics

TDVP for Thermofield Double

$$\hat{\rho} = \sum_{\alpha} \gamma_{\alpha} |\alpha\rangle\langle\alpha| \quad \Leftrightarrow \quad |\psi\!\!\psi\rangle = \sum_{\alpha} \gamma_{\alpha}^{1/2} |\alpha, \alpha\rangle$$

$$\mathcal{H}\!\!\mathcal{H} = \mathcal{H} \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{H}$$

Thermofield MPS:

Doubled physical index

- **MPS approximation to** $|\psi\!\!\psi\rangle$ **with** $A\!\!A_{i\,j}^{\sigma\,\delta}$
- **TDVP optimizes fidelity** $\mathrm{Max}_X Tr\left[\sqrt{\hat{\rho}(\mathbf{X}+d\mathbf{X})}\sqrt{e^{i\mathcal{H}dt}\hat{\rho}(\mathbf{X})e^{-i\mathcal{H}dt}}\right]$
- **Optimizes a set of observations**
- **Pure states evolve to mixed states**

Mapping to Classical Dynamics

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$$\mathcal{H\!\!H} = \mathcal{H} \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{H}$$

Aside

Thermofield MPS:

- Expectations should be same on two sub-spaces
- Numerical drift fixed by symmetrising

$$\mathbb{A}^{\sigma\delta}_{i\otimes i', j\otimes j'} = \mathbb{A}^{\delta\sigma}_{i'\otimes i, j'\otimes j}$$

- Achieve by tangent space gauge fixing

[Haegeman et al PRL107, 070601 (2011)] [Hallam,Morley,Green arXiv:1806.05204]

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- Mapping Quantum to Classical Dynamics
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- **Extracting the Lyapunov Spectrum**
- Results: Ising with tilted field
  - Entanglement growth vs Kolmogorov-Sinai
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Lyapunov Spectrum

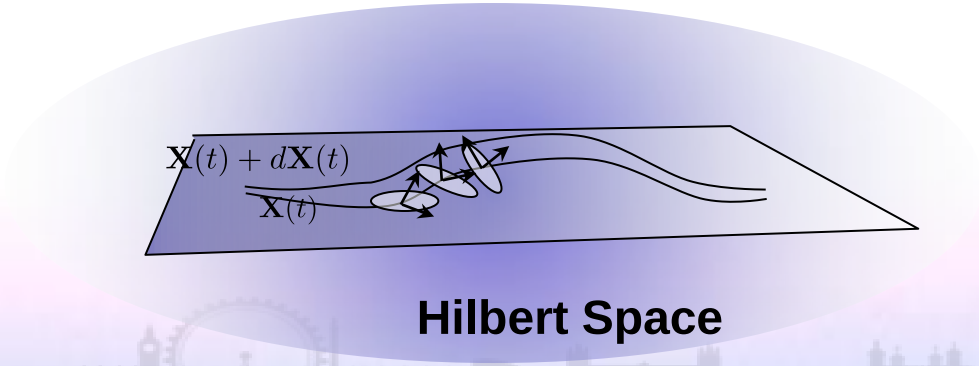
Calculation

- **Trajectory – TDVP**

$$\langle \partial_{X_i}\psi|\partial_{X_j}\psi\rangle \dot{X}_j = i\langle \partial_{X_i}\psi|\mathcal{H}|\psi\rangle$$

- **Distance measure** $dS^2 = 1 - |\langle \psi(\mathbf{X}+d\mathbf{X})|\psi(\mathbf{X})\rangle|^2$
- **Hamiltonian dynamics on classical phase space …**
  … use tools from classical dynamical systems

[Geist et al, Prog Theor Phys 83, 875 (1990)]



Lyapunov Spectrum

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- **Trajectory – TDVP**

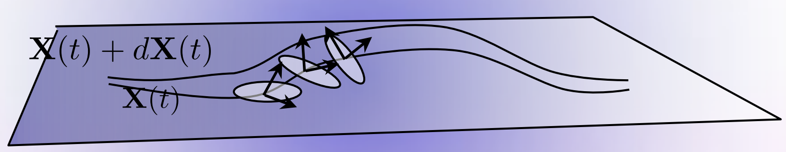
$$\langle \partial_{X_i}\psi|\partial_{X_j}\psi\rangle \dot{X}_j = i\langle \partial_{X_i}\psi|\mathcal{H}|\psi\rangle$$

- **Divergence of $\mathbf{X}(t)$ and $\mathbf{X}(t) + d\mathbf{X}(t)$ -linearised TDVP**

$$-i\langle \partial_{X_i}\psi|\partial_{X_j}\psi\rangle dX_j = \underbrace{\langle \partial_{X_i}\partial_{X_j}\psi|\mathcal{H}|\psi\rangle}_{\text{Generates non-zero Lyapunovs}} d\bar{X}_j + \langle \partial_{X_i}\psi|\mathcal{H}|\partial_{X_j}\psi\rangle dX_j$$

Generates non-zero Lyapunovs

- **Growth in some direct's**
- **Reduce in others**
- **Direct's change**
- **Average along trajectories**



[Haegemann, Osborne, Vestraete PRB (2013)]

Lyapunov Spectrum

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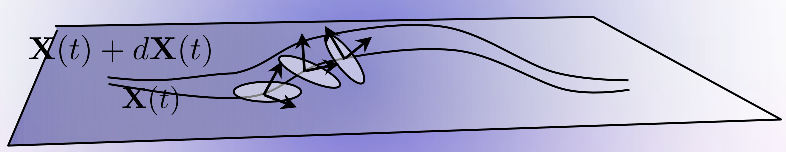
- **Instantaneous Lyapunov Exponents**

$$d\dot{X}_i = M_{ij} dX_j \quad \Rightarrow \quad \{dX_i(t=0)\} = Q(t=0)$$

$$Q(t+dt)R(t+dt) = \exp[M(t)dt]Q(t) \quad \Rightarrow \quad \lambda_i = \lim_{T\to\infty} \frac{1}{T}\int_0^T \lambda_i(t)$$

- **Lyapunov Exponents** $\lambda_i(t) = \lim_{dt\to 0} \frac{1}{dt} \log R_{ii}(t)$



Hilbert Space

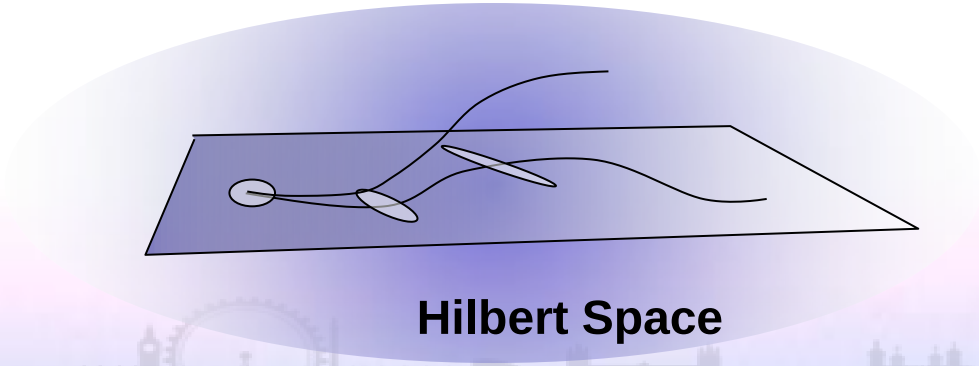
Aside

Relationship to Zero-point Fluctuations

- **Initial wavepacket – evolve with truncated Wigner**
- **Path integral over MPS [arXiv:1607.01778]** $e^{i\int \mathcal{H}dt} = \int DAe^{i\mathcal{S}[A]}$

$$\mathcal{S}[A] = \mathcal{S}[A_0] + Tr\left[\begin{array}{c} i\bar{Y}\dot{Y} + \bar{Y}Y\langle\partial_{\bar{Y}}\psi|\mathcal{H}|\partial_Y\psi\rangle \\ +\bar{Y}\bar{Y}\langle\partial_{\bar{Y}}\partial_{\bar{Y}}\psi|\mathcal{H}|\psi\rangle + YY\langle\psi|\mathcal{H}|\partial_{\bar{Y}}\partial_{\bar{Y}}\psi\rangle \end{array}\right]$$

Anomalous terms: Generate zero-point fluctuations about saddle point



Lyapunov Spectrum

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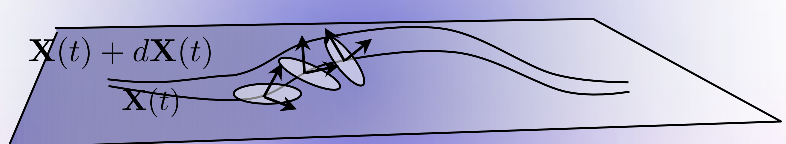

[Haegemann, Osborne, Vestraete PRB (2013)]

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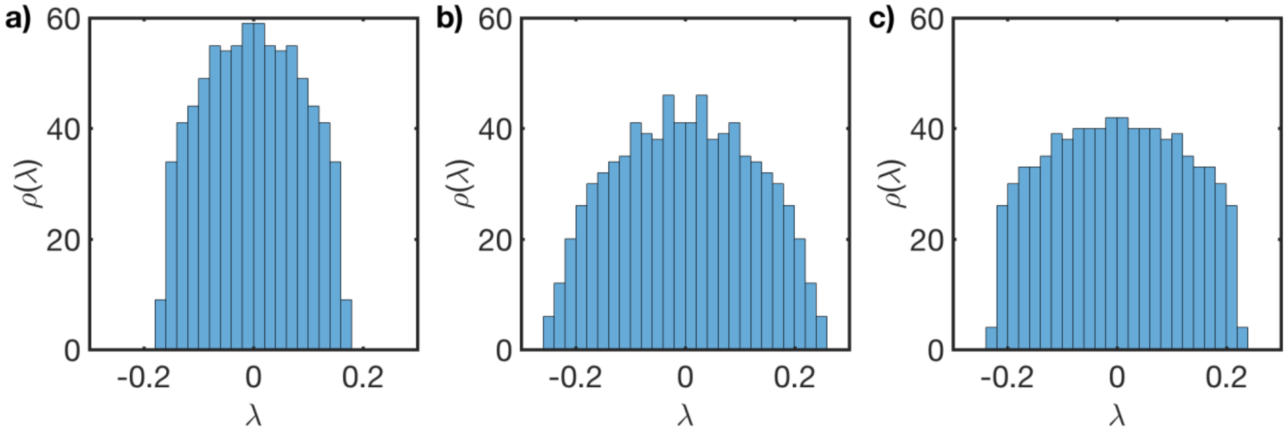
Results: Ising Lyapunov Spectrum

Wavefunction MPS

$$\mathcal{H} = \sum_i \left[J\sigma_i^z \sigma_{i+1}^z + h^x \sigma_i^x + h^z \sigma_i^z \right]$$

Spectra: **Evolve from product state** $|\psi(t=0)\rangle_i = 0.540|\uparrow\rangle + 0.841|\downarrow\rangle$



Non-integrable	**Integrable**	**Nearly - Integrable**
$J=1,\ h^z=1,\ h^x=0.5$	$J=1,\ h^z=1,\ h^x=0$	$J=1,\ h^z=1,\ h^x=0.1$

Results: Ising Lyapunov Spectrum

Wavefunction MPS

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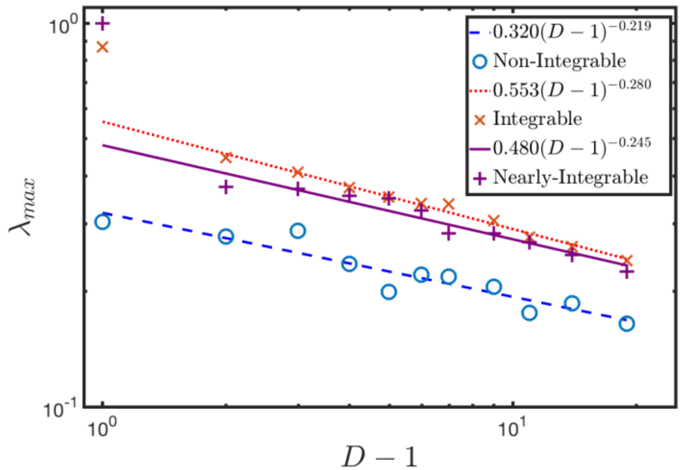
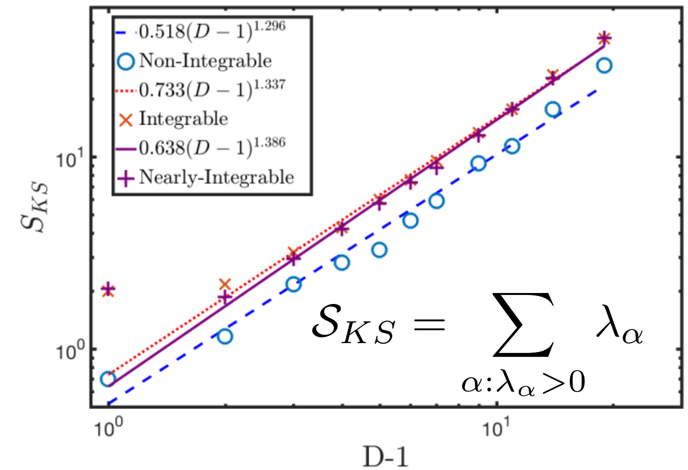
Bond Order Dependence:





- **Spectra do not converge with D**
- **How then do we relate to physical properties?**

Results: Ising Lyapunov Spectrum

Wavefunction MPS

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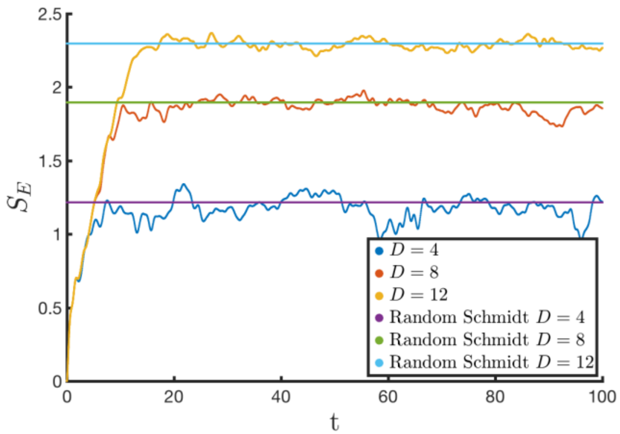
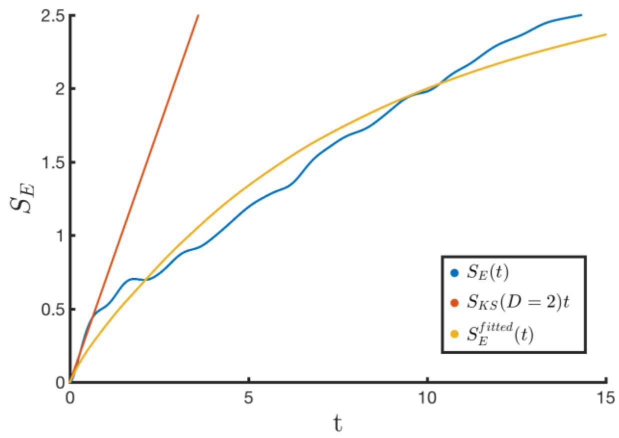
Entanglement growth:





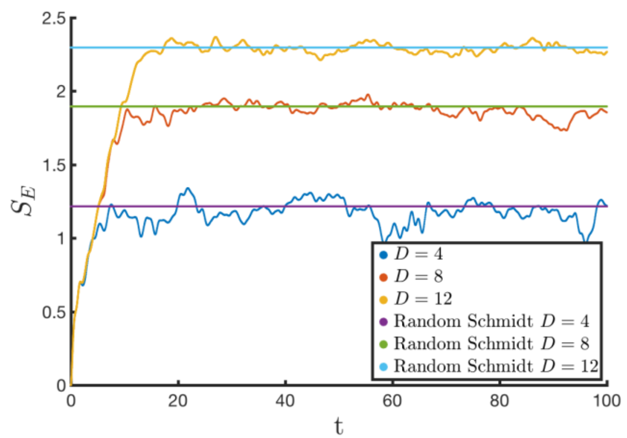
- **Initial entanglement growth:** $\dot{\mathcal{S}}_E(t=0) = \mathcal{S}_{\mathcal{KS}}(D=2)$
- **Bond order grows with time:** $\mathcal{S}_E(t) = \mathcal{S}_E^*(D) \quad \Rightarrow \quad D(t)$

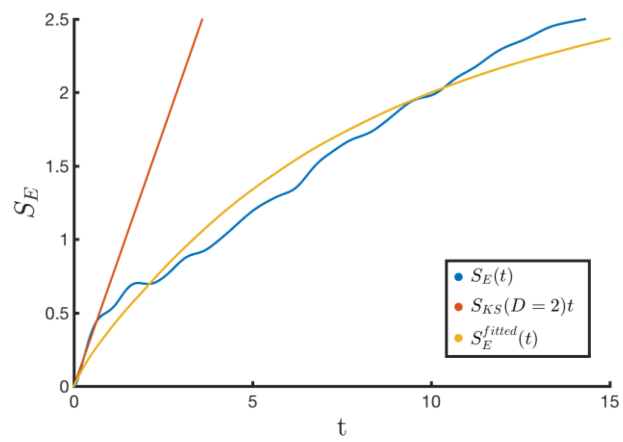
[Zurek and Paz, Physica D (1995)] [Miller and Sarkar PRE (1999)]

Wavefunction MPS

$$\mathcal{H} = \sum_i \left[J\sigma_i^z \sigma_{i+1}^z + h^x \sigma_i^x + h^z \sigma_i^z \right]$$

Entanglement growth:





Generalise:

$$\dot{\mathcal{S}}_E(t) = \frac{\mathcal{S}_{\mathcal{KS}}(D(t))}{(D(t)-1)^2}$$

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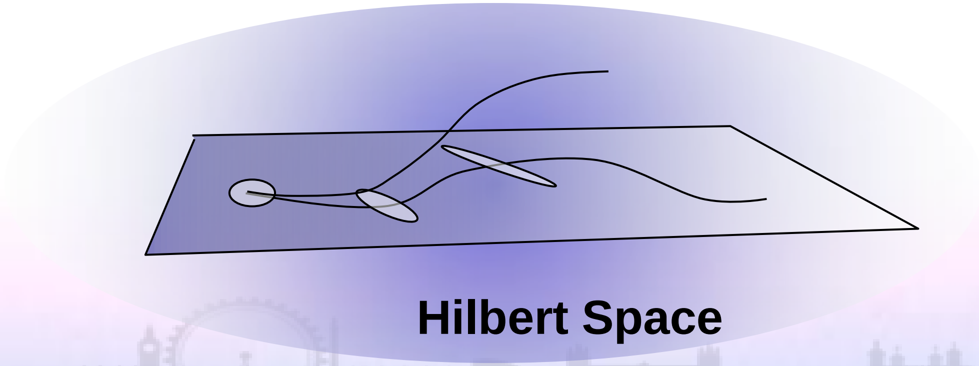
- $\mathcal{S}_{\mathcal{KS}}$ **related to fluctuation determinant averaged on trajectory?**



Results: Ising Lyapunov Spectrum

Wavefunction MPS

$$\mathcal{H} = \sum_i \left[J\sigma_i^z \sigma_{i+1}^z + h^x \sigma_i^x + h^z \sigma_i^z \right]$$

Comments:

- **Crossover between complementary pictures**
- **Low bond order – chaotic thermalisation**
- **High bond order – dephasing thermalization**

- **Lyapunov Spectra don't converge**
- **Effective bond order grows D(t)**

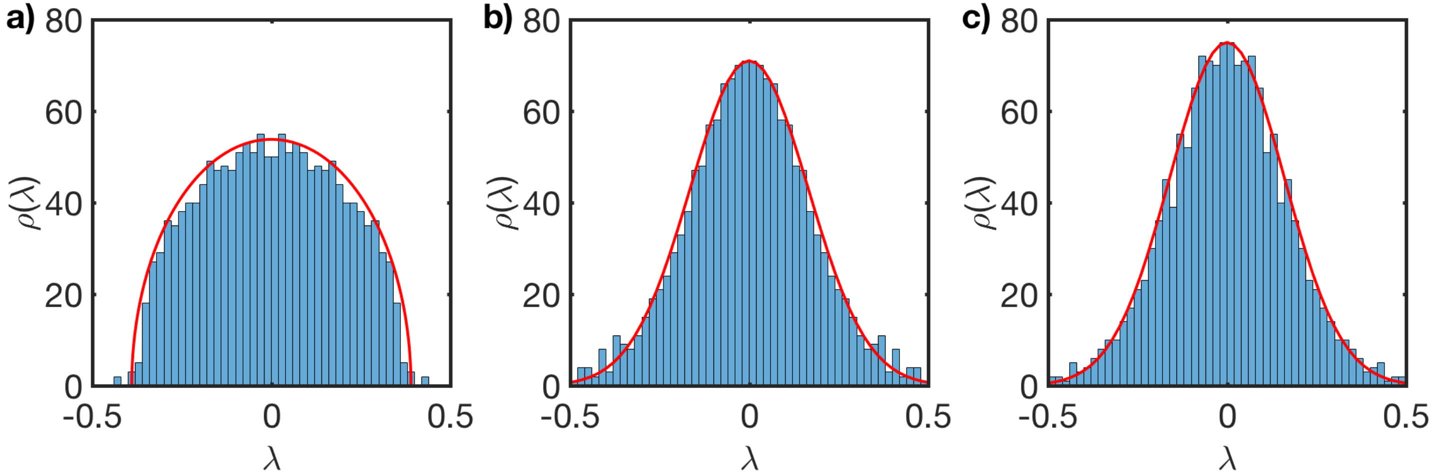
$$\dot{\mathcal{S}}_E(t) = \frac{\mathcal{S}_{\mathcal{KS}}(D(t))}{(D(t)-1)^2}$$

Results: Ising Lyapunov Spectrum

Thermofield MPS

$$\mathcal{H} = \sum_i \left[J\sigma_i^z \sigma_{i+1}^z + h^x \sigma_i^x + h^z \sigma_i^z \right]$$

Lyapunov Spectra: Start from product state near centre of spectrum



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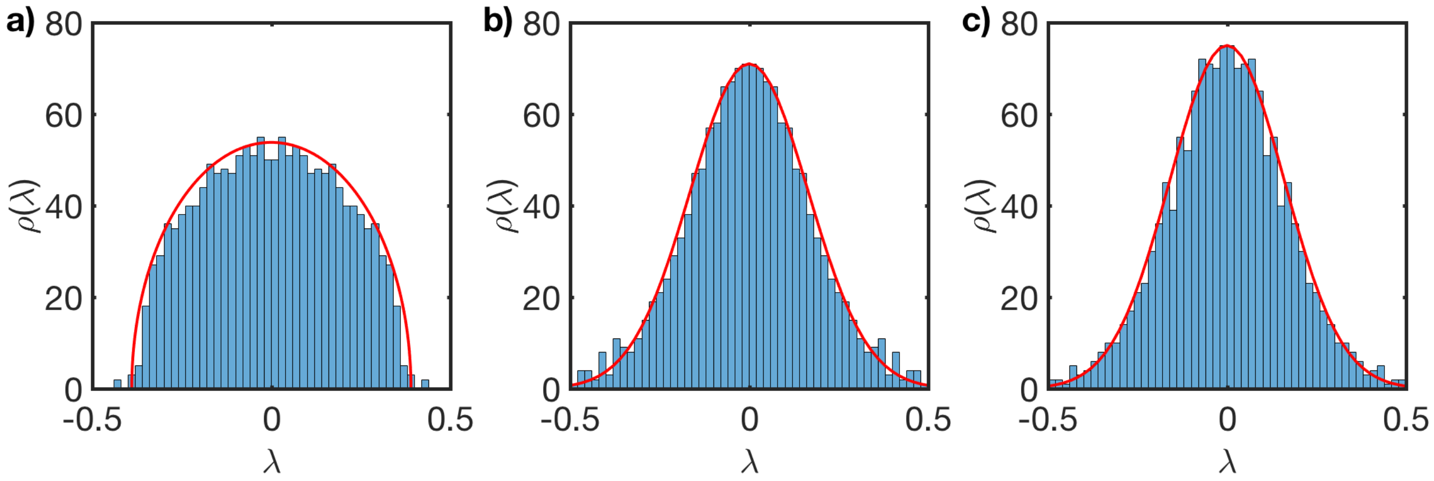

- **Semi-circle distribution also in semiclassical limit of matrix model**
  [Gur-Ari, Hanada, and Shenker, JHEP('16), Hanada, Shimada, and Tezuka, PRE ('18)]

Results: Ising Lyapunov Spectrum

Thermofield MPS

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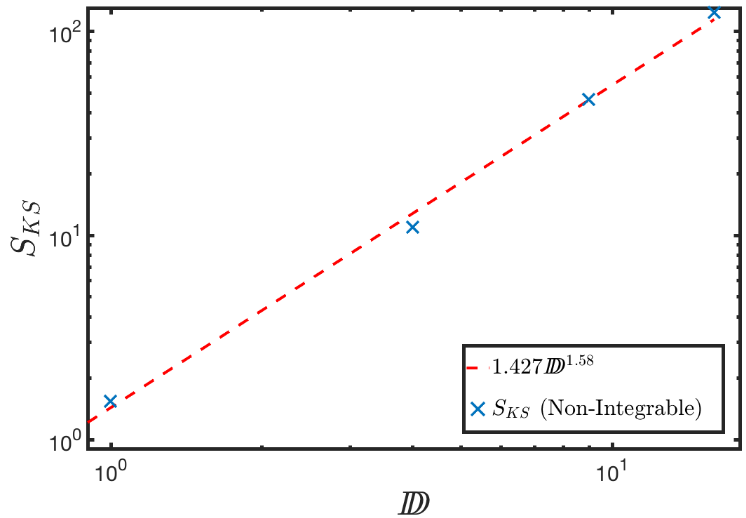
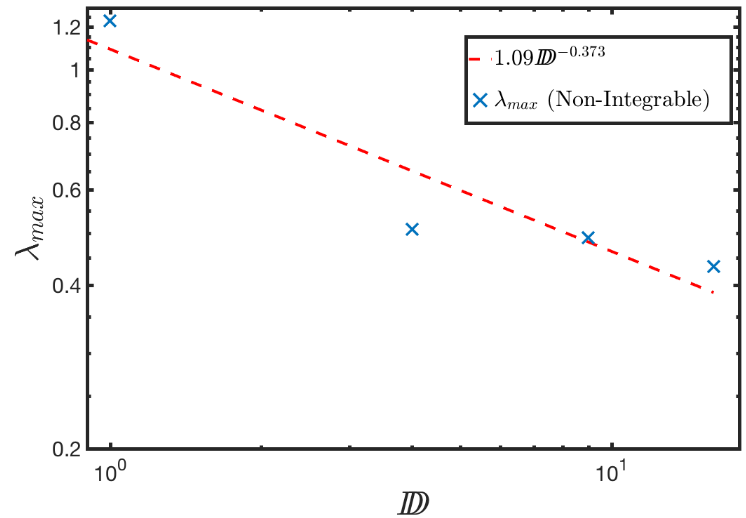
Bond Order Dependence:



Kolmogorov-Sinai Entropy

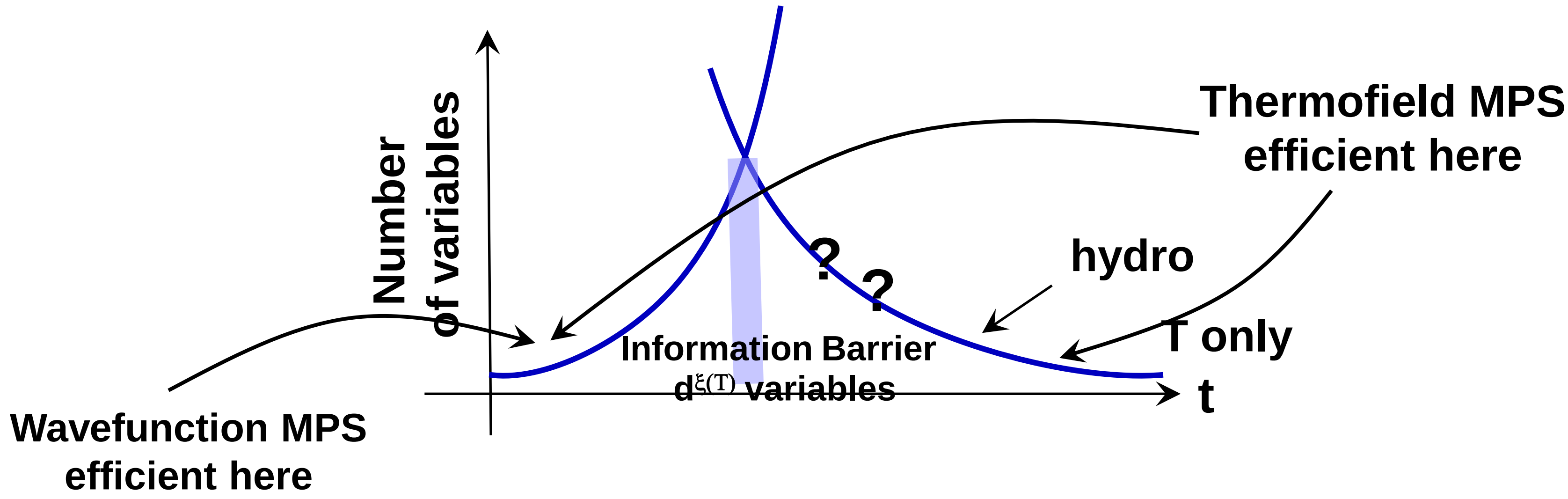


Max Lyapunov

Results: Ising Lyapunov Spectrum

Information Paradox of Eigenstate Thermalisation

- Cf Frank's talk Fri 10th



Results: Ising Lyapunov Spectrum

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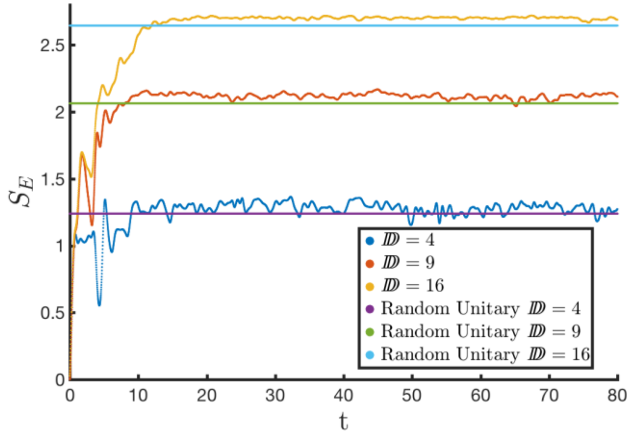

$$\mathbb{D} = 1: \qquad A^{\sigma\delta} = \delta^{\sigma\delta}/\sqrt{2}$$

$$\mathbb{D} > 1: \qquad A^{\sigma\delta}_{IJ} = \frac{1}{\sqrt{d}} \sum_{\gamma=1}^{d} U_{(\sigma i),(\gamma j)} U_{(\delta i'),(\gamma j')}$$

$$U \in SU(d\sqrt{\mathbb{D}})$$

Apply arbitrary unitary to each half of Hilbert space

- **Operator entanglement: saturates near random unitary values**
- **Equivalent representations of infinite temperature state**
- **Seed of how to compress thermofield MPS**

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Conclusions:

- Eigenstate Thermalisation vs Dynamical Chaos
- Map quantum to classical Hamiltonian dynamics
  - TDVP of Wavefunction MPS
  - TDVP of Thermofield MPS
- Extract the full Lyapunov spectrum

Wavefunction MPS

$$\dot{\mathcal{S}}_E(t) = \frac{\mathcal{S}_{\mathcal{KS}}(D(t))}{(D(t)-1)^2}$$

Thermofield MPS