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GORDON AND BETTY MOORE FOUNDATION

Operator dynamics in chaotic long-range interaction systems

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Operator dynamics

- For a quantum wave function, it can be expanded into a linear combination of basis in the Hilbert space,

$$|\psi\rangle = \sum_n c_n |n\rangle$$

- For a quantum operator, it can be treated as a wave function living in an operator Hilbert space.

- Under unitary time evolution, it can be expanded as

$$\hat{O}(t) = \sum_j \alpha_j(t) \hat{\mathcal{B}}_j \qquad \sum_j |\alpha_j(t)|^2 = 1$$

$\{\hat{\mathcal{B}}_j\}$ is a set of operator basis satisfying $\mathrm{Tr}\hat{\mathcal{B}}_i^\dagger \hat{\mathcal{B}}_j = \delta_{ij}$

- The evolution of $\alpha_j(t)$ determines the operator dynamics.

Chaotic operator dynamics

- In a chaotic system, under unitary time evolution, a simple operator becomes increasingly complicated.

- The quantum information encoded in this operator is delocalized and this phenomenon is called scrambling.

- It is impossible to keep track of $\alpha_j(t)$

- The universal features

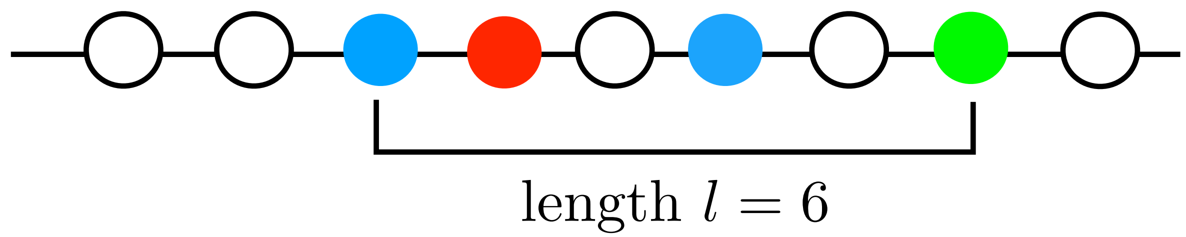
The development of random matrix physics XC and Ludwig, 2017

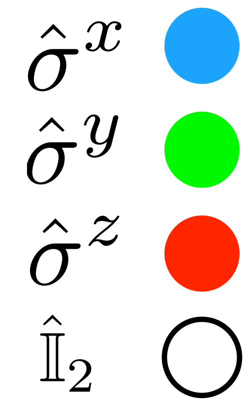
The emergent hydrodynamics

Nahum, Vijay and Haah, 2017, Keyserlingk, Rakovszky, Pollmann and Sondhi, 2017
Khemani, Vishwanath and Huse, 2017, Rakovszky, Pollmann and Keyserlingk, 2017
Roberts, Stanford and Streicher, 2018, Xu and Swingle, 2018, XC and Zhou, 2018

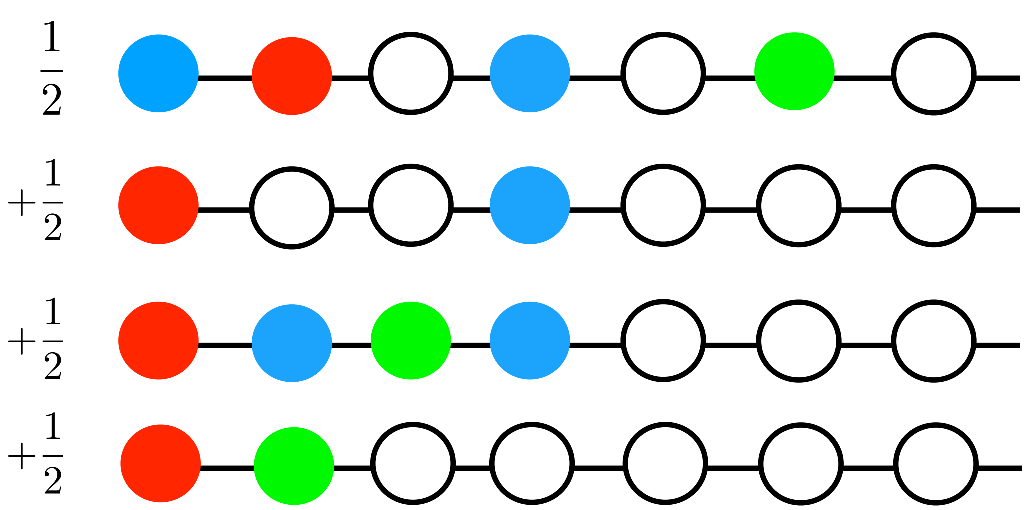
The operator length distribution

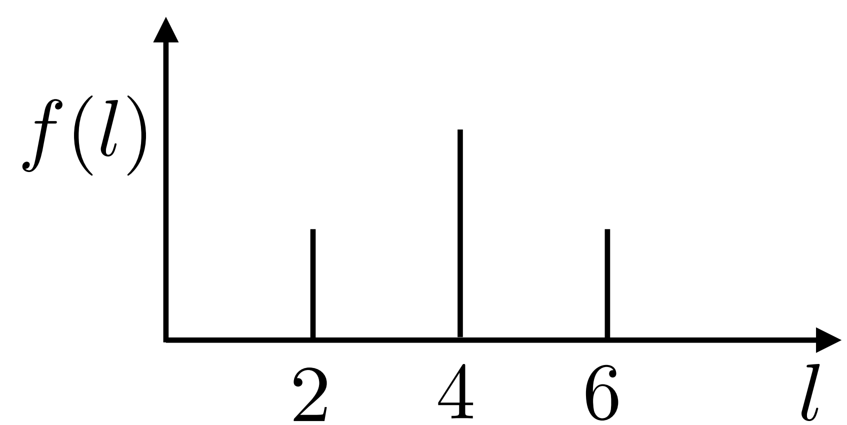
The length of the Pauli string basis operator





The length distribution $$f(l,t) = \sum_j |\alpha_j(t)|^2 \delta(\text{length } (\hat{\mathcal{B}}_j) = l)$$

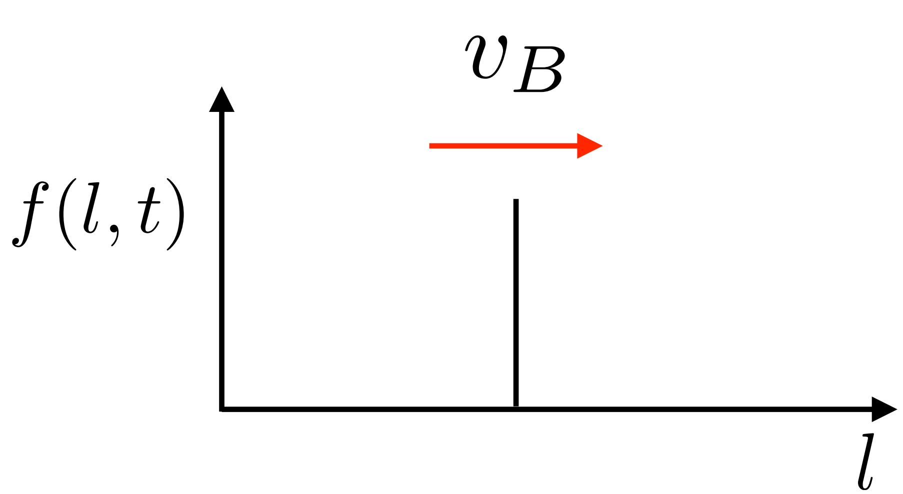




Useful in systems with local interaction

Operator dynamics in spin-1/2 chain with local interaction

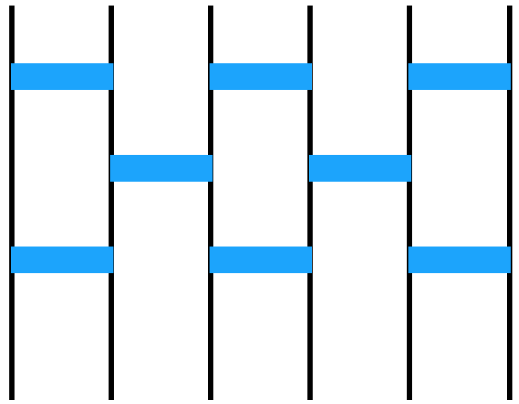
- For spin-1/2 chain, a natural basis operator is the Pauli string operator
- Under unitary time evolution, a local operator $\hat{O}(x, t=0)$ will become increasingly non-local and is a complicated superposition of the basis operators
- It is reasonable to assume the coefficients $\alpha_j(t)$ are uniformly distributed among the Pauli string operators with the same length
- The zeroth order solution $f(l,t) = \delta(l - v_B t)$ neglects the possible dispersion as it moves with group velocity v_B



The hydrodynamics of operator front

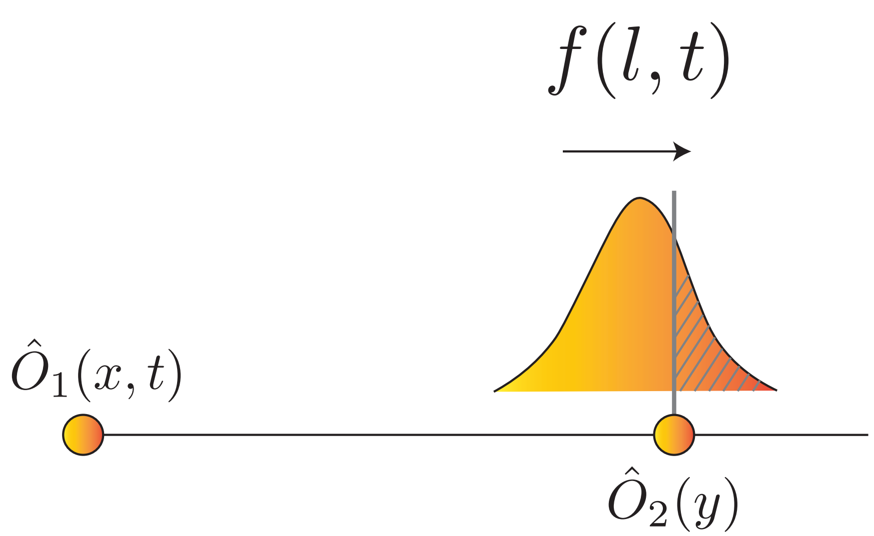
The Haar random circuit provides a simple solution of the distribution function $f(l,t)$

Nahum, Vijay and Haah, 2017
Keyserlingk, Rakovszky, Pollmann, Sondhi, 2017

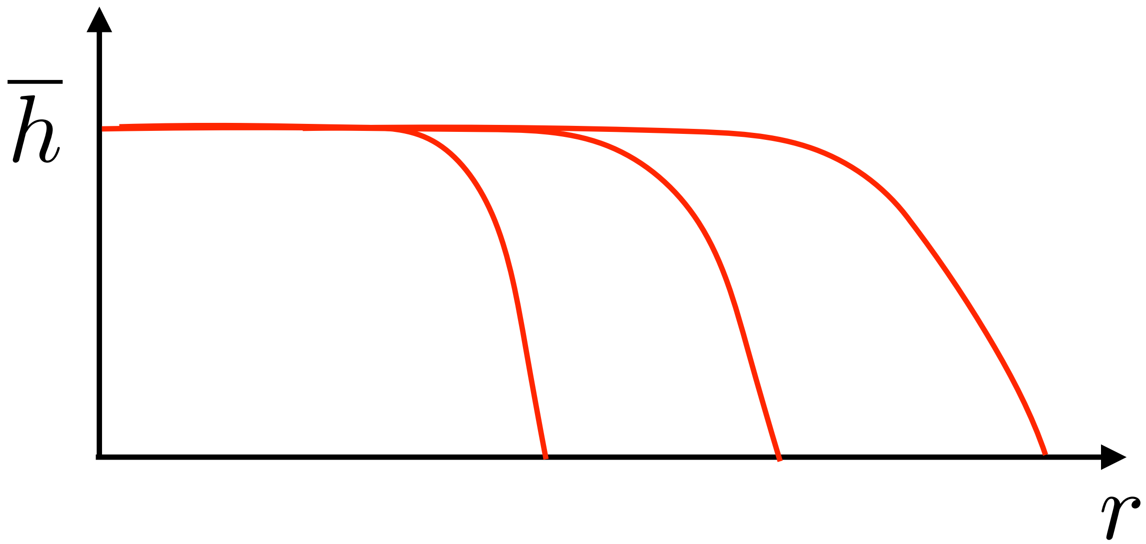


Talk by Vijay in the conference: Dynamics in quantum circuit

(1) The mean length grows linearly with the time
(2) Diffusive wavefront (biased random walk)
(3) OTOC is measuring the area of the wavefront



A moving Gaussian packet



Fast scrambler

- Sachdev-Ye-Kitaev (SYK) model

$$H_{\mathrm{SYK}_4} = \sum_{i,j,k,l=1}^{N} \frac{J_{ijkl}}{4!} \chi_i \chi_j \chi_k \chi_l \qquad \overline{J_{ijkl}} = 0, \quad \overline{J_{ijkl}^2} = \frac{3! J^2}{N^3}$$

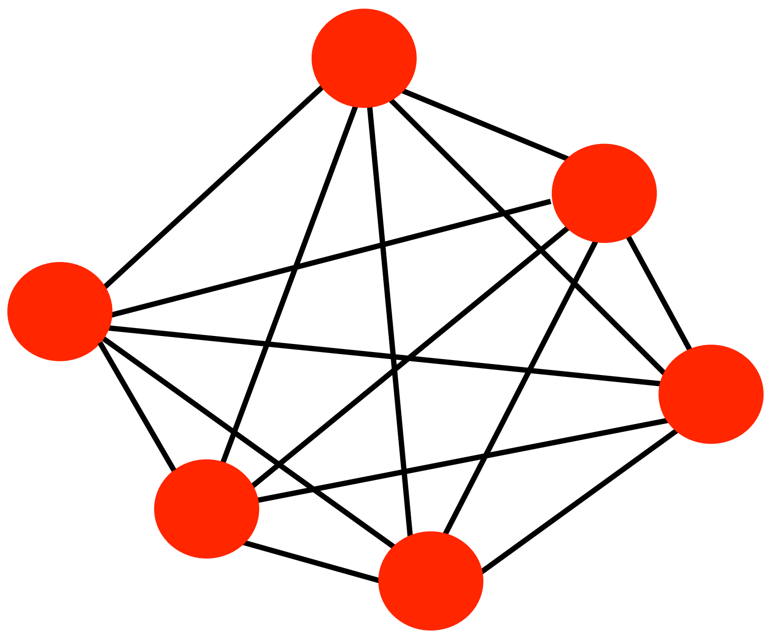
(1) SYK model can be analytically solved in the large N limit
(2) Extensive T=0 residual entropy
(3) Maximally chaotic & saturate the chaos bound at low temperature

Sachdev and Ye, 1993, Kitaev, 2015, Maldacena, Shenker and Stanford 2016
Sekino and Susskind, 2008, Maldacena, Shenker and Stanford 2016

- Two-local qubit model

$$H = \frac{1}{\sqrt{9N}} \sum_{1 \le i < j \le N} \sum_{a,b=1}^{3} \alpha_{a,b,(i,j)} \sigma_i^a \sigma_j^b$$

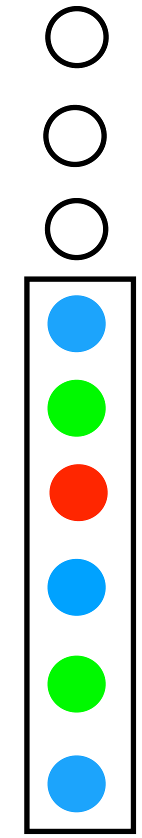
Scrambling time scales as log N



The height of the operator

The height distribution is a weighted sum $f(h,t) = \sum_{\text{height}(B_j)=h} |\alpha_j(t)|^2$

height h=6



The short time evolution h to $h+1$:

$$\hat{\mathcal{B}}(t) \sim \hat{\mathcal{B}}(0) + i[\hat{H}, \hat{\mathcal{B}}(0)]t$$

$$\sigma_i^a \sigma_j^b$$

$2h$ $\quad$ $3(N-h)$

(1) The number of terms which changes the height (does not commute with B) from h to $h+1$ is $6h(N-h)$

(2) The transition time from h to $h+1$ is inversely proportional to the number of terms and scales as $1/h$ when $h<<N$

$$\Delta t \frac{\mathcal{N}(1 \to 2)}{\mathcal{N}(h \to h+1)} = \Delta t \frac{(N-1)}{h(N-h)}$$

(3) The time to grow to the height h is the sum of the transition times in each step that increases the height by 1

$$t = \sum_{l=1}^{h} \Delta t \frac{N-1}{l(N-l)} \approx \Delta t \log \frac{h(N-1)}{N-h} \quad \Rightarrow \quad h(t) = \frac{N e^{\frac{t}{\Delta t}}}{N + e^{\frac{t}{\Delta t}} - 1}$$

The mean height satisfies the logistic differential equation and is linearly proportional to OTOC

$$\frac{dh}{d(t/\Delta t)} = h(1 - \frac{h}{N})$$

Assume that the operator has a typical height with no fluctuation

XC, Zhou, 2018

The height distribution

The Brownian circuit approach

$$e^{-iH_s\Delta t}e^{-iH_{s-1}\Delta t}\dots$$ where $$H_s = J\sum_{i<j}\sum_{\mu_i,\mu_j=0}^{q^2-1}\sigma_i^{\mu_i}\otimes\sigma_j^{\mu_j}\Delta B^s_{i,j,\mu_i,\mu_j}$$

The height distribution $\mathbf{f}(h,t)$ is determined by the master equation:

$$\frac{d\mathbf{f}(t)}{dt} = A_f\mathbf{f}(t)$$

A is tri-diagonal matrix

$$(A_f)_{k,k} = \frac{4}{n}k\big[-(n-k)+\frac{1}{q^2}(n-2k+1)\big]$$
$$(A_f)_{k-1,k} = \frac{4}{n}\frac{k(k-1)}{q^2}$$
$$(A_f)_{k+1,k} = \frac{4}{n}k(n-k)\big[1-\frac{1}{q^2}\big].$$

In the continuum limit,

$$\partial_t f(h,t) = -\frac{4}{N}\partial_h[h(h_{\text{sat}}-h)f]$$ → $$\partial_t\langle h\rangle = \frac{4}{N}(h_{\text{sat}}-\langle h\rangle)\langle h\rangle - \frac{4}{N}(\langle h^2\rangle - \langle h\rangle^2)$$

logistic differential equation

In the large N limit, $$\frac{df_k(t)}{dt} = -\lambda_q k f_k + \lambda_q(k-1)f_{k-1}$$

$$\lambda_q = 4(1-\frac{1}{q^2})$$

Similar equation was found in SYK model in some limit

Zhou and XC, 2018

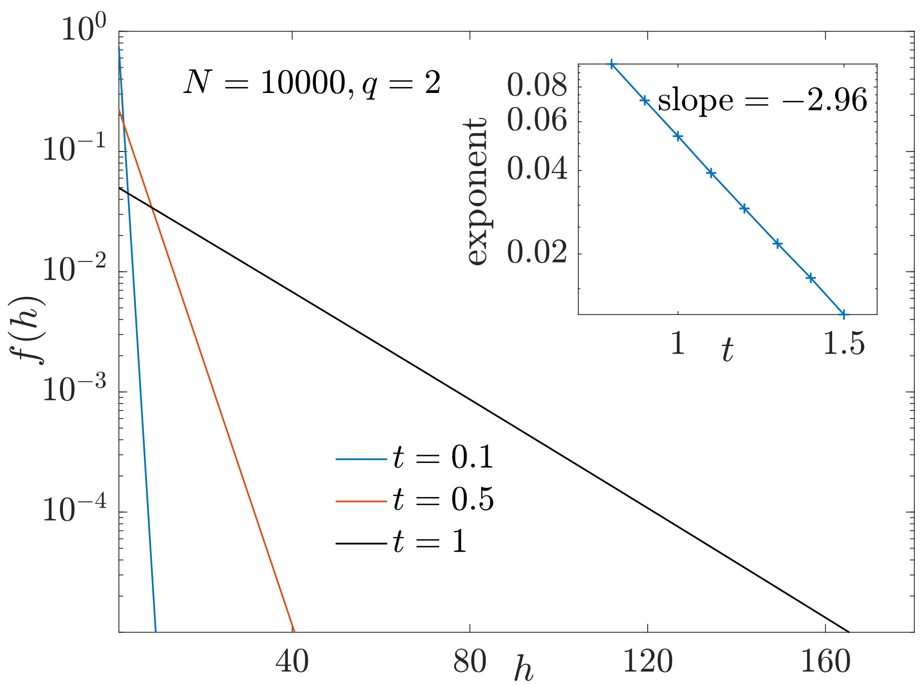
Roberts, Stanford and Streicher, 2018

Early time behavior

$$\frac{df_k(t)}{dt} = -\lambda_q k f_k + \lambda_q (k-1) f_{k-1}$$

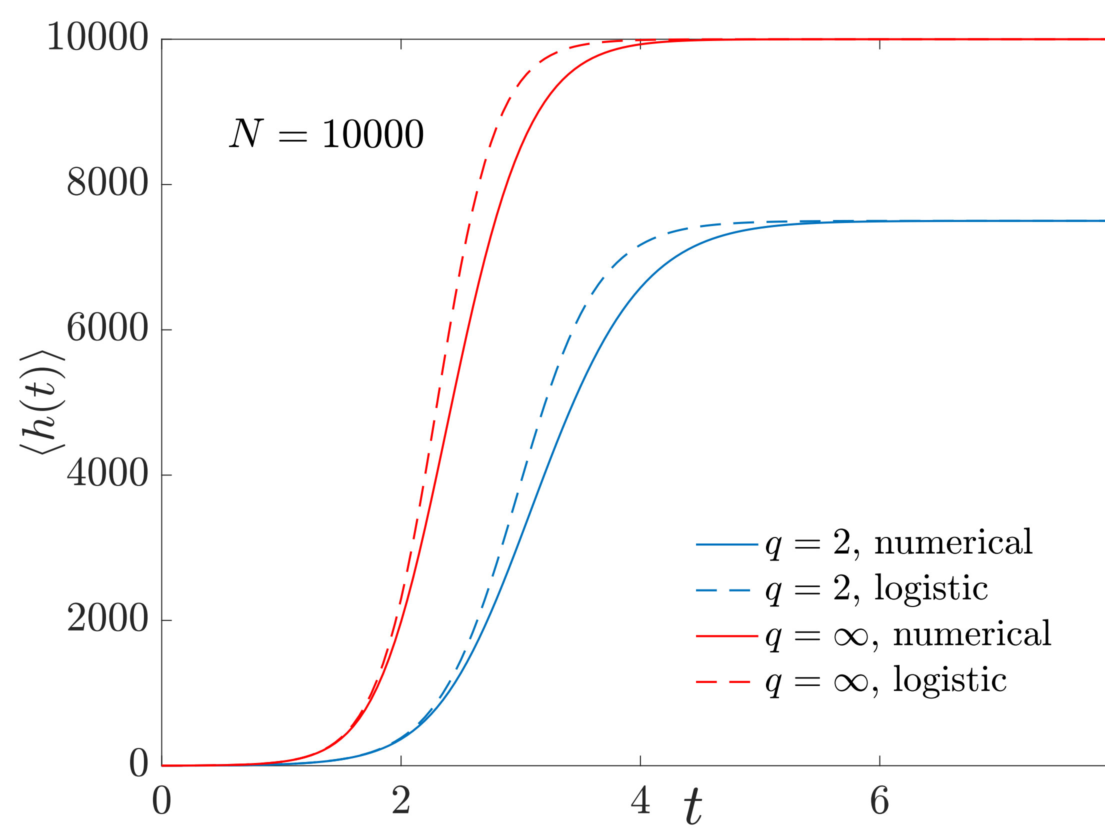
$$f_k(t) = e^{-\lambda_q t}[1 - e^{-\lambda_q t}]^{(k-1)}$$

$$\langle h(t) \rangle = e^{\lambda_q t} \langle h(t=0) \rangle \quad \lambda_q = 4(1 - \frac{1}{q^2})$$



The distribution function looks like a collapsing sandpile whose surface is exponential decreasing in space.

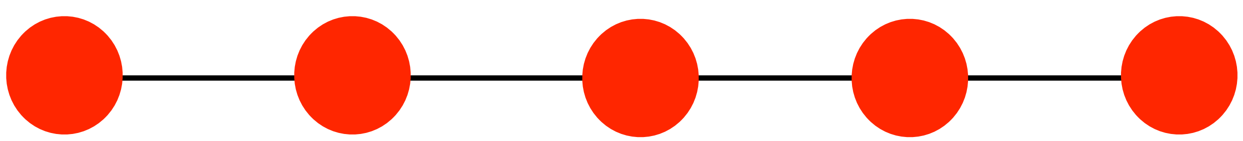
Comparison with logistic function



Zhou and XC, 2018

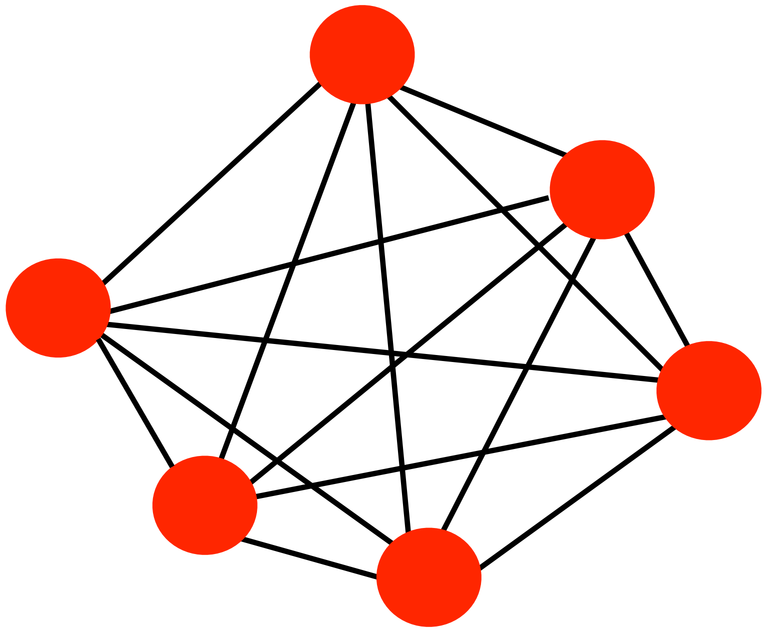
Two pictures of operator dynamics

Local interaction



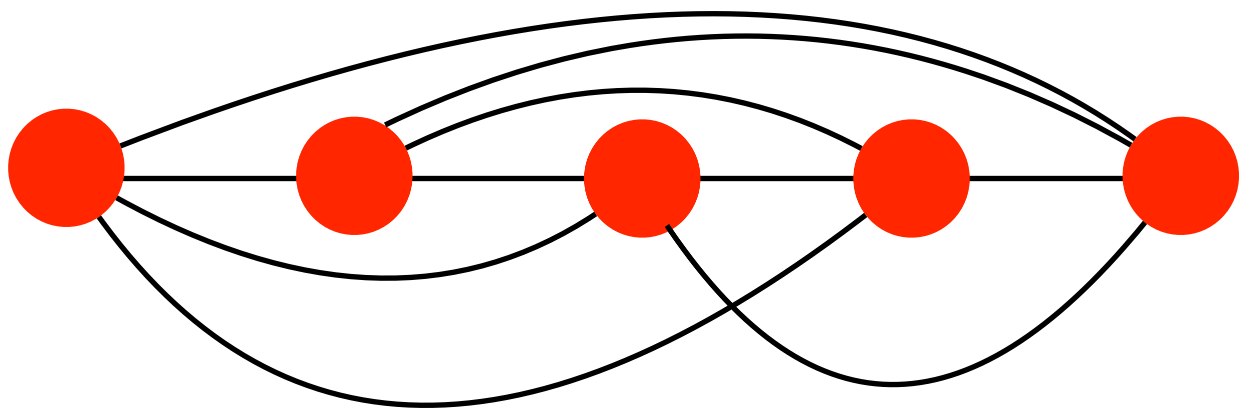
(1) The length distribution (end point distribution) is a moving Gaussian packet with a constant velocity.
(2) OTOC is measuring the area of the wavefront/ the mean height at each site.

All-to-all interaction



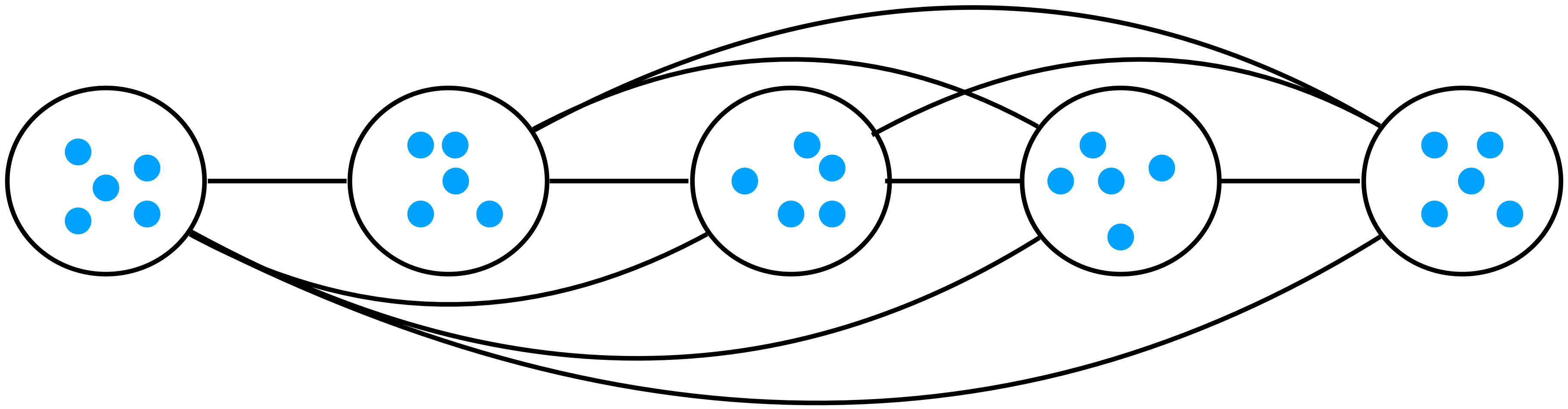
(1) The height distribution is determined by a master equation and can be explained by the collapsing sandpile picture.
(2) OTOC is measuring the mean height of the operator.

Operator dynamics in long range interaction systems



- The emergent light cone behavior
- The OTOC (mean height) dynamics

Brownian circuit with power law interaction



Three parameters:
N, L and α

$$H = \sum_{i \neq j} \sum_{n,m=1}^{N} \sum_{\mu,\nu=0}^{3} J_{\mu\nu}(i,j,m,n,t)\sigma^{\mu}(i,m)\sigma^{\nu}(j,n)$$

$$J \sim \frac{1}{|i-j|^{\alpha}}$$

The complete unitary time evolution is generated in the continuum limit of $e^{-iH_s\Delta t}e^{-iH_{s-1}\Delta t}\ldots$

The statistical average of the operator spreading is analytical tractable

Lashkari, Stanford, Hastings, Osborne and Patrick Hayden, 2011
Xu and Swingle, 2018, Zhou and XC, 2018

The operator height distribution $f(\mathbf{h},t) = \sum_{\mathrm{height}(B_\mu)=\mathbf{h}} |\alpha_\mu(t)|^2$ where **h** is a L-component vector

The joint distribution f(h) is governed by this master equation:

$$\frac{\partial f(\mathbf{h},t)}{\partial t} = \sum_{j\neq i} 3D_{ij}h_j f(\mathbf{h}-\mathbf{e}_i,t) + \sum_{j\neq i} D_{ij}h_j f(\mathbf{h}+\mathbf{e}_i,t) - \left\{\sum_{j\neq i} 3D_{ij}h_j(N-h_i) + \sum_{j\neq i} D_{ij}h_j h_i\right\} f(\mathbf{h},t)$$

Operator dynamics → Classical stochastic problem → Classical simulation

The update rule in the N=1 case and the connection with kinetic Ising model

$\uparrow$

before	after	rate($q=2$)	rate($q=\infty$)
$\uparrow$	$\downarrow$	D_{ij}	0
$\downarrow$	$\uparrow$	$3D_{ij}$	$4D_{ij}$

One spin facilitated Fredrickson-Andersen model

Talk by Garrahan in the program: Kinetically Constrained Models

Initial condition 1 0 0 0 0 0

Final steady state 1 1 1 1 1 1

$(q=\infty)$

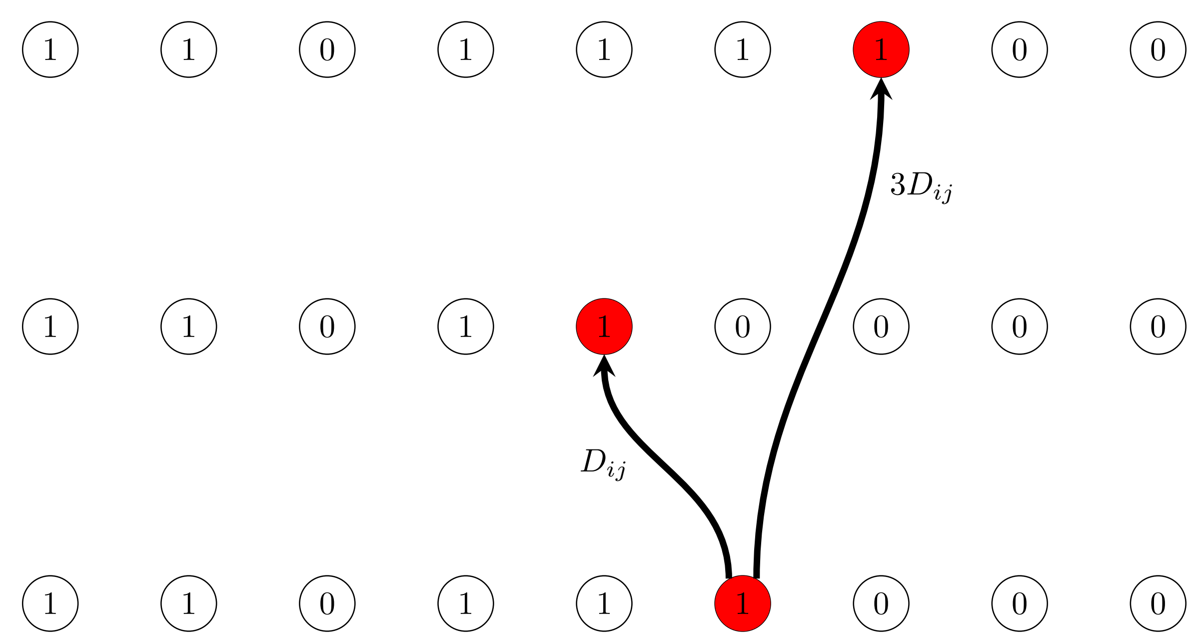
- Local interaction

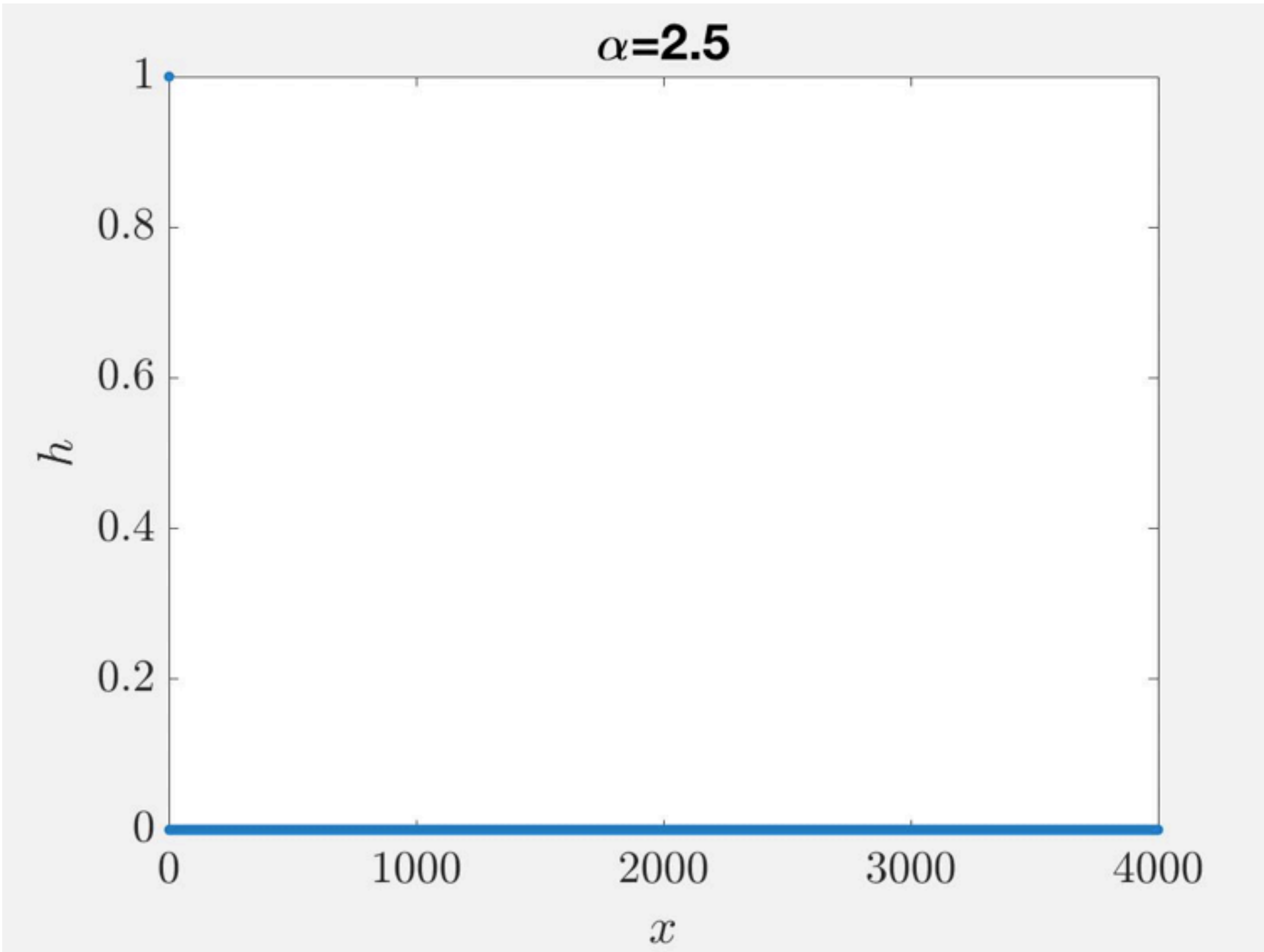
$$D_{ij} \neq 0 \text{ only when } i = j \pm 1$$

(1) The end point is performing biased random walk
(2) The wavefront interpolates between the left h=1 and the right h=0 domains
(3) The physics is the same as Haar random circuit

- All-to-all interaction

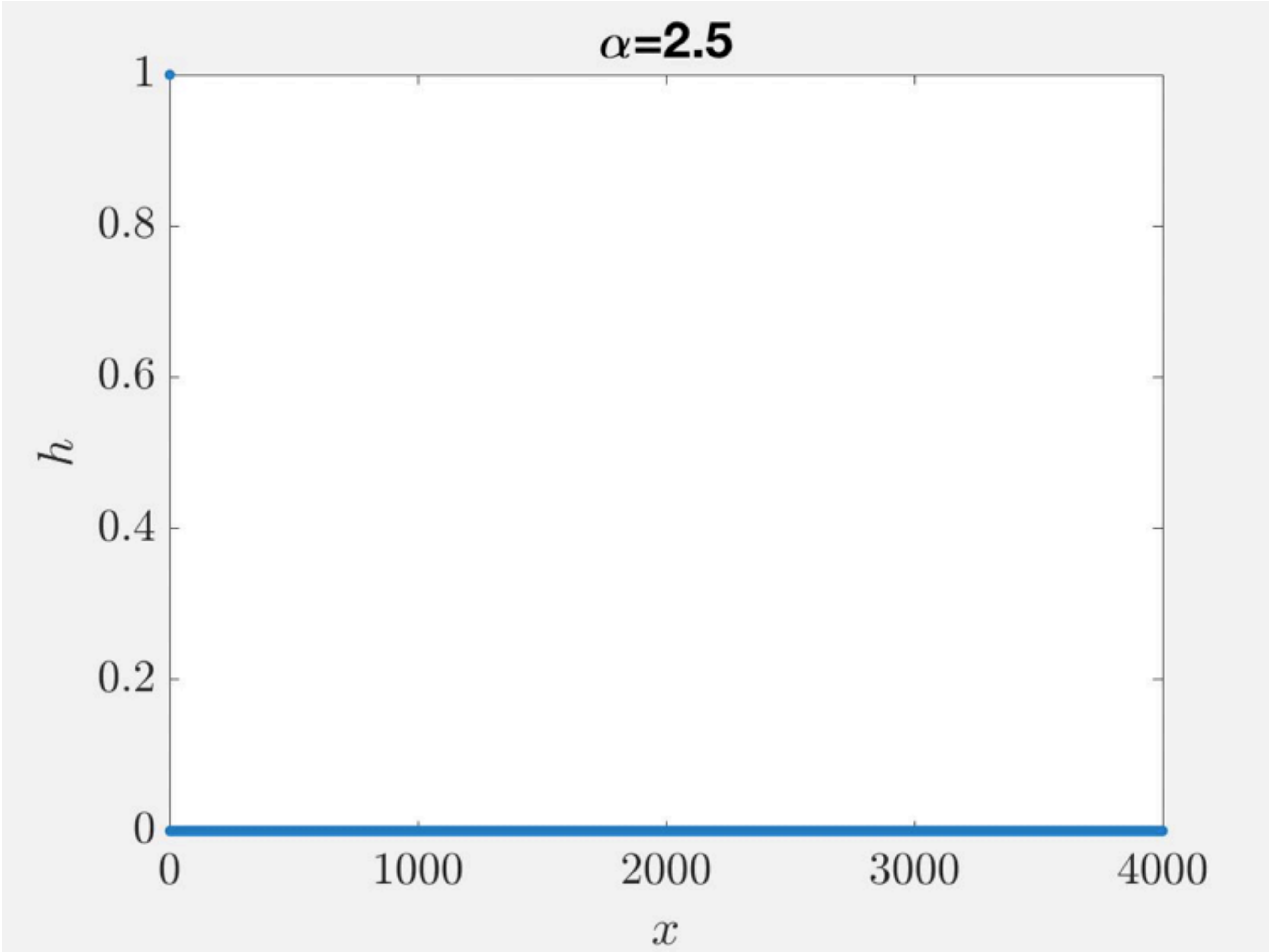
- Non-local power law interaction $D_{ij} \sim \dfrac{1}{|i-j|^{\alpha}}$





(1) The growth of the h=1 domain

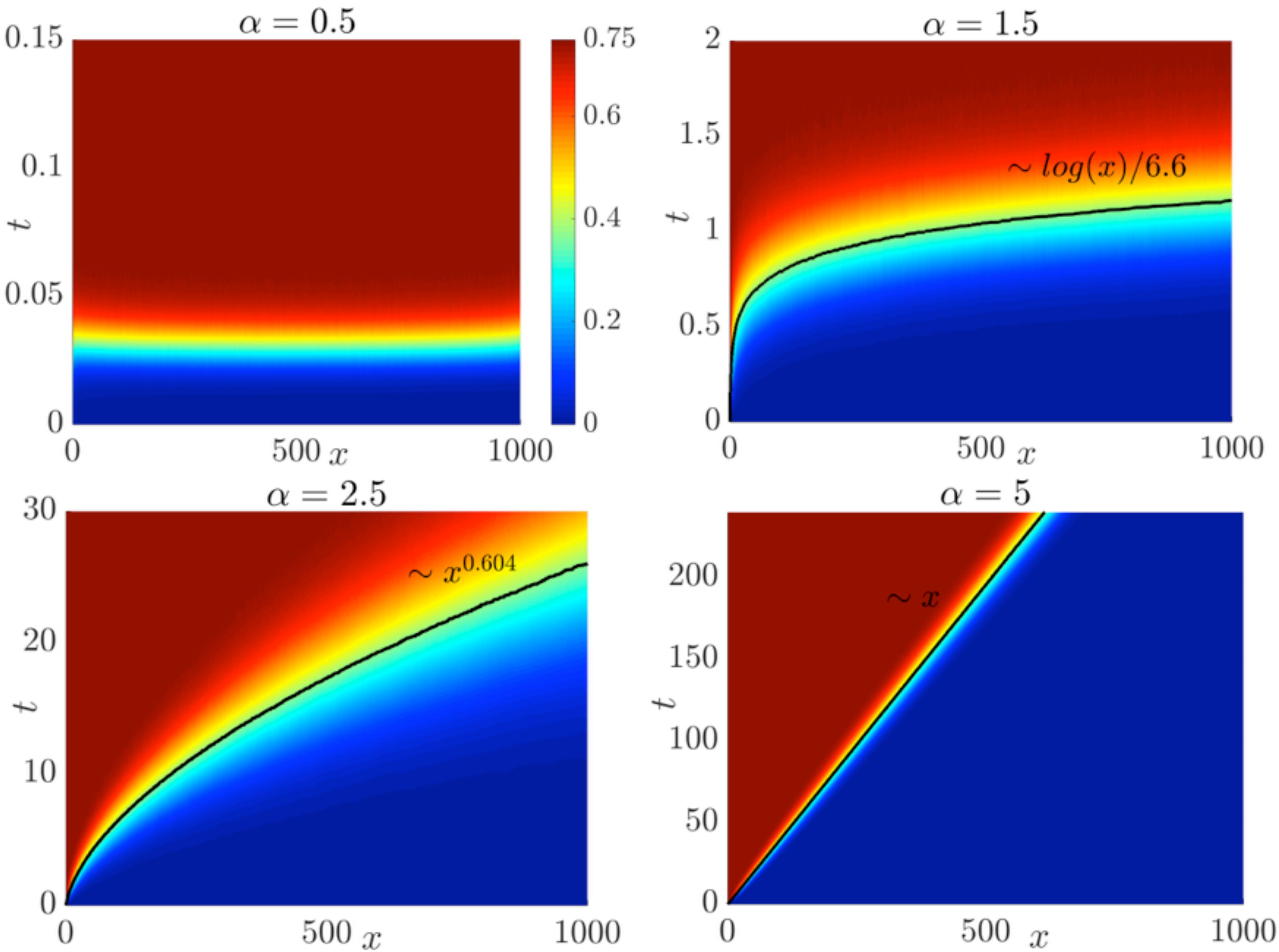
(2) The non-local flipping process



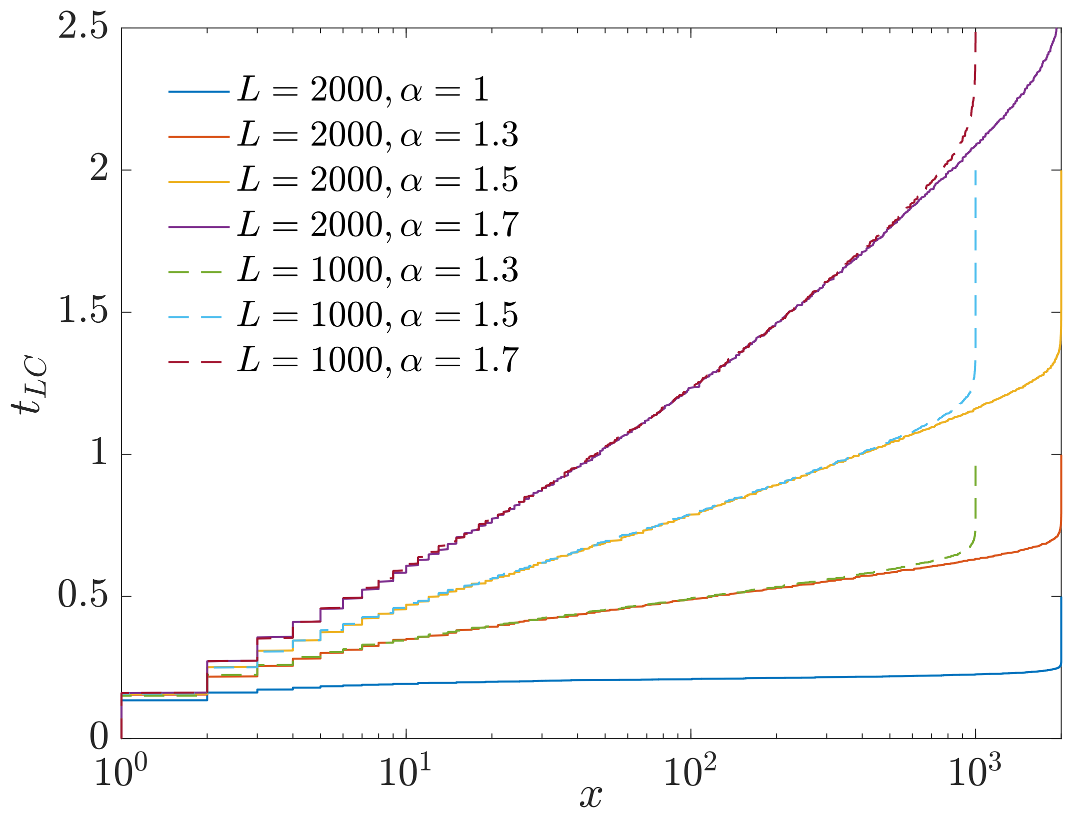
(1) The growth of the h=1 domain

(2) The non-local flipping process

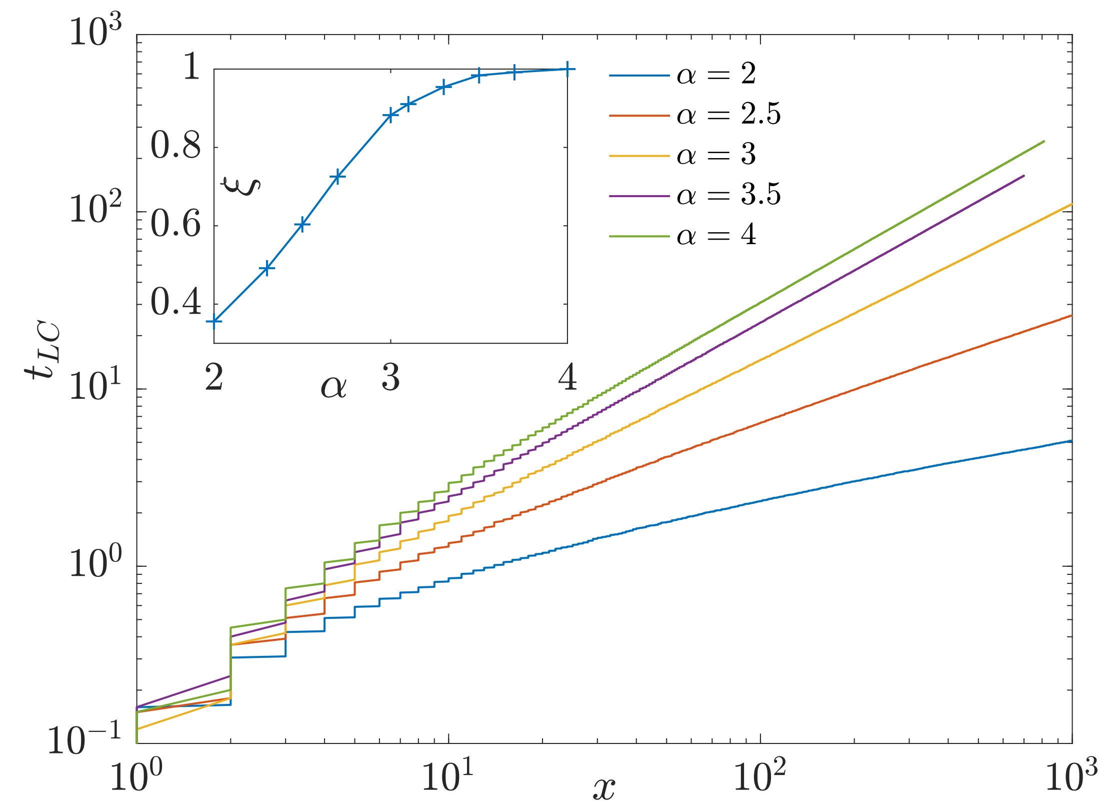
The formation of effective light cone



Light cone and butterfly velocity



$t \sim x^{\xi}$



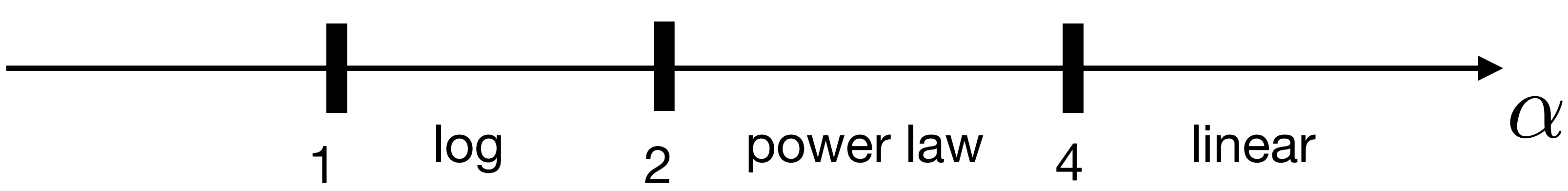
Local interaction

v_B is a constant and is bounded by Lieb-Robinson velocity

$$v_B = x/t_{LC}$$

Power law interaction

v_B can be (1) a constant, (2) grows algebraically or (3) grows exponentially in time

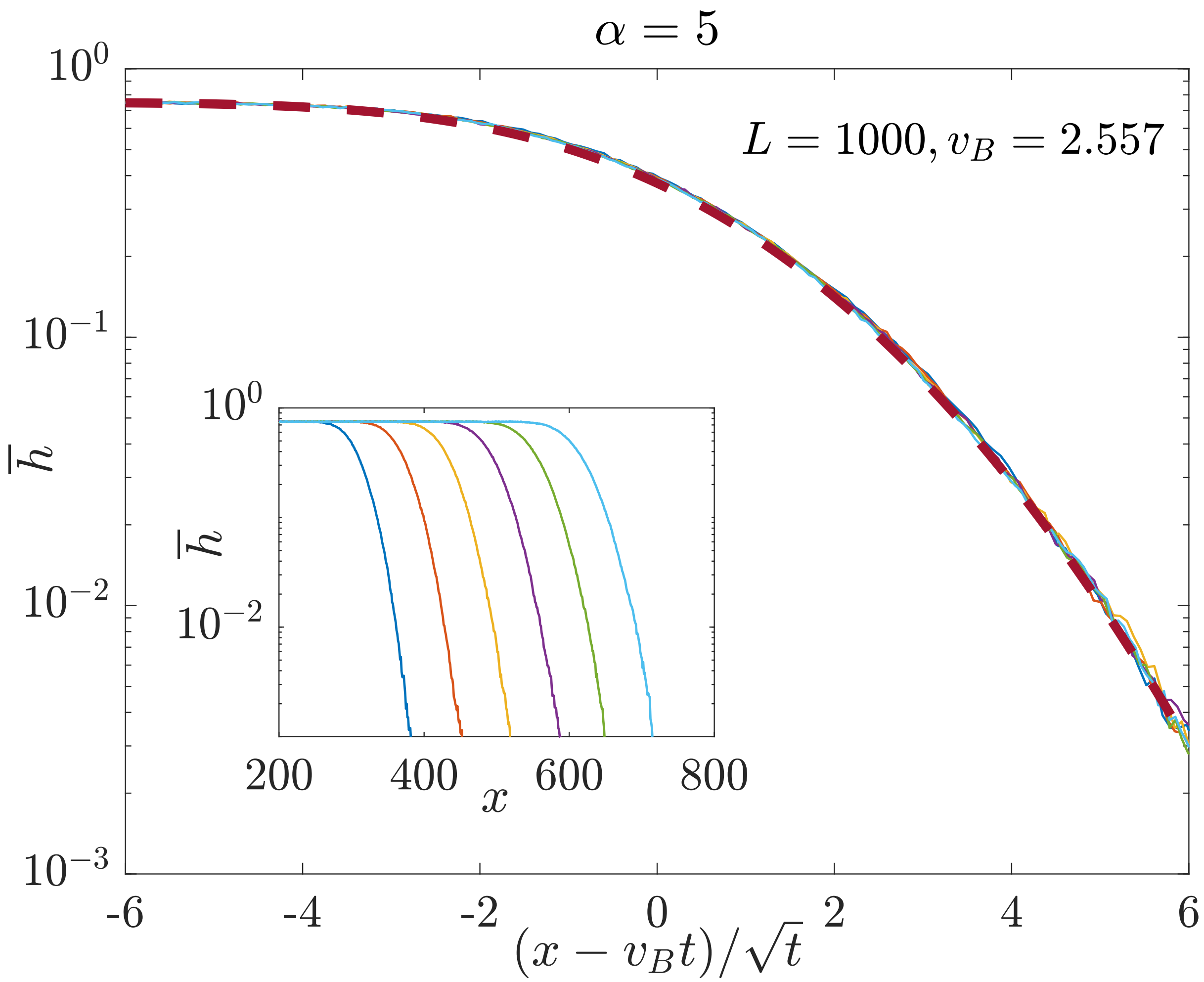


Hastings and Koma, 2006, Foss-Feig, Gong, Clark and Gorshkov, 2015

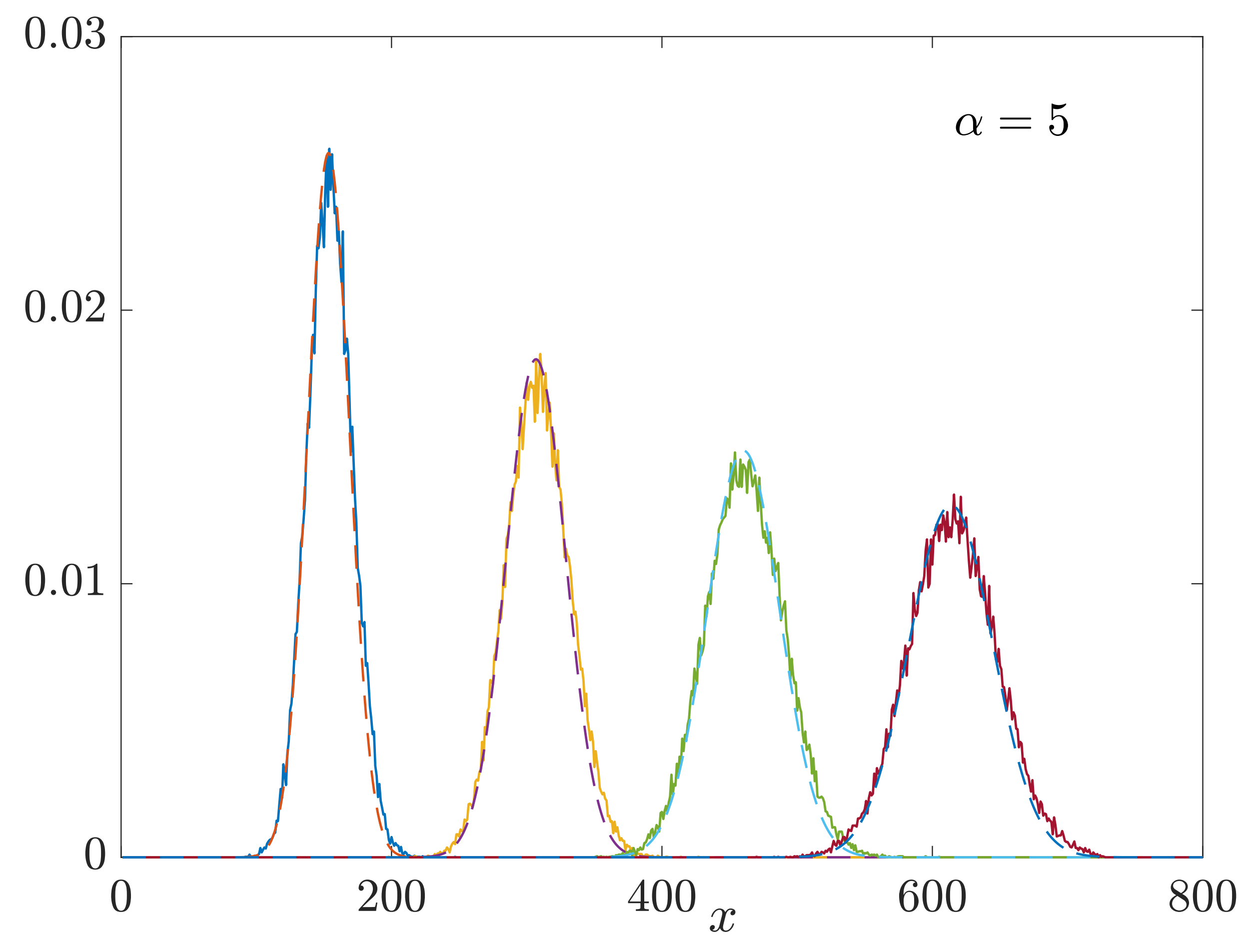
Talk by Gorshkov in the program: Information Propagation and Entanglement Generation with Long-Range Interactions

Emergent linear light cone and locality

$$h(x,t) \to h\left(\frac{x - v_B t}{g(t)}\right) \longrightarrow \text{scaling argument}$$

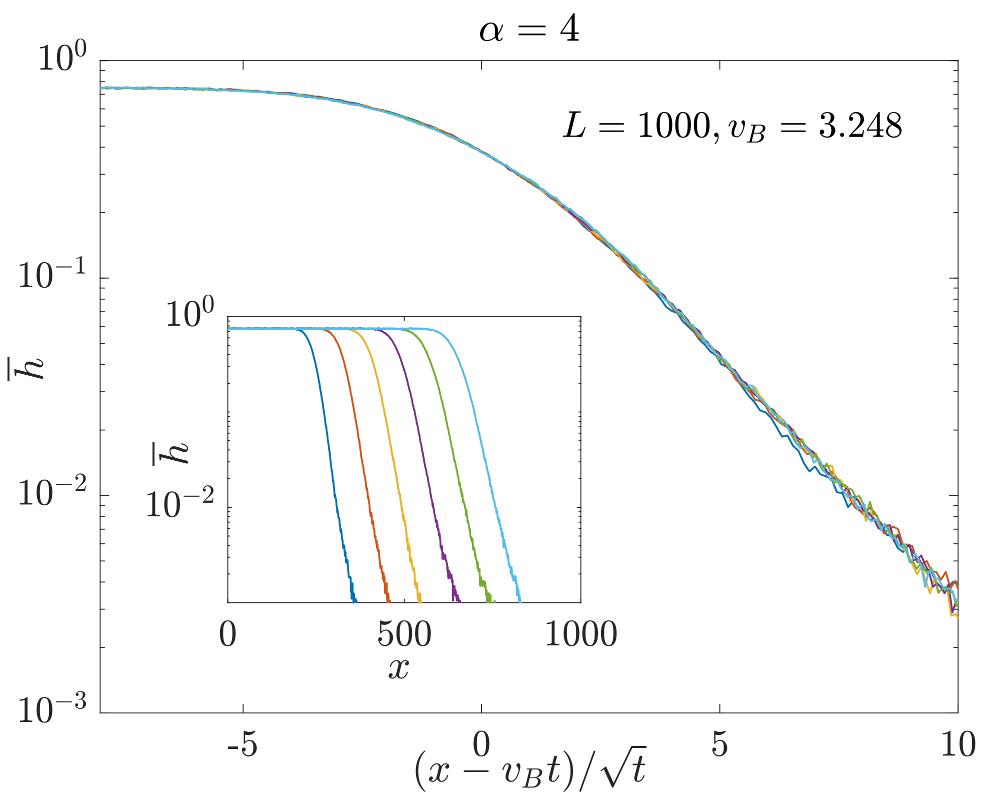


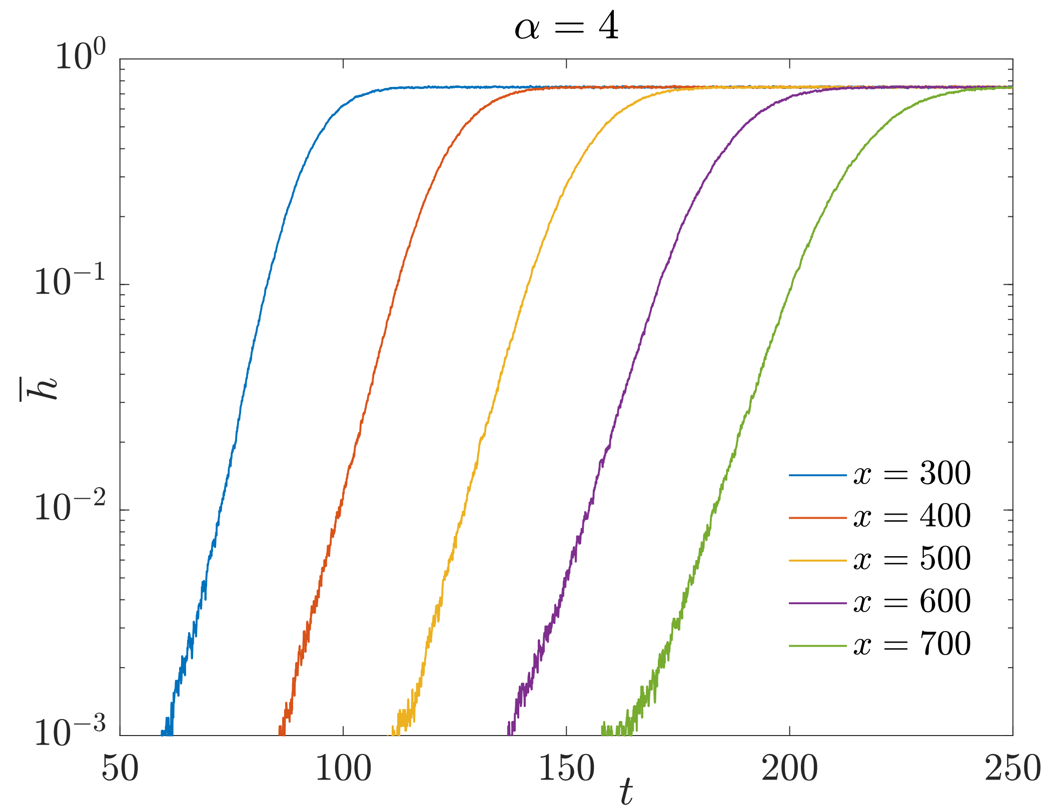
Data collapse of mean height



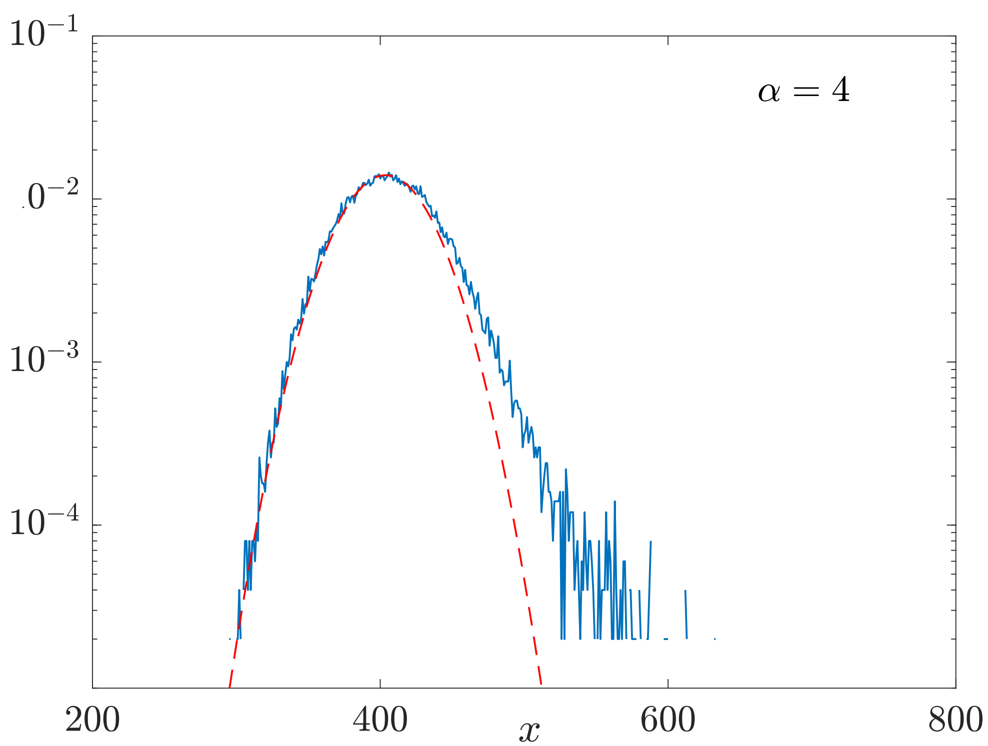
End point distribution

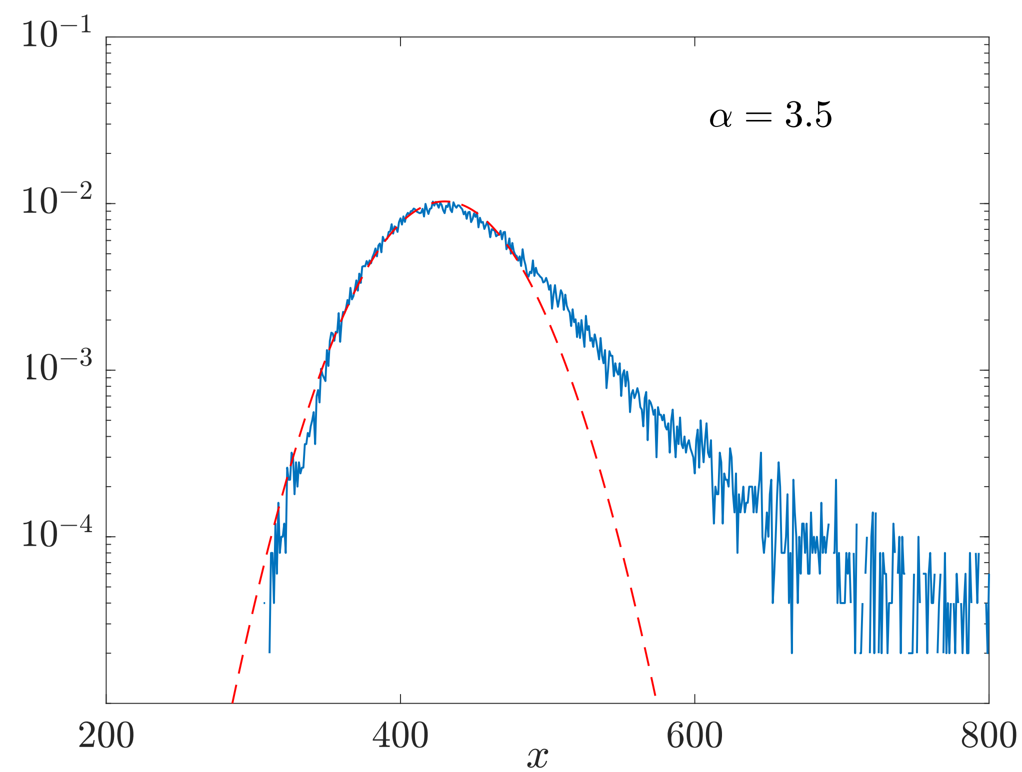
Non-Gaussian tail



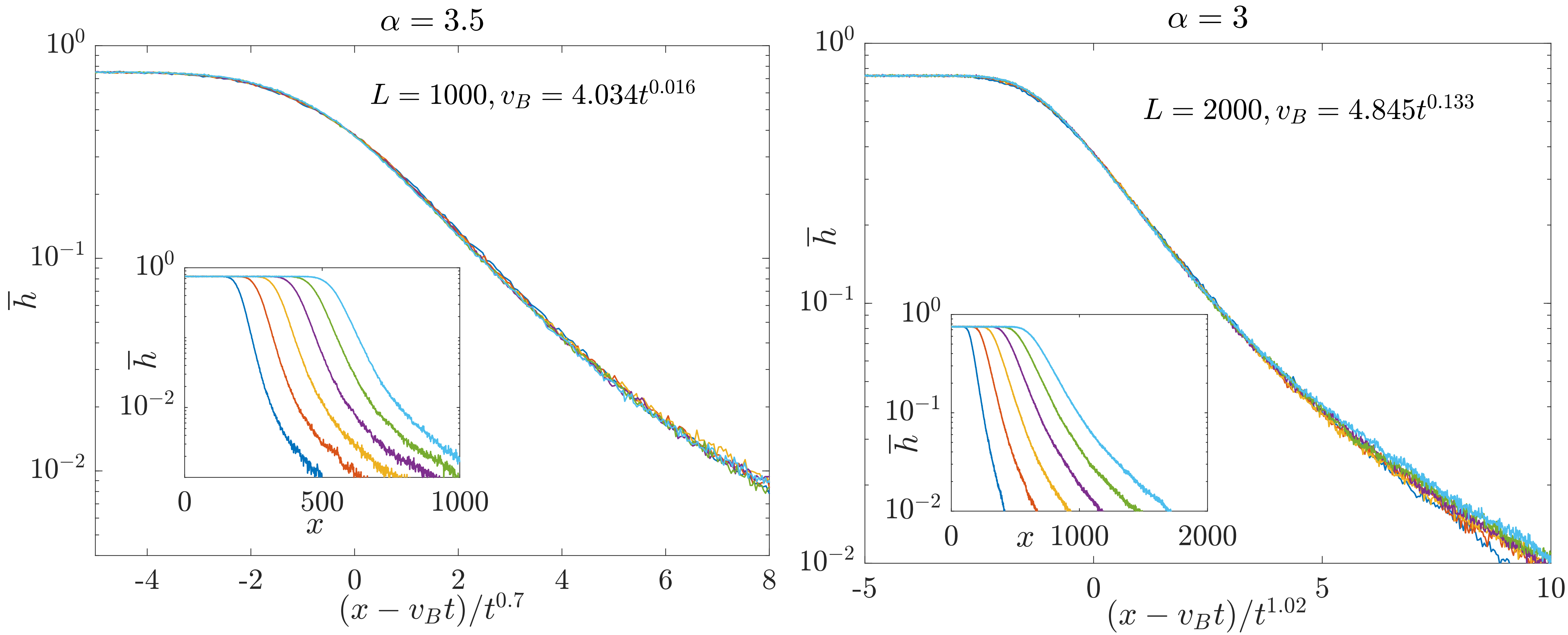


XC, Zhou and Xu, 2017





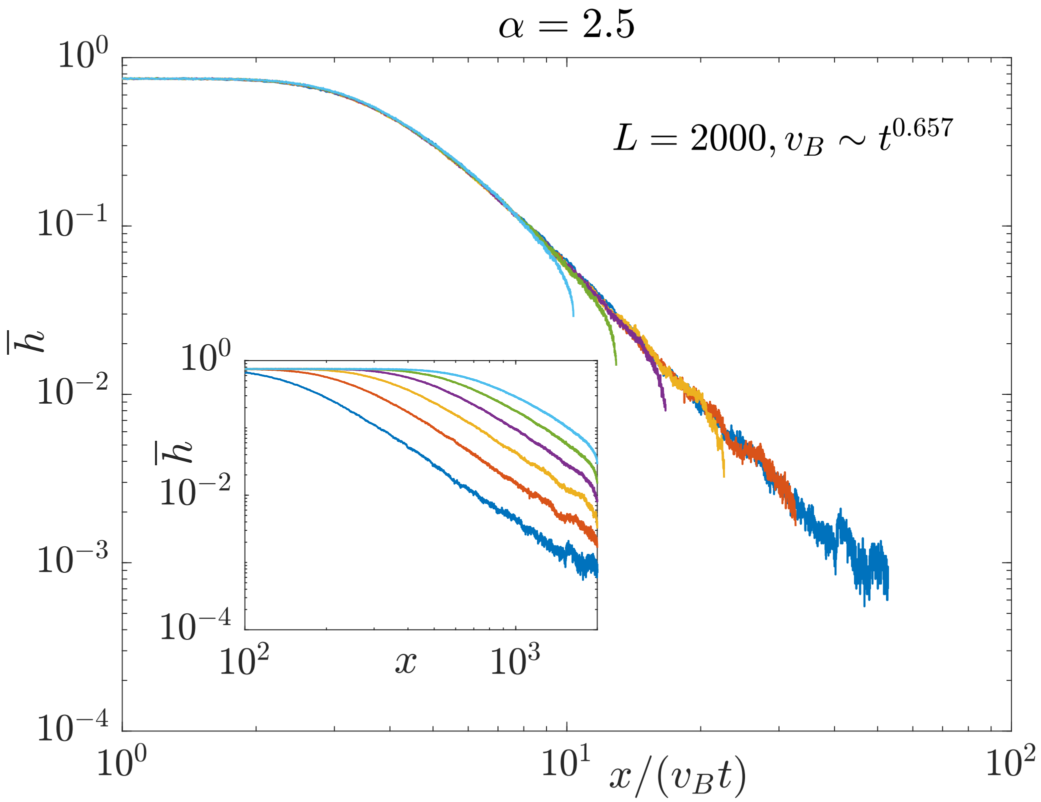
The end point distribution cannot be directly connected to mean height when α is small

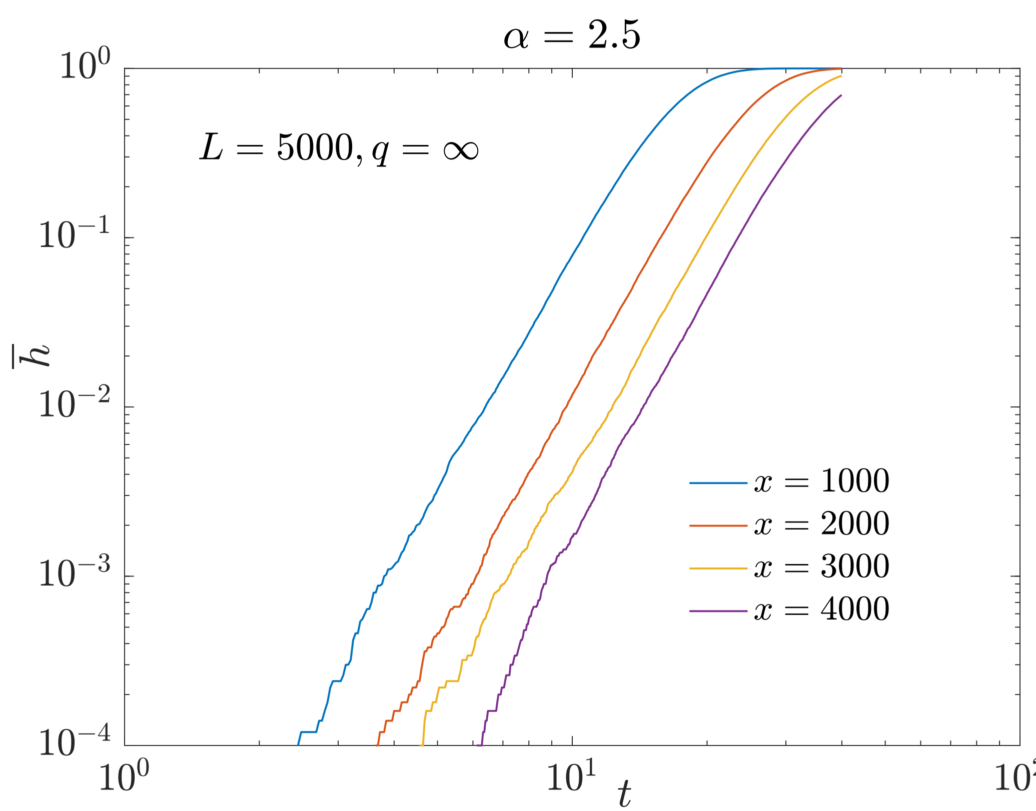


$g(t) \sim t^b$ with $b > 0.5$

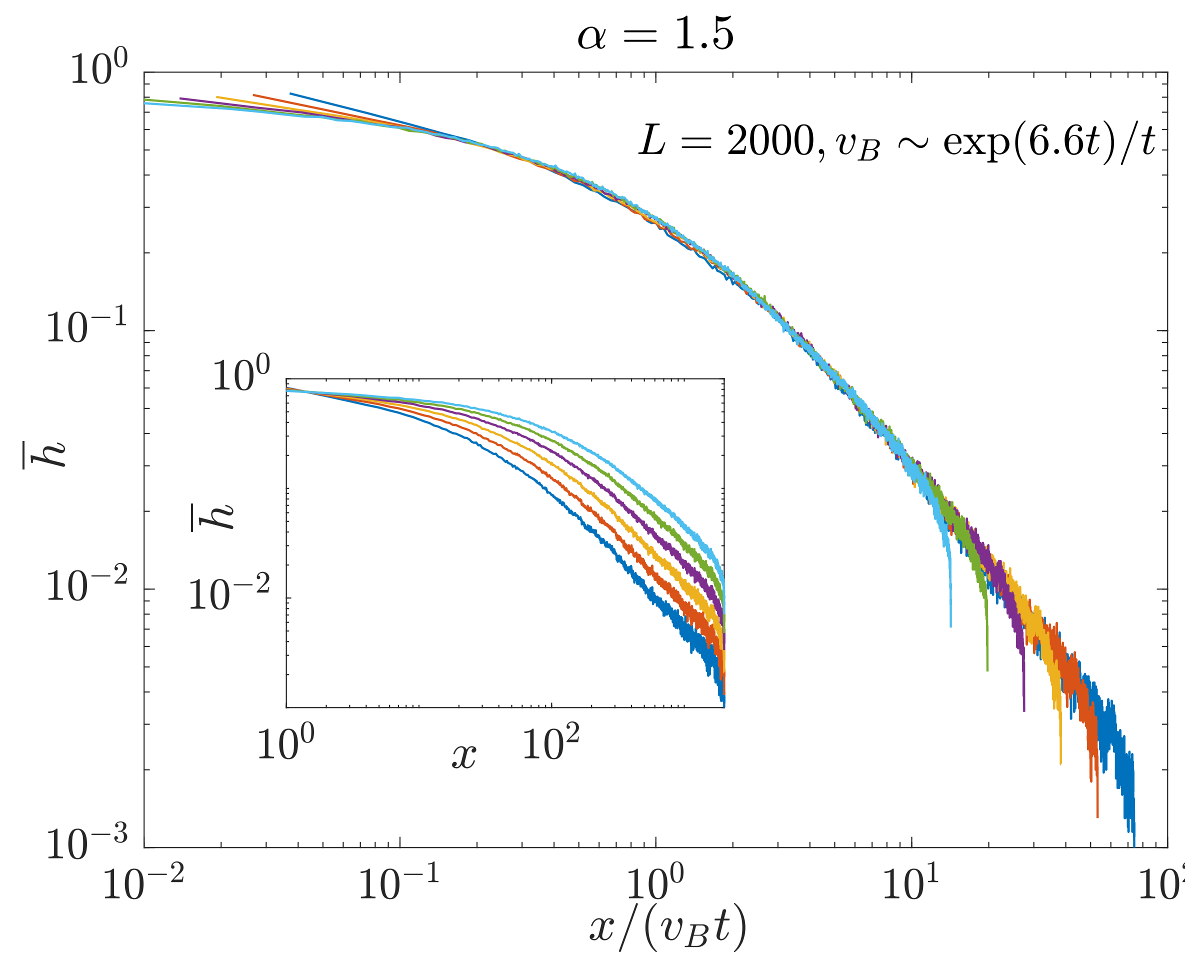
$$\overline{h(x,t)} \sim \left(\frac{t^b}{x}\right)^c$$

Power law in both directions

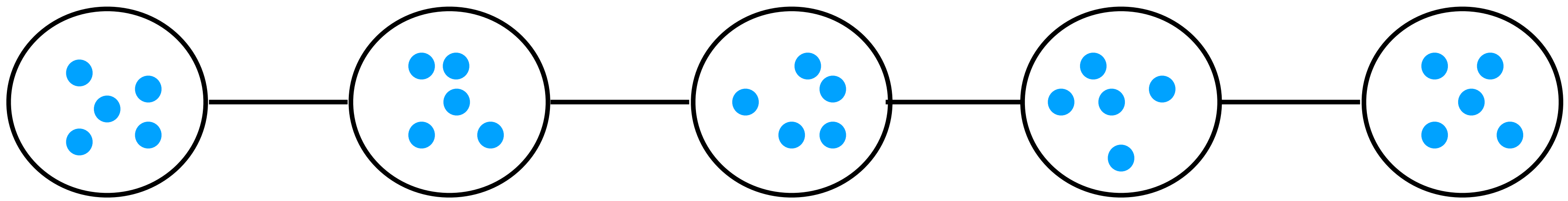




$$\overline{h(x,t)} \sim \frac{e^{\lambda t}}{x^c}$$



Large N limit with local interaction



Similar to SYK chain

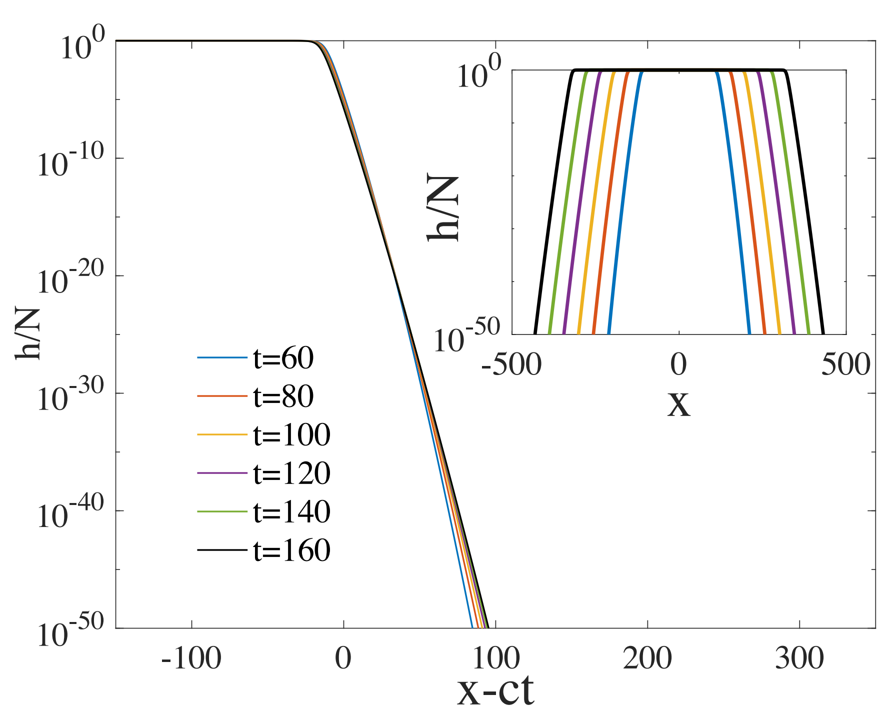
Gu, Qi and Stanford, 2016

Operator dynamics in two directions:
(1) the local onsite Hilbert space
(2) the spatial direction

Logistic differential equation + Spatial diffusion → Fisher-KPP equation

$$\frac{\partial h}{\partial t} = D\frac{\partial^2 h}{\partial x^2} + \lambda h(1 - \frac{h}{N})$$

XC and Zhou, 2018



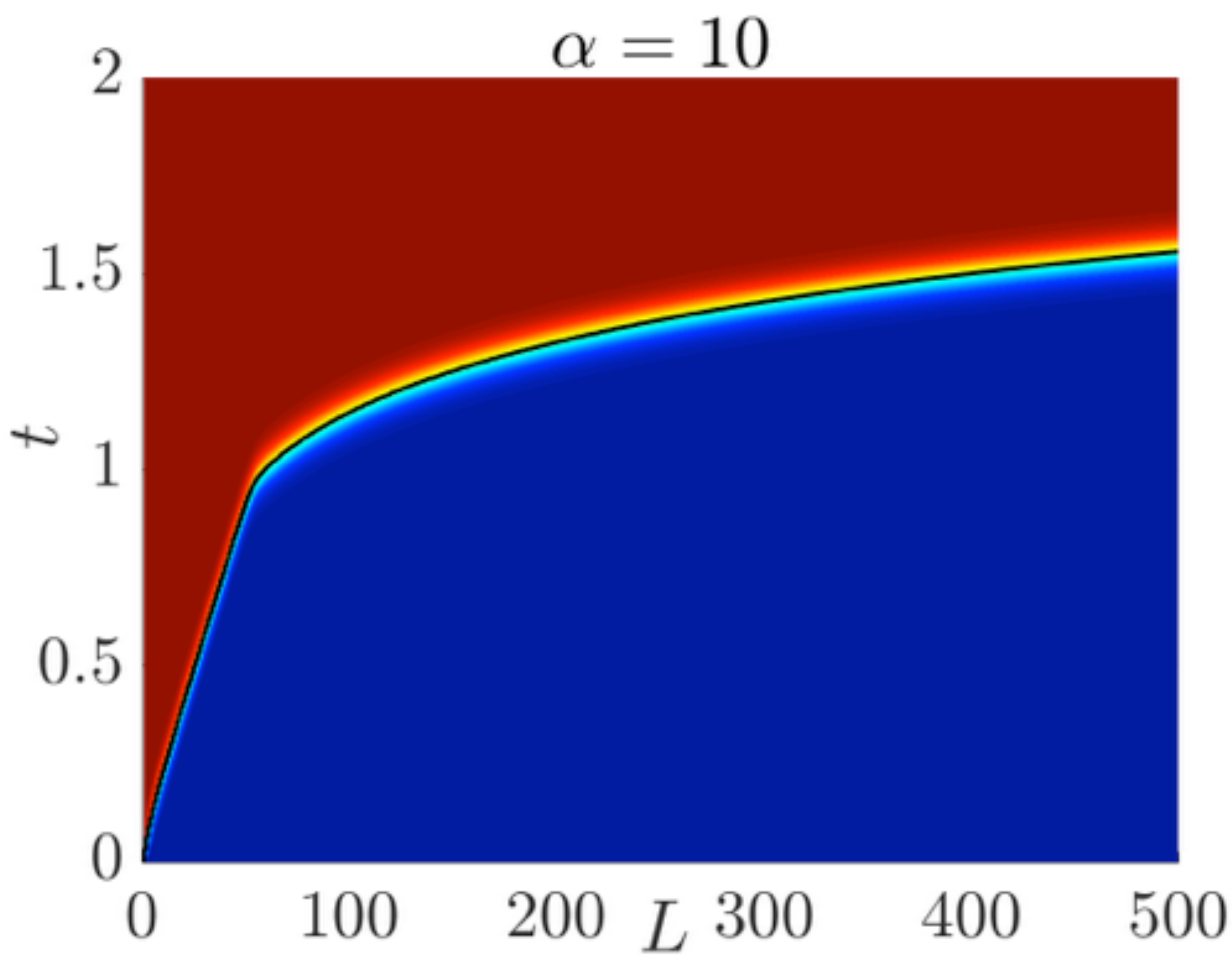
(1) Stable traveling wave solution & no dispersion
(2) Crossover to the diffusive wavefront picture at finite N. This is done by Xu and Swingle, where they also used Brownian circuit technique

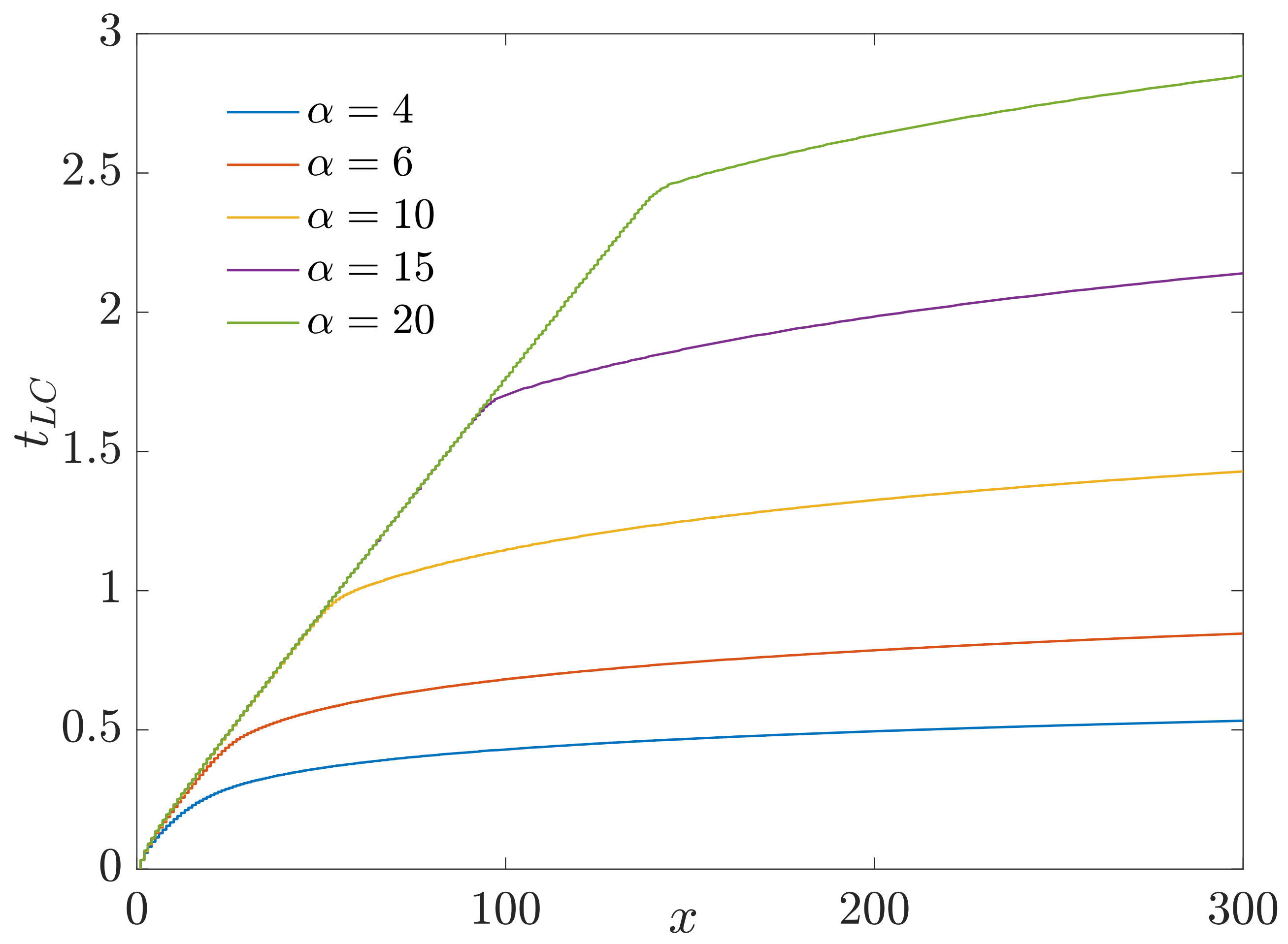
Xu and Swingle, 2018

Large N limit with power-law interaction

$$\frac{\partial \underline{h}(x,t)}{\partial t} = \int dy \underline{h}(y,t) D(y,x)(1 - \frac{1}{h_{sat}} \underline{h}(x,t))$$

The power law kernel $D(x,y) = \frac{1}{|x-y|^{\alpha}}$





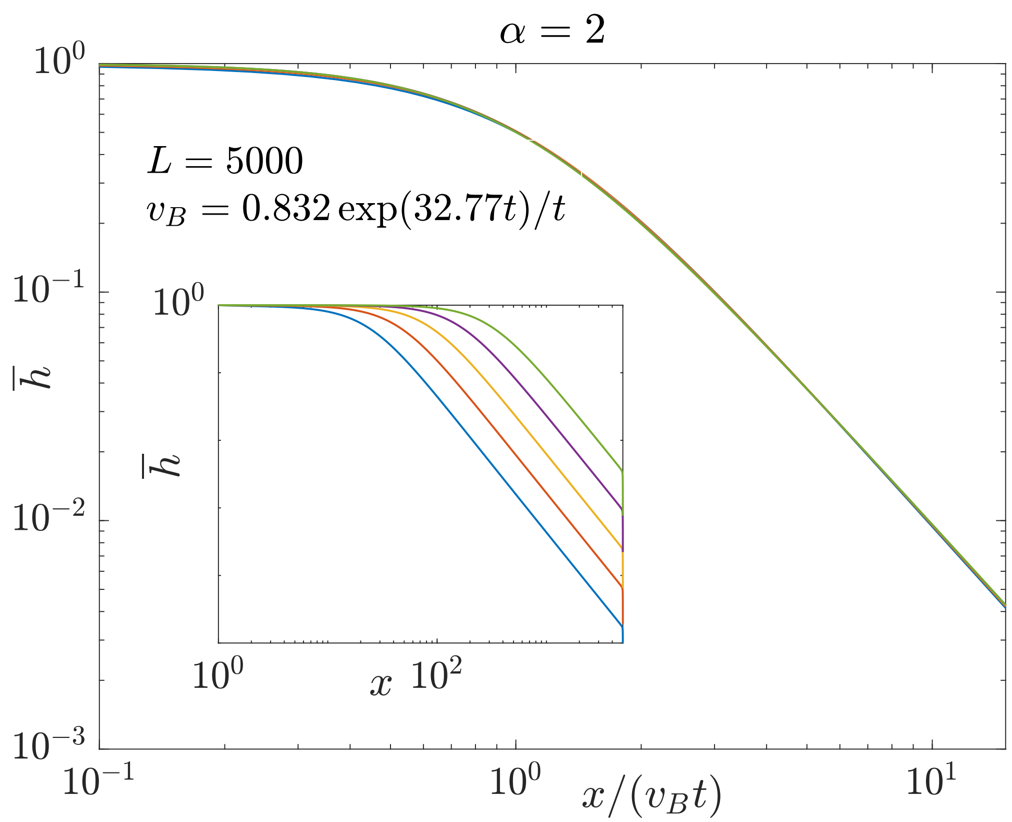
Three observations:

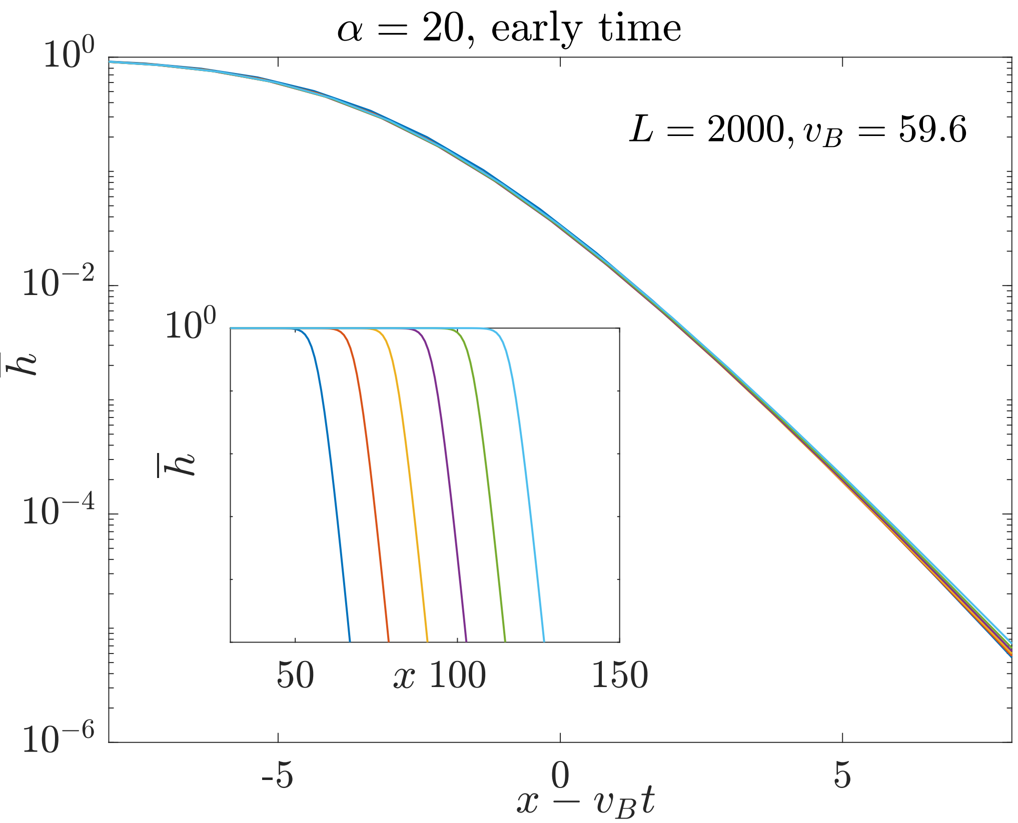
(1) The log light cone scales as $t_{LC} \sim \alpha \log x$

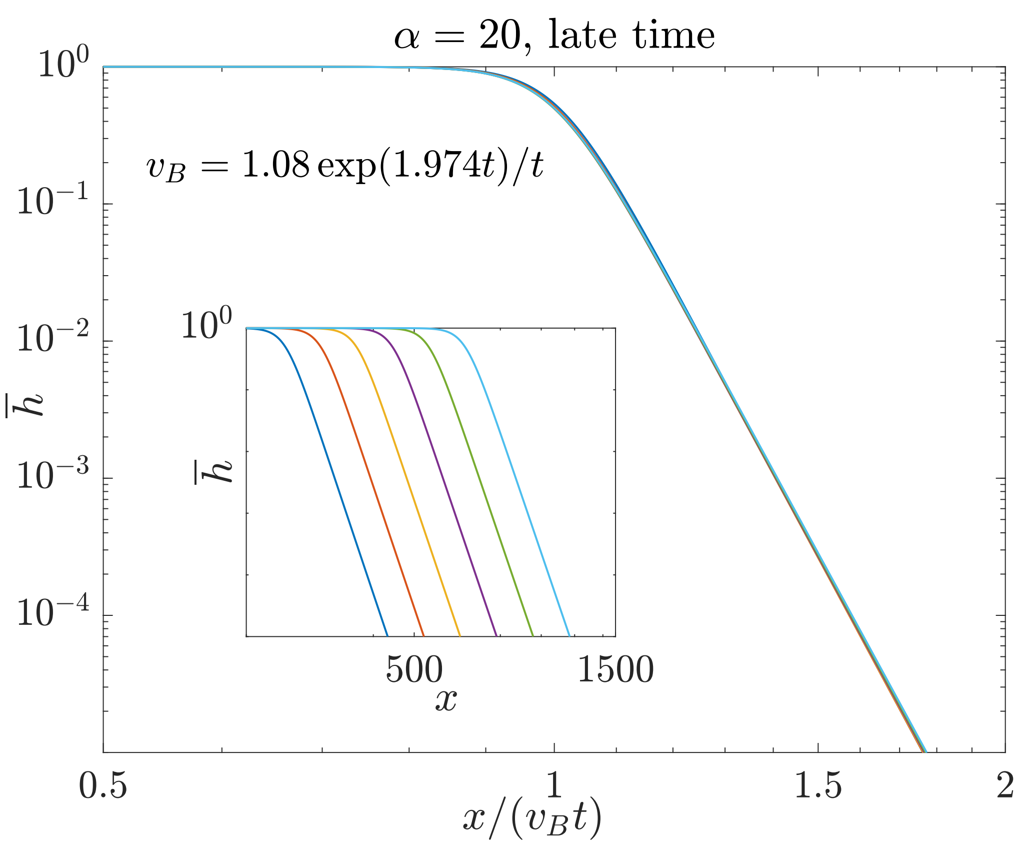
(2) The butterfly velocity in the linear light cone regime is independent of α

(3) The transition from linear to log light cone occurs at the intersection $x \sim \alpha \log x$

Data collapse





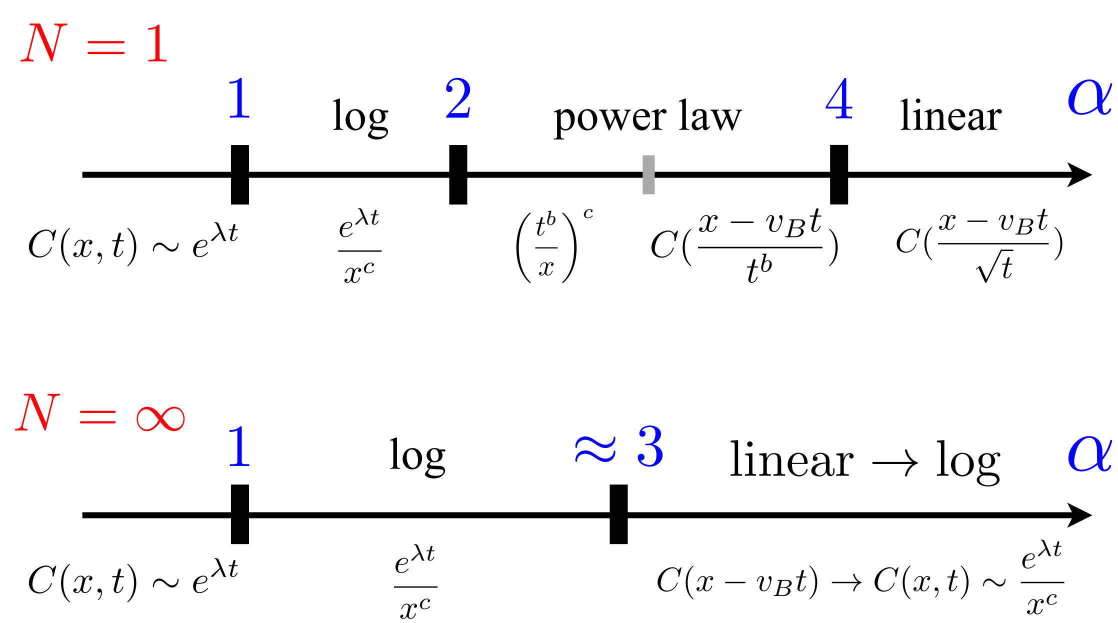


$$\overline{h(x,t)} \sim \frac{e^{\lambda t}}{x^c}$$

Summary and Outlook

Operator dynamics → (Brownian circuit approach) → Classical stochastic problem

Power law interaction systems



Possible future directions:

(1) The regime $2<\alpha<3$

(2) 1/N correction

(3) Entanglement dynamics after quench

(4) Applying this Brownian circuit technique to other interacting systems

(5) Operator dynamics at finite temperature